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A directed version of Deza graphs—Deza digraphs*

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Abstract

As a generalization of Deza graphs, we introduce Deza digraphs and describe the basic theory of these graphs. We also prove the necessary and sufficient conditions when a weakly distance-regular digraph is a Deza digraph.

1 Introduction

In [2], Erickson, Fernando, Haemers, Hardy and Hemmeter introduced Deza graphs as a generalization of strongly regular graphs. They introduced several ways to construct Deza graphs, and developed some basic theory.

Definition 1.1 Suppose Γ is an undirected graph with n vertices, and A is its adjacency matrix. Γ is called an (n, k, b, c)-*Deza graph* if

$$A^2 = bB + cC + kI,$$

$$AJ = JA = kJ,$$

for some $(0, 1)$-matrices B and C such that $B + C + I = J$, the all ones matrix.

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Note that Γ is a strongly regular graph if and only if B or C is A.

In this paper, we consider the directed version of Deza graphs and develop some basic theory. Moreover, we discuss the connections to weakly distance-regular digraphs.

Definition 1.2 Let Γ be a digraph with n vertices and let A be the adjacency matrix of Γ. Γ is said to be an (n, k, b, c, t)-*Deza digraph* if

$$A^2 = bB + cC + tI,$$

$$AJ = JA = kJ$$

for some $(0, 1)$-matrices B and C such that $B + C + I = J$, the all ones matrix.

Note that if $t = k$, then an (n, k, b, c, t)-Deza digraph is an (n, k, b, c)-Deza graph.

It is easy to see that we can get an equivalent definition of Deza digraphs from a combinatorial view point.

Definition 1.3 A digraph Γ with n vertices is an (n, k, b, c, t)-*Deza digraph* if for $u, v \in V(\Gamma)$,

$$|N_{u,v}| = \begin{cases} b \text{ or } c, & \text{if } u \neq v, \\ t, & \text{if } u = v, \end{cases}$$

where $N_{u,v} = \{w \in V(\Gamma) \mid \partial(u, w) = \partial(w, v) = 1\}$.

We next give some elementary constraints on the parameters.

Proposition 1.1 *Let Γ be an (n, k, b, c, t)-Deza digraph. Define, for a vertex u,*

$$\alpha = |\{v \in V(\Gamma) \mid |N_{u,v}| = b\}|, \quad \beta = |\{v \in V(\Gamma) \mid |N_{u,v}| = c\}|.$$

Then α and β do not depend on u and

$$\alpha = \begin{cases} \frac{k^2-t}{b}, & \text{if } b = c, \\ \frac{k^2-t+c-nc}{b-c}, & \text{if } b \neq c, \end{cases}$$

$$\beta = \begin{cases} \frac{k^2-t}{b}, & \text{if } b = c, \\ \frac{k^2-t+b-nb}{c-b}, & \text{if } b \neq c. \end{cases}$$

Proof. Let N be the number of ordered triples (u, w, v) with $\partial(u, w) = \partial(w, v) = 1$ and $u \neq v$. That is,

$$N = |\{(u, w, v) \mid \partial(u, w) = \partial(w, v) = 1, u \neq v\}|.$$

If $\partial(w, u) \neq 1$, then the number of triples (u, w, v) is $(k - t)k$, while if $\partial(w, u) = 1$, then the number of triples (u, w, v) is $(k - 1)t$. Thus

$$N = k^2 - t.$$

If $b = c$, then

$$N = \alpha b = \beta c.$$

Otherwise, by the definition of α and β, we have

$$N = \alpha b + \beta c.$$

If we equate these two expressions for N and use $\alpha + \beta = n - 1$, then α and β are obtained. ■

Corollary 1.2 *Suppose $b < c$. Then the following hold:*

(*i*) *$c - b$ divides $nc - c - k^2 + t$;*

(*ii*) *if $\alpha\beta \neq 0$, then $(n-1)b < k^2 - t < (n-1)c$.*

2 Constructions

Firstly we will give a construction of Deza digraphs using Cayley digraphs.

