AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **28** (2003), Pages 209–215

Helly-type problems for convex quadrilaterals in the plane*

Liping Yuan Ren Ding

College of Mathematics
Hebei Normal University
Shijiazhuang 050016
China
lpyuan@heinfo.net rending@heinfo.net

Abstract

A family of sets in the plane is said to have a 3-transversal if there exists a set of 3 points such that each member of the family contains at least one of them. A conjecture of Grünbaum's says that a planar family of translates of a convex compact set has a 3-transversal provided that any two of its members intersect. Recently the conjecture has been proved affirmatively (see [1]). We provide a straightforward proof for the conjecture for the family of translates of a closed convex quadrilateral without parallel sides in the plane. Moreover, in our proof we obtain exactly the 3-transversals, i.e. the concrete 3-point sets the conjecture claims. The proof is valid for some other convex polygons and it is likely that we can prove the conjecture in the same straightforward way.

For brevity's sake, a family of sets is said to be Π^3, or to have a 3-*transversal* if there exists a set of 3 points such that each member of the family contains at least one of them. The family is said to be Π_2^1 if every two sets of the family have a nonempty intersection. Grünbaum's conjecture says

Conjecture *For a family of translates of a compact convex set in the plane,* Π_2^1 *implies* Π^3.

In a recent paper by M. Katchalski and D. Nastir (see [2]) the above conjecture of Grünbaum was mentioned again. Karasev [1] gives an affirmative answer to the conjecture. We provide a straightforward proof for the conjecture for the case of a quadrilateral instead of a general compact convex set. In the same way we can prove the conjecture for triangles, parallelograms and trapezoids, etc. Accordingly, it is

* This research was supported by NSFH and SFHEM

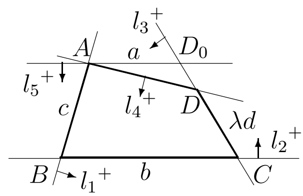

Figure 1: Quadrilateral $ABCD$ and its related trapezoid $ABCD_0$

likely to prove the conjecture in a similar straightforward way. The proof is completely different from [1]. Moreover, in our proof we obtain exactly the 3-transversals, i.e. the concrete 3-point sets the conjecture claims.

First, we introduce some terminology.

Let x^+ and x^- denote half-planes bounded by the line x with $x^+ \cap x^- = x$.

Two half-planes are *related* if one of them is a translate of the other. Related half-planes are ordered by inclusion so that $x^+ < y^+$ implies that x^+ is contained in y^+ and $x^+ \neq y^+$.

Let l_{AB} denote the line determined by points A and B. Let *relint* AB denote the relative interior of the line segement AB, that is, the line segement without its endpoints. Let *int* $ABCD$ denote the interior of the polygon $ABCD$.

Theorem 1. *For a family* $\mathbb{K}$ *of translates of a closed convex quadrilateral* $ABCD$ *with* $AB \nparallel CD$ *and* $AD \nparallel BC$, Π_2^1 *implies* Π^3.

Proof. See Fig. 1. Note that convex quadrilateral $ABCD = \cap_{i=1}^4 l_i^+$, where $l_1 \cap l_4 = A$, $l_1 \cap l_2 = B$, $l_2 \cap l_3 = C$ and $l_3 \cap l_4 = D$. Assume without loss of generality that both l_3 and l_4 intersect the two open rays $l_1 \backslash l_2^-$ and $l_2 \backslash l_1^-$. Through A draw the line $l_5 \parallel l_2$, intersecting l_3 at D_0. Note that the trapezoid $ABCD_0 = \cap_i l_i^+ (i = 1, 2, 3, 5)$ and we define it as the *related trapezoid* of the convex *quadrilateral* $ABCD$. Let $|AD_0| = a$, $|BC| = b$, $|CD_0| = d$ and $|AB| = c$. Assume without loss of generality that $|CD| = \lambda d$ where $\frac{1}{2} < \lambda < 1$.

