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Minimum degree conditions for the Overfull Conjecture for odd order graphs

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Abstract

The Overfull Conjecture states that a graph G with $3\Delta(G) \geq n(G)$ is Class 2 if and only if it has a $\Delta(G)$-overfull subgraph. M. Plantholt showed that the Overfull Conjecture is true for graphs with even order and high minimum degree. In this paper we look at graphs with odd order and show under which restrictions the Overfull Conjecture holds true for these.

1 Introduction

All graphs considered are finite and simple. We use standard graph terminology. By τ_d we denote the number of vertices of degree d in a graph. If $S \subset V(G)$ is a set of vertices of G, then S^C denotes the vertex set $V(G) \setminus S$ and $G[S]$ denotes the subgraph of G induced by S. The chromatic index $\chi'(G)$ of a graph G denotes the minimal number of colours needed to colour the edges of the graph such that incident edges receive different colours. It is obvious that $\chi'(G) \geq \Delta(G)$ always, where $\Delta(G)$ is the maximum degree of G. The classical result of Vizing [10] states that $\chi'(G) \leq \Delta(G) + 1$. Vizing's result gave rise to a classification of graphs into Class 1 if $\chi'(G) = \Delta(G)$, and Class 2 if $\chi'(G) = \Delta(G) + 1$. For regular graphs, being Class 1 is equivalent to the graph being 1-factorable. The well known 1-*Factorization Conjecture* states that every Δ-regular graph G is 1-factorable if $2\Delta \geq |V(G)|$.

The best known result concerning this conjecture has been proved independently by Chetwynd and Hilton [2] as well as Niessen and Volkmann [7].

Theorem 1.1 (Chetwynd and Hilton [2], Niessen and Volkmann [7]) *Let G be a Δ-regular graph with even order n. If $2\Delta \geq (\sqrt{7} - 1)n$, then G is 1-factorable.*

In the following we only consider odd order subgraphs of G. If H is a subgraph of G such that $|V(H)| \geq 3$ is odd and $|E(H)| \geq 1$, then any matching in H can contain at most $\frac{|V(H)|-1}{2}$ edges. Consequently, $\chi'(H) \geq \frac{2|E(H)|}{|V(H)|-1}$. As $\chi'(G) \geq \chi'(H)$ for any subgraph H of G, we know that G is Class 2, if $\frac{2|E(H)|}{|V(H)|-1} > \Delta(G)$. Such a subgraph H is called $\Delta(G)$-overfull. In the case that we have equality, the subgraph is called $\Delta(G)$-full. It is easy to see that $\Delta(H) = \Delta(G)$ for any $\Delta(G)$-overfull subgraph. Niessen [5] showed that the odd order of a $\Delta(G)$-overfull subgraph is necessary. Chetwynd and Hilton [1] conjectured that a graph G with $\Delta(G) > \frac{|V(G)|}{3}$ is Class 2 if and only if G has a $\Delta(G)$-overfull subgraph. This conjecture is known as the *Overfull Conjecture*.

In [5] and [6] Niessen presented algorithms which find a $\Delta(G)$-overfull subgraph in polynomial time. As a corollary of his algorithms, Niessen has been able to prove the following theorem.

Theorem 1.2 (Niessen [5], [6]) *A graph G with $3\Delta(G) \geq |V(G)|$ contains at most three different $\Delta(G)$-overfull subgraphs. If G meets $2\Delta(G) \geq |V(G)|$, then G contains at most one $\Delta(G)$-overfull subgraph.*

For graphs of even order with sufficiently high minimum degree, Plantholt [8] was able to give the following structural result concerning overfull subgraphs.

Lemma 1.3 (Plantholt [8]) *Let G be a graph with even order n and minimum degree $\delta(G) > \frac{n}{2}$. If $S \subseteq V(G)$, with $|S|$ odd, induces a $\Delta(G)$-full or $\Delta(G)$-overfull subgraph in G, then $S = V(G) - x$ for a vertex $x \in V(G)$.*

The equivalent for graphs with odd order is the following lemma.