Proposition 2.1 *Let G be a finite group of order n and let S be a k-subset of G not containing the identity element e of G. If*

$$S^2 = bB \cup cC \cup t\{e\},$$

where B, C and $\{e\}$ partition G, then the Cayley digraph $Cay(G, S)$ is an (n, k, b, c, t)-Deza digraph.

Proof. The proof is obvious and will be omitted. ■

Let Γ_1 and Γ_2 be digraphs. The *lexicographic product* $\Gamma_1[\Gamma_2]$ of Γ_1 and Γ_2 is a digraph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and adjacency defined by

$$\partial((u_1, u_2), (v_1, v_2)) = 1 \text{ if and only if } \partial(u_1, v_1) = 1 \text{ or } u_1 = v_1, \partial(u_2, v_2) = 1.$$

Let Γ be a digraph with adjacency matrix A and n vertices. Γ is called a *strongly regular digraph with parameters* (n, k, μ, λ, t), if

$$A^2 = tI + \lambda A + \mu(J - I - A),$$

$$JA = AJ = kJ.$$

The parameters are related by the equation

$$k(k + (\mu - \lambda)) = t + (n-1)\mu.$$

These graphs were first investigated by Duval in [1]. For more information about strongly regular digraphs, see [3], [4].

Note that if B or C is A, then a Deza digraph is a strongly regular digraph.

The next theorem tells us how to derive a Deza digraph using a strongly regular digraph.

Theorem 2.2 *Let Γ_1 be a strongly regular digraph with parameters (n,k,λ,μ,t) and let Γ_2 be an (n',k',b,c,t')-Deza digraph. Then $\Gamma_1[\Gamma_2]$ is a Deza digraph if and only if*

$$|\{b+tn', c+tn', \mu n', \lambda n'+2k'\}| \le 2.$$

Proof. Let $u=(u_1,u_2)$ and $v=(v_1,v_2)$ be vertices of $\Gamma_1[\Gamma_2]$. Then

$$|N_{u,v}| = \begin{cases} b+tn', & \text{if } u_1=v_1 \text{ and } |N_{u_2,v_2}|=b,\\ c+tn', & \text{if } u_1=v_1 \text{ and } |N_{u_2,v_2}|=c,\\ \lambda n'+2k', & \text{if } \partial(u_1,v_1)=1,\\ \mu n', & \text{if } \partial(u_1,v_1)>1. \end{cases}$$

Hence $\Gamma_1[\Gamma_2]$ is a Deza digraph if and only if these numbers take on at most two values. ■

Two corollaries follow easily from the theorem.

Corollary 2.3 *Let Γ be a strongly regular digraph with parameters $(n,k,\lambda,\mu,0)$, and let K_m be a complete graph on m vertices. Then $\Gamma[K_m]$ is a Deza digraph if and only if*

$$\mu-\lambda=1 \text{ and } m=2.$$

Corollary 2.4 *Let Γ_1 be a strongly regular digraph with parameters (n,k,λ,λ,t), and let $\overline{K_{n'}}$ be a coclique on n' vertices. Then $\Gamma_1[\overline{K_{n'}}]$ is an $(nn',kn',\lambda n',\lambda n',tn')$-Deza digraph.*

Theorem 2.5 *Let Γ_1 and Γ_2 be two digraphs. The product $\Gamma_1\times\Gamma_2$ of Γ_1 and Γ_2 is a Deza digraph if and only if it is in the list below.*

(*i*) $\Gamma_1=\overline{K}_n$ *for some* $n\ge 2$ *and* Γ_2 *is an* (n',k,b,c,t)*-Deza digraph with* $b=c$ *or* $c=0$.

(*ii*) Γ_1 *is an* (n,k,b,c,t)*-Deza digraph and* Γ_2 *is an* (n',k',b',c',t')*-Deza digraph, where* $(b,c),(b',c')\in\{(2,2),(2,0)\}$.