For any $K \in \mathbb{K}$, let K' be the related trapezoid of K, and note that $K' = \cap_i l_i^+(K')$ $(i = 1,\ 2,\ 3,\ 5)$ where $l_i^+(K')$ is related to l_i^+ for $i = 1, 2, 3, 5$. Let $\mathbb{K}' = \{K' : K'$ is a related trapezoid of K, $K \in \mathbb{K}\}$. Then $\mathbb{K}'$ is Π_2^1 since $\mathbb{K}$ is Π_2^1. We denote $l_i^+(\mathbb{K}') = \max\left\{l_i^+(K') : K' \in \mathbb{K}'\right\}$ and suppose $l_i^+(K_i') = l_i^+(\mathbb{K}')$ $(i = 1, 2, 3, 5)$, and that the trapezoid $A_1B_1C_1D_1 = \cap_i l_i^+(\mathbb{K}')$ $(i = 1, 2, 3, 5\ )$; we have $\cup_{K' \in \mathbb{K}'} K' \subseteq A_1B_1C_1D_1$, and therefore $\cup_{K \in \mathbb{K}} \subseteq A_1B_1C_1D_1$. Let $|A_1D_1| = a_1,\ |B_1C_1| = b_1,\ |A_1B_1| = c_1$ and $|C_1D_1| = d_1$ and let $\frac{c_1}{c} = \frac{d_1}{d} = \beta$. Then $1 \leq \beta \leq 2$ since $\mathbb{K}'$ is Π_2^1.

$\boxed{\text{Case 1}}$ $\beta = 1$

See Fig. 2. Note that $\beta = 1$ implies $b_1 \leq 2b$ since $\mathbb{K}$ is Π_2^1. Then $a_1 = b_1 - (b - a) \leq a + b$. Let M_1, N_1 be midpoints of A_1B_1 and C_1D_1 respectively and let

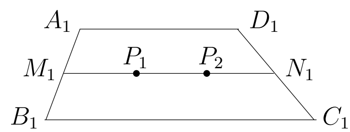

Figure 2: $\beta = 1$

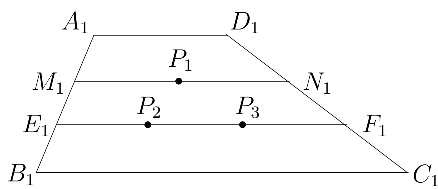

Figure 3: $1 < \beta \le 2$, $b < b_1 \le 2b - a$

$|M_1P_1| = |P_1P_2| = |P_2N_1|$.

For any $K \in \mathbb{K}$, let $MN = K \cap M_1N_1$; then $|MN| = \frac{a+b}{2}$ since $l_2(K) = l_{B_1C_1}$ and $\lambda > \frac{1}{2}$. However, $|M_1P_1| = |P_1P_2| = |P_2N_1| = \frac{1}{3}|M_1N_1| = \frac{a_1+b_1}{6} \le \frac{a+3b}{6}$. Thus $|MN| > |M_1P_1| = |P_1P_2| = |P_2N_1|$. It follows that $MN \cap \{P_1, P_2\} \neq \emptyset$ and therefore $K \cap \{P_1, P_2\} \neq \emptyset$, which implies that $\mathbb{K}$ is Π^3.

$\boxed{\text{Case 2}}$ $1 < \beta \le 2$.

It is easy to see that $b_1 > b$.

<u>Subcase 2.1</u> $b < b_1 \le 2b - a$.

See Fig. 3. $|B_1M_1| = c$. $M_1N_1 \parallel B_1C_1$. So $|C_1N_1| = d$. Let P_1 be the midpoint of M_1N_1 and let E_1, F_1 be midpoints of B_1M_1, C_1N_1. On the line segment E_1F_1, set $|E_1P_2| = |P_2P_3| = |P_3F_1|$.

For any $K \in \mathbb{K}$, it is easy to see that $l_2(K)$ must lie between $l_{B_1C_1}$ and $l_{M_1N_1}$.

(a) $l_2(K)$ lies between $l_{B_1C_1}$and $l_{E_1F_1}$.