Lemma 1.4 *Let G be a graph with odd order $n \geq 5$ and minimum degree $\delta(G) \geq \frac{n+1}{2}$. If $S \subseteq V(G)$, with $|S|$ odd, induces a $\Delta(G)$-full or $\Delta(G)$-overfull subgraph in G, then $S = V(G)$.*

Proof. Assume that $S \subseteq V(G)$, with $|S| \leq n - 2$ odd, induces a $\Delta(G)$-full or $\Delta(G)$-overfull subgraph in G. Then

$$\Delta(G[S]) \leq \Delta(G) \leq \frac{2|E(G[S])|}{|S|-1} \leq \frac{\Delta(G[S])|S|}{|S|-1} \leq \Delta(G[S]) + 1.$$

It follows that $|S| \geq \Delta(G) \geq \delta(G) \geq \frac{n+1}{2}$ and for the complement S^C of S we get $2 \leq |S^C| \leq \frac{n-1}{2}$. Let us take a look at the number of edges leading from S to S^C. On the one hand we have

$$e_G(S, S^C) \leq \Delta(G)|S| - \Delta(G)(|S| - 1) \leq |S| = n - |S^C|.$$

On the other hand

$$e_G(S, S^C) \geq \delta(G)|S^C| - 2|E(G[S^C])| \geq |S^C|\frac{n + 3 - 2|S^C|}{2}.$$

Combining these two inequalities leads to $0 \leq |S^C|^2 - |S^C|\frac{n+5}{2} + n$ which gives a contradiction as the right-hand-side is negative for $2 \leq |S^C| \leq \frac{n-1}{2}$ and $n \geq 5$. Thus our assumption was wrong and the statement follows as $|S|$ is odd. □

Lemma 1.5 *Let G be a graph of order n with $\delta(G) \geq \lceil n/2 \rceil$. If G has a $\Delta(G)$-overfull subgraph or is $\Delta(G)$-overfull, then the following hold:*

- *if n is even, then $\tau_\delta = 1$;*
- *if n is odd, then $\tau_\Delta > n - \delta$.*

Proof. The first statement is an immediate consequence of Lemma 1.3, Theorem 1.2 and the definition of a $\Delta(G)$-overfull subgraph if n is even. In the other case G is $\Delta(G)$-overfull by Lemma 1.4. Assume that $\tau_\Delta \leq n - \delta$. Add a vertex x to G and connect x to δ vertices of degree $< \Delta$ such that at least one vertex of degree δ, which we denote with y, is not connected to x. The resulting graph G' has maximum degree $\Delta(G') = \Delta(G)$, minimum degree $\delta(G') = \delta(G)$ and even order $n + 1$. As G is a $\Delta(G)$-overfull subgraph of G', the existence of y and x with $d_{G'}(y) = d_{G'}(x) = \delta(G')$ contradicts the first case of this corollary. □

The Overfull Conjecture has only been validated in very restricted cases so far, with the following results of Plantholt [8] being among the most recent ones.

Theorem 1.6 (Plantholt [8]) *Let $c^* \geq \frac{3}{4}$ be a real number such that any regular graph with even order n and degree at least nc^* is 1-factorable. Let G be a graph with even order n, maximum degree Δ and minimum degree δ. If $3\delta - \Delta \geq 2c^*n$, then G is Class 2 if and only if G contains a $\Delta(G)$-overfull subgraph.*

Corollary 1.7 (Plantholt [8]) *Let G be a graph with even order n such that $\delta(G) + 1 \geq (\sqrt{7}/3)n$. Then G is Class 2 if and only if G contains a $\Delta(G)$-overfull subgraph.*

The object of our work is to provide analogous results for graphs with odd order. We will show that the Overfull Conjecture holds true if the graph is almost-regular or if it has at most two vertices of sufficiently low degree $\leq \Delta - 2$. Furthermore we will show that the Overfull Conjecture holds true if the number of vertices of minimum degree is large enough. We will, however, not be able to show an equivalent result to Theorem 1.6. This is due to the fact that the proof of Theorem 1.6 relies on a deep result of Seymour [9] which only holds for graphs with even order, and does not extend to odd order.