Proof. First note that $\Gamma_1\times\Gamma_2$ is regular if and only if both of Γ_1 and Γ_2 are regular. Moreover, the degree of $\Gamma_1\times\Gamma_2$ is the sum of the degrees of Γ_1 and Γ_2.

Now suppose $u=(u_1,u_2)$ and $v=(v_1,v_2)$ are two distinct vertices of $\Gamma_1\times\Gamma_2$. It is easy to check that

$$|N_{u,v}| = \begin{cases} |N_{u_2,v_2}|, & \text{if } u_1=v_1,\\ |N_{u_1,v_1}|, & \text{if } u_2=v_2,\\ 2, & \text{if } \partial(u_1,v_1)=\partial(u_2,v_2)=1,\\ 0, & \text{otherwise}. \end{cases}$$

Case 1. $\Gamma_1=\overline{K}_n$ for some $n\ge 2$. Then the third case for the size of $N_{u,v}$ given above does not occur. So $\Gamma_1\times\Gamma_2$ is a Deza digraph if and only if Γ_2 is an (n',k,b,c,t)-Deza digraph with $b=c$ or $c=0$. Thus (i) occurs.

Case 2, Both 0 and 2 appear as the value of $|N_{u,v}|$, so (ii) follows. ■

3 Connection to weakly distance-regular digraphs

In this section, we will discuss the connections to weakly distance-regular digraphs.

For any two vertices $x, y \in V(\Gamma)$, define $\tilde{\partial}(x,y) = (\partial(x,y), \partial(y,x))$.

Definition 3.1 A connected digraph Γ is said to be *weakly distance-regular* if

$$p^{\tilde{k}}_{\tilde{i},\tilde{j}}(x,y) = |\{z \in V(\Gamma) \mid \tilde{\partial}(x,z) = \tilde{i} \text{ and } \tilde{\partial}(z,y) = \tilde{j}\}|$$

depends only on $\tilde{k}, \tilde{i}, \tilde{j}$ and does not depend on the choices of x and y with $\tilde{\partial}(x,y) = \tilde{k}$.

As a natural generalization of distance-regular graphs, weakly distance-regular digraphs were introduced in [5].

Theorem 3.1 *Let Γ be a weakly distance-regular digraph of diameter d. Let*

$$F = \{(1,i) \mid \tilde{\partial}(x,y) = (1,i) \text{ for some } x, y \in V(\Gamma)\}.$$

Then Γ is a Deza digraph if and only if

$$\sum_{\tilde{i},\tilde{j} \in F} p^{\tilde{k}}_{\tilde{i},\tilde{j}}$$

takes on at most two values as $\tilde{k}$ ranges over $\{\tilde{\partial}(x,y) \mid x, y \in V(\Gamma)\}$.

Proof. Let u and v be two vertices of Γ with $\tilde{\partial}(u,v) = \tilde{k}$. Then

$$|N_{u,v}| = \sum_{\tilde{i},\tilde{j} \in F} p^{\tilde{k}}_{\tilde{i},\tilde{j}}.$$

Hence, Γ is a Deza digraph only when these numbers take on at most two values. ■

We know that $\Gamma = \mathrm{Cay}(Z_n \times Z_2, \{(1,0),(0,1)\})$ is a weakly distance-regular digraph. By the above theorem, it is a Deza digraph.

Note that a weakly distance-regular digraph is *distance-regular* if $\partial(x,y) = \partial(x',y')$ implies $\partial(y,x) = \partial(y',x')$ for all $x, y, x', y' \in V(\Gamma)$.

Corollary 3.2 *A distance-regular digraph Γ of diameter d is a Deza digraph if and only if one of the following holds.*

(*i*) $d = 2$,

(*ii*) $p^1_{1,1} = 0$,

(*iii*) $p^2_{1,1} = p^1_{1,1}$.

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