Let $EF = K \cap E_1F_1$. It is easy to see that $|EF| \ge \frac{a+b}{2}$ since $\lambda > \frac{1}{2}$. However, $|E_1F_1| = \frac{2b_1-b+a}{2}$ implies that $|E_1P_2| = |P_2P_3| = |P_0F_1| = \frac{2b_1-b+a}{6}$, so we have $|EF| \ge |P_2E_1| = |P_2P_3| = |P_3F_1|$. It follows that $EF \cap \{P_2, P_3\} \neq \emptyset$, and therefore $K \cap \{P_1, P_2, P_3\} \neq \emptyset$.

(b) $l_2(K)$ lies between $l_{E_1F_1}$and $l_{M_1N_1}$.

Let $MN = K \cap M_1N_1$. It is easy to see that $|MN| \ge \frac{a+b}{2}$ and $|M_1N_1| = b_1 - b + a$. Then we have $|M_1P_1| = |P_1N_1| = \frac{b_1-b+a}{2}$, which implies that $|MN| > |M_1P_1| = |P_1N_1|$. As a result, $P_1 \in MN$ and therefore $K \cap \{P_1, P_2, P_3\} \neq \emptyset$.

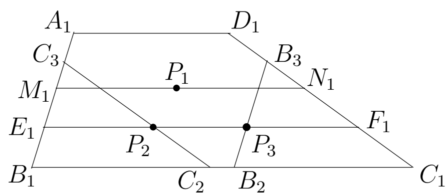

Figure 4: $1 < \beta \leq 2$, $2b - a < b_1 \leq 2b$

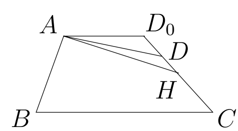

Figure 5: Quadrilateral $ABCD$ and its related trapezoid $ABCD_0$

<u>Subcase 2.2</u> $2b - a < b_1 \leq 2b$.

See Fig. 4. Here $|B_1B_2| = |C_1C_2| = b$, $B_2B_3 \parallel A_1B_1$, $C_2C_3 \parallel C_1D_1$, $|B_1M_1| = c$, $M_1N_1 \parallel B_1C_1$. Let E_1, F_1 be midpoints of B_1M_1 and C_1N_1. Draw E_1F_1, meeting C_2C_3 at P_2 and meeting B_2B_3 at P_3. Then $\{P_2, P_3\} \subseteq relint\ E_1F_1$ since $|E_1F_1| = \frac{2b_1-b+a}{2} > b$, $P_2 \in relint\ E_1P_3$ since $|E_1F_1| < 2b$. Let P_1 be the midpoint of M_1N_1.

For any $K \in \mathbb{K}$, it is easy to see that $l_2(K)$ lies between $l_{B_1C_1}$ and $l_{M_1N_1}$ since $K \cap K_2 \neq \emptyset$.

(a) $l_2(K)$ lies between $l_{B_1C_1}$ and $l_{E_1F_1}$.

Let $EF = K \cap E_1F_1$, then $|EF| \geq \frac{a+b}{2}$ since $\lambda > \frac{1}{2}$. Notice that $|P_2P_3| = \frac{5b-2b_1-a}{2} > 0$ and $|EF| > |P_2P_3|$. Moreover, $l_{C_2C_3} \subseteq l_3^+(K)$ and $l_{B_2B_3} \subseteq l_1^+(K)$ since $\mathbb{K}$ is Π_2^1, and it is clear to see that $P_2 \in EF$ when $l_3(K) = l_{C_2C_3}$ and $P_3 \in EF$ when $l_1(K) = l_{B_2B_3}$. Then we conclude that $EF \cap \{P_2, P_3\} \neq \emptyset$ which implies $K \cap \{P_1, P_2, P_3\} \neq \emptyset$.

(b) $l_2(K)$ lies between $l_{E_1F_1}$ and $l_{M_1N_1}$.

Let $MN = K \cap M_1N_1$, then $|MN| \geq \frac{a+b}{2}$ since $\lambda > \frac{1}{2}$. However, $|M_1P_1| = |P_1N_1| = \frac{b_1-b+a}{2}$ implies that $|MN| \geq |M_1P_1| = |P_1N_1|$, then we have $P_1 \in MN$ and therefore $K \cap \{P_1,\ P_2,\ P_3\} \neq \emptyset$.