2 Graphs with odd order

Analogously to Theorem 1.6, let $c^* \geq \frac{3}{4}$ denote a real number such that any regular graph with even order n and degree at least nc^* is 1-factorable. We call a graph G degree-bounded if it has maximum degree Δ and minimum degree δ such that $3\delta - \Delta \geq 2c^*|V(G)|$. With Lemma 1.4 we know that a degree-bounded graph G with odd order has a $\Delta(G)$-overfull subgraph if and only if G is $\Delta(G)$-overfull. This fact will be used from now on without further reference to Lemma 1.4.

It is obvious that a Δ-regular graph of odd order n is $\Delta(G)$-overfull and thus Class 2, fulfilling the Overfull Conjecture. The next theorem shows that the Overfull Conjecture also holds true in the almost-regular case, meaning $\delta = \Delta - 1$, if the graph is degree-bounded.

Theorem 2.1 *Let G be a graph with odd order n, maximum degree Δ and minimum degree δ such that $\delta = \Delta - 1$. If G is degree-bounded, then G is Class 2 if and only if G is $\Delta(G)$-overfull.*

Proof. First suppose that G is $\Delta(G)$-full. Then the following holds

$$2e(G) = \tau_\Delta \Delta + (n - \tau_\Delta)(\Delta - 1) = \Delta(n-1).$$

Hence $\Delta(G) = n - \tau_\Delta = \tau_\delta$. Add a vertex v_0 to G and connect it to all vertices of minimum degree in G, thus constructing a new graph G'. It is easy to verify that G' is a $\Delta(G)$-regular graph with even order $n(G') = \frac{3\delta(G') - \Delta(G')}{2} \geq c^* n$. It follows that $\Delta(G') \geq c^*(n+1)$ and Theorem 1.6 is applicable to G', telling us that G' is Class 1. Since $\Delta(G') = \Delta(G)$, we know that G is Class 1, too.
Now suppose that G is neither $\Delta(G)$-full nor $\Delta(G)$-overfull. If $\tau_\delta = 1$, then G would be $\Delta(G)$-overfull, in contradiction to our assumption. If $\tau_\delta = 2$, it follows that $\Delta < 2$, as G is not $\Delta(G)$-overfull. Thus $\delta = 0$, contradicting $\delta > 0$. It remains the case that $\tau_\delta \geq 3$. Let H be the subgraph of G induced by the vertices of minimum degree δ. We are going to show that H is not complete, if G is not $\Delta(G)$-overfull. Assume that the opposite holds. Then we have $2e(H) = \tau_\delta(\tau_\delta - 1)$. Let us consider the edge-cut $(V(H), V(H)^c)$. Then

$$e_G(V(H), V(H)^c) = \tau_\delta \delta - 2e(H) = \tau_\delta(\Delta - 1) - 2e(H) = \tau_\delta(\Delta - \tau_\delta). \qquad (1)$$

As G is neither $\Delta(G)$-full nor $\Delta(G)$-overfull we know that $2e(G) = \tau_\Delta \Delta + (n - \tau_\Delta)(\Delta - 1) < \Delta(n-1)$. Since $\tau_\delta = n - \tau_\Delta$, it follows that $\Delta - \tau_\delta < 0$ yielding a contradiction in (1). As a consequence, H is not complete. Thus we can add an edge in H, giving us the graph G^1. Note that G^1 cannot be $\Delta(G^1)$-overfull, as G was neither $\Delta(G)$-full nor $\Delta(G)$-overfull. However, G^1 may be $\Delta(G^1)$-full. Using the same argument for G^1 as for G, we can recursively add i edges to G until G^i is $\Delta(G)$-full. But then the first part of the proof applies to G^i, giving us that G^i is Class 1. As a consequence, G is also Class 1. □

The following result on graphs with at most one vertex of degree $\leq \Delta(G) - 2$ shows that under the condition of the degree-boundedness, the Overfull Conjecture again holds.