<u>Subcase 2.3</u> $2b < b_1 < \frac{5b-a}{2}$.

See Fig. 5. In the trapezoid $ABCD_0$, let H be the midpoint of side CD_0. Then $H \in relint\ CD$ since $\lambda > \frac{1}{2}$. Draw AH, then $AH \subseteq int\ ABCD$.

See Fig. 6. In the trapezoid $A_1B_1C_1D_1$, $|B_1M_1| = c$, $M_1N_1 \parallel B_1C_1$, $|B_1B_2| = |C_1C_2| = b$. So, $B_2 \in relint\ B_1C_2$ since $b_1 > 2b$. $B_2B_3 \parallel A_1B_1$, $C_2C_3 \parallel C_1D_1$. Then

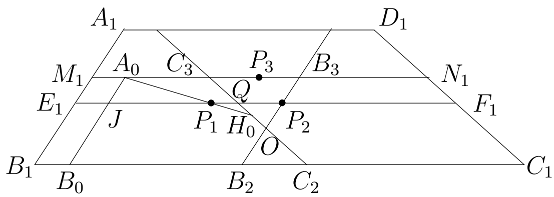

Figure 6: $1 < \beta \leq 2$, $2b < b_1 \leq \frac{5b-a}{2}$

$\{B_3, C_3\} \subseteq relint\ M_1N_1$ since $|M_1N_1| = b_1-b+a > b+a > b$. Let $O = B_2B_3 \cap C_2C_3$, it is easy to see that $O \in int\ A_1B_1C_1D_1$ since $B_2 \in relint\ B_1C_2$. Let P_2 be the midpoint of OB_3. Through P_2 draw $E_1F_1 \parallel B_1C_1$, meeting C_2C_3 at Q. On the line segment B_1C_2 set $|B_0C_2| = b$, then $B_0 \in relint\ B_1C_2$ since $|B_1C_2| > b$. Through B_0 construct $B_0A_0 \parallel A_1B_1$ meeting E_1F_1 at J. Then the trapezoid $A_0B_0C_2C_3 \backsimeq$ the trapezoid $ABCD_0$. Let H_0 be the midpoint of C_2C_3. Since $|B_2C_2| = b_1 - 2b$, we have $|OC_2| = \frac{b_1-2b}{b-a}d < \frac{\frac{5b-a}{2}-2b}{b-a}d = \frac{d}{2}$, and therefore $O \in relint\ C_2H_0$. However, $|OC_3| = \frac{3b-a-b_1}{b-a}d$, so $|C_2Q| = |OC_2| + \frac{1}{2}|OC_3| = \frac{b_1-b-a}{2(b-a)}d > \frac{d}{2}$. It follows that $H_0 \in relint\ C_2Q$. Let $P_1 = A_0H_0 \cap E_1F_1$, then $P_1 \in relint\ JQ$. Let P_3 be the midpoint of M_1N_1.

For any $K \in \mathbb{K}$, it is easy to see that $l_2(K)$ must lie between $l_{B_1C_1}$ and $l_{M_1N_1}$ since $\mathbb{K}$ is Π_2^1.

(a) $l_2(K)$ lies between $l_{B_1C_1}$and $l_{E_1F_1}$.