Theorem 2.2 *Let G be a graph of odd order n with at most one vertex x of degree $\leq \Delta(G) - 2$. If G is degree-bounded, then G is Class 2 if and only if G is $\Delta(G)$-overfull.*

Proof. If G does not have a vertex with degree less than or equal to $\Delta(G) - 2$, then G is regular or almost regular and the statement follows directly or from Theorem 2.1. Otherwise let x be the vertex with $d(x, G) \leq \Delta(G) - 2$. Then $\delta(G) = d(x, G)$. As we only need to show the sufficiency of the statement, suppose that G is not $\Delta(G)$-overfull. The proof will be by induction over $k := \Delta(G) - \delta(G)$. Since G is degree-bounded, we have $\delta(G) \geq \frac{n+1}{2}$. By the classical result of Dirac [3] we can find a perfect matching M in $G - x$. We delete the edges of M in G and denote the resulting graph by G'. For G' we have $\Delta(G') - \delta(G') = k - 1$ and $2|E(G')| = 2|E(G)| - 2|M| = 2|E(G)| - (n - 1)$. As G is not $\Delta(G)$-overfull, it follows that $2|E(G')| \leq (n - 1)\Delta(G) - (n - 1) = |V(G')|\Delta(G')$. Hence G' is not $\Delta(G')$-overfull.
If $k = 2$, then G' is almost-regular and Theorem 2.1 yields that G' is Class 1. Then G is Class 1, since we can colour the edges of M with an extra colour. If $k \geq 3$, then G' is Class 1 by induction. Again it follows that G is also Class 1, proving our theorem. □

In the case that G has more than one vertex of minimum degree the methods of Theorem 2.1 and Theorem 2.2 are not applicable. However, if G has exactly two vertices of degree $\leq \Delta - 2$, we can prove the following result.

Theorem 2.3 *Let G be a graph of odd order n which is degree-bounded such that there exist exactly two vertices x, y with $d(x, G) < d(y, G) < \Delta(G) - 1$. If $d(y, G) = \Delta(G) - p$ and $d(x, G) = \Delta(G) - q$ with $q \geq 2p - 1$, then: G is Class 2 if and only if G is $\Delta(G)$-overfull.*

Proof. Since G is Class 2 if G is $\Delta(G)$-overfull, suppose that G is not $\Delta(G)$-overfull. We want to show that G is Class 1. Let x and y be as in the assumption. As G is degree-bounded it holds that $2\delta(G) > 3\delta(G) - \Delta(G) \geq 2c^*n \geq \frac{3}{2}n$ and thus $\delta(G) = d(x, G) = \Delta(G) - q > \frac{3}{4}n$. As $n > \Delta(G)$, it holds that $q \leq \frac{1}{4}n$.
Let $G_1 := G$. If the graph G_l is given and has two vertices of degree $\leq \Delta(G_l) - 2$, then proceed as follows. In the case that $l \equiv 1, 2 \pmod 3$, delete a perfect matching of $G_l - x$ in G_l to get to G_{l+1}. If $l \equiv 0 \pmod 3$, then delete a perfect matching of $G_l - y$ in G_l to get to G_{l+1}. We will show later that these matchings exist. When does this procedure terminate? For $l = 3(p - 1)$,

- $\Delta(G_l) = \Delta(G) - 3p + 3$,
- $d(y, G_l) = \Delta(G) - p - \lceil \frac{2l}{3} \rceil = \Delta(G) - 3p + 2$ and
- $d(x, G_l) = \Delta(G) - q - \lfloor \frac{l}{3} \rfloor \leq \Delta(G) - 3p + 2.$

Thus the graph G_l, with $l = 3(p-1)$, has at most one vertex of degree $\leq \Delta(G_l) - 2$ and the procedure terminates (we leave it to the reader to show that the procedure does not terminate for any $l < 3(p-1)$). As G was not overfull,

$$2|E(G_l)| = 2|E(G)| - l(n-1) \leq (n-1)\Delta(G) - 3(p-1)(n-1) = (n-1)\Delta(G_l),$$

and G_l is not overfull. Furthermore, for $l = 3(p-1)$ we have

$$3\delta(G_l) - \Delta(G_l) = 3(\Delta(G) - q - \lfloor \frac{l}{3} \rfloor) - (\Delta(G) - l) \geq 3(\Delta - q) - \Delta \geq c^* n.$$