Let $EF = K \cap E_1F_1$; then $|EF| \geq |JP_1|$. Since $|E_1F_1| = \frac{b_1+b+a}{2}$, we have $|JQ| = \frac{3b-b_1+a}{2}$. Since $|H_0Q| = \frac{b_1-2b}{2(b-a)}d$, it follows that $|P_1Q| = \frac{a(b_1-2b)}{b-a}$. So we have $|JP_1| = |JQ| - |P_1Q| = \frac{(b-a)(3b-b_1+a)-2(b_1-2b)a}{2(b-a)}$ and $|P_1P_2| = |JP_2| - |JP_1| = 3b - b_1 - |JP_1|$. Then $|EF| - |P_1P_2| \geq |JP_1| - |P_1P_2| = 2|JP_1| - (3b - b_1) > 0$. It follows that $|EF| > |P_1P_2|$. Moreover, $l_{C_2C_3} \subseteq l_3^+(K)$ and $l_{B_2B_3} \subseteq l_1^+(K)$ since $\mathbb{K}$ is Π_2^1, $P_1 \in EF$ when $l_3(K) = l_{C_2C_3}$ and $P_2 \in EF$ when $l_1(K) = l_{B_2B_3}$, so we conclude that $EF \cap \{P_1, P_2\} \neq \emptyset$ which implies $K \cap \{P_1, P_2, P_3\} \neq \emptyset$.

(b) $l_2(K)$ lies between $l_{E_1F_1}$ and $l_{M_1N_1}$.

Let $MN = K \cap M_1N_1$; then $|MN| \geq \frac{b_1-b+a}{2}$ since $|F_1N_1| = \frac{3b-a-b_1}{2(b-a)}d < \frac{d}{2} < \lambda d$. However, $|M_1P_3| = |P_3N_1| = \frac{b_1-b+a}{2}$ implies that $|MN| \geq |M_1P_3| = |P_3N_1|$, so it follows that $P_3 \in MN$ and therefore $K \cap \{P_1, P_2, P_3\} \neq \emptyset$.

<u>Subcase 2.4</u> $\frac{5b-a}{2} \leq b_1 < 3b - a$.

See Fig. 7. Here $|B_1M_1| = c$, $M_1N_1 \parallel B_1C_1$, $|B_1B_2| = |C_1C_2| = b$, $B_2P_2 \parallel A_1B_1$, $C_2P_1 \parallel C_1D_1$. Let E_1, F_1 be the midpoints of B_1M_1 and C_1N_1. Draw E_1F_1 intersecting B_2P_2 at Q and C_2P_1 at P_3. Let $O = l_{P_1C_2} \cap l_{P_2B_2}$. Then $O \in relint\ P_1C_2$ since $|OC_2| = \frac{b_1-2b}{b-a}d < d$ and $P_1 \in relint\ M_1P_2$ since $|M_1N_1| = b_1 - b + a < 2b$. We also have $|P_3C_2| = \frac{d}{2}$ and $|OC_2| \geq |P_3C_2|$, it follows that $P_3 \in relint\ OC_2 \cup \{O\}$ and

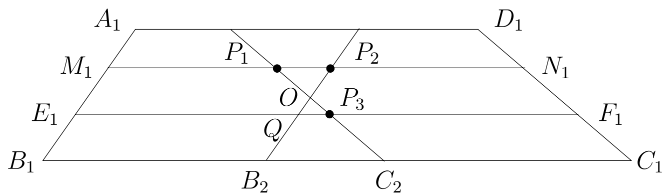

Figure 7: $1 < \beta \leq 2$, $\frac{5b-a}{2} \leq b_1 < 3b - a$

$Q \in relint\ E_1P_3 \cup \{P_3\}$.

For any $K \in \mathbb{K}$, $l_2(K)$ lies between $l_{B_1C_1}$ and $l_{M_1N_1}$ since $\mathbb{K}$ is Π_2^1.

(a) $l_2(K)$ lies between $l_{B_1C_1}$ and $l_{E_1F_1}$.

Let $EF = K \cap E_1F_1$; then $|EF| \geq \frac{a+b}{2}$ since $\lambda > \frac{1}{2}$. At the same time, we have $|EF| > |QP_3|$ since $|QP_3| = \frac{2b_1-5b+a}{2}$. Noticing that $l_{P_1C_2} \subseteq l_3^+(K)$ and $l_{B_2P_2} \subseteq l_1^+(K)$ since $\mathbb{K}$ is Π_2^1, and $P_3 \in EF$ when $l_3(K) = l_{P_1C_2}$ since $|P_3C_2| < \lambda d$, we conclude that $P_3 \in EF$, which implies that $K \cap \{P_1, P_2, P_3\} \neq \emptyset$.

(b) $l_2(K)$ lies between $l_{E_1F_1}$and $l_{M_1N_1}$.

Let $MN = K \cap M_1N_1$; then $|MN| \geq \frac{a+b}{2}$. Combining $|P_1P_2| = 3b - a - b_1 > 0$, we find that $|MN| \geq |P_1P_2|$. It is easy to see that $l_{P_1C_2} \subseteq l_3^+(K)$ and $l_{P_2B_2} \subseteq l_1^+(K)$. Moreover, we have $P_1 \in MN$ if $l_3(K) = l_{P_1C_2}$ and $P_2 \in MN$ if $l_2(K) = l_{P_2B_2}$. It follows that $MN \cap \{P_1, P_2\} \neq \emptyset$, and therefore $K \cap \{P_1, P_2, P_3\} \neq \emptyset$.

<u>Subcase 2.5</u> $b_1 = 3b - a$.

See Fig. 8. Now $|B_1M_1| = c$. $M_1N_1 \parallel B_1C_1$. Hence $|M_1N_1| = b_1 - b + a = 2b$. So we have $l_2(K_1) = l_2(K_3) = l_{M_1N_1}$ since $K_1 \cap K_3 \neq \emptyset$. Thus $\beta = 2$. Let P be the midpoint of M_1N_1; then $P \in l_1(K_2)$ since $K_1 \cap K_2 \neq \emptyset$ and $K_2 \cap K_3 \neq \emptyset$.

For any $K \in \mathbb{K}$, it is easy to see that $l_2(K)$ lies between $l_{B_1C_1}$ and $l_{M_1N_1}$.

(a) $l_2(K) = l_{B_1C_1}$.

It is easy to see that $K = K_2$ since $K \cap K_1 \neq \emptyset$ and $K \cap K_3 \neq \emptyset$. Thus $P \in K$.

(b) $l_2(K)$ lies between $l_{B_1C_1}$ and $l_{M_1N_1}$ (excluding $l_{B_1C_1}$ and $l_{M_1N_1}$).

Let $MN = K \cap M_1N_1$. If $P \notin MN$, then either $MN \subseteq M_1P \backslash \{P\}$ or $MN \subseteq PN_1 \backslash \{P\}$. Therefore either $K \cap K_3 = \emptyset$ or $K \cap K_1 \neq \emptyset$, a contradiction since $\mathbb{K}$ is Π_2^1. So we have $P \in MN$, which implies $P \in K$.

(c) $l_2(K) = l_{M_1N_1}$.

Let $MN = K \cap M_1N_1$; then $|MN| = b$. However, $|M_1P| = |PN_1| = b$. So we have $P \in MN$, and therefore $P \in K$.

By (a), (b), (c), we can conclude that $\mathbb{K}$ is Π^1. Therefore $\mathbb{K}$ is Π^3 if $b_1 = 3b - a$.

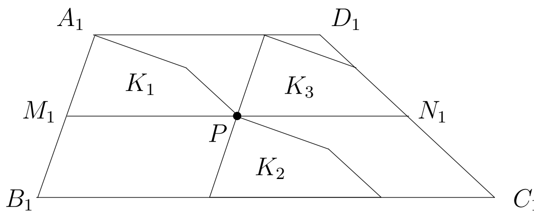

Figure 8: $1 < \beta \leq 2$, $b_1 = 3b - a$

Subcase 2.6 $b_1 > 3b - a$.

It is easy to see that $K_1 \cap K_3 = \emptyset$ if $b_1 > 3b - a$, a contradiction since $\mathbb{K}$ is Π_2^1. So b_1 cannot be greater than $3b - a$.

The proof is complete. $\square$

References

[1] R. N. Karasev, Transversals for families of translates of a two-dimensional convex compact set, *Discrete Comput. Geom.* **24** (2000), 345–353.

[2] M. Katchalski and D. Nashtir, A Helly type conjecture, *Discrete Comput. Geom.* **21** (1999), 37–43.

(Received 10 June 2002)