Thus by Theorem 2.2 we know that G_l is Class 1. Due to the construction of G_l it follows that G is Class 1, too.
It remains to show that the perfect matchings in $G_l - x$ and $G_l - y$ exist as long as $l < 3(p-1)$. If $l < 3(p-1)$ we have in G_l

- $\delta(G_l - x) = d(y, G_l - x) \geq \Delta(G) - p - \lceil \frac{2l}{3} \rceil - 1$ and
- $\delta(G_l - y) = d(x, G_l - y) \geq \Delta(G) - q - \lfloor \frac{l}{3} \rfloor - 1$.

As $q \geq 2p - 1$, we have $\min\{\delta(G_l - x), \delta(G_l - y)\} \geq \Delta(G) - 2q$. Since $\Delta(G) - q \geq \frac{3}{4}n$ and $q \leq \frac{1}{4}n$, as mentioned in the beginning, both $G_l - x$ and $G_l - y$ have a perfect matching, by Dirac [3]. This completes the proof of the theorem. □

In Lemma 1.5 we showed that a $\Delta(G)$-overfull graph G has at most $\delta(G)$ vertices which do not have maximum degree. In view of the Overfull Conjecture one would need to show that a graph G of odd order n with $\tau_\Delta \leq n - \delta(G)$ is Class 1. This is beyond our reach in the general case, but we can show that $\tau_\delta \geq \delta(G)$ implies that the graph is Class 1, if the minimum degree δ is large enough. For our first condition we need the following theorem.

Theorem 2.4 (Niessen and Volkmann [7]) *Let G be a graph of odd order n such that*

$$\delta(G) \geq \frac{n-1}{2} + \tau_\Delta + \left\lfloor \frac{\tau_\Delta \Delta(G)}{n} \right\rfloor.$$

Then G is Class 2 if and only if G is $\Delta(G)$-overfull.

Theorem 2.5 *Let G be a graph of odd order n. If $\delta(G) \geq \frac{5}{6}n$ and $\tau_\delta \geq \delta$, then G is Class 1.*

Proof. As $\tau_\delta \geq \delta$ we have $\tau_\Delta \leq n - \delta$ and Lemma 1.5 tells us that G is not overfull. Furthermore it follows that

$$\frac{n-1}{2} + \tau_\Delta + \left\lfloor \frac{\tau_\Delta \Delta}{n} \right\rfloor \leq \frac{4n + \Delta - 1}{6} \leq \frac{5}{6}n,$$

and Theorem 2.4 tells us that G is Class 1. □

We can lower the bound $\frac{5}{6}n$ of Theorem 2.5 further, in the case that the graph is degree-bounded.

Theorem 2.6 *Let G be a graph of odd order n which is degree-bounded. If $\delta(G) < \tau_\delta$, then G is Class* 1.

Proof. As $\tau_\delta > \delta(G)$, Lemma 1.5 tells us that G is not $\Delta(G)$-overfull. If $\Delta(G) \leq \delta(G) + 1$ then the statement has been shown to hold true. So let $\Delta(G) \geq \delta(G) + 2$.
Case 1. Let $\delta(G) < \tau_\delta \leq \Delta(G)$. In this case add a vertex x_0 to G and connect it with all vertices of degree δ. For the resulting graph G' it holds that $\Delta(G') = \Delta(G)$, $d(x_0, G') = \tau_\delta \leq \Delta(G)$ and $\delta(G') = \delta(G) + 1$. We now have

$$\frac{3\delta(G') - \Delta(G')}{2} = \frac{3\delta(G) + 3 - \Delta(G)}{2} \geq c^* n + \frac{3}{2} \geq c^*(n + 1)$$

and it follows that G' is degree-bounded. Furthermore, G' does not have a $\Delta(G')$-overfull subgraph. Assume that G' has a $\Delta(G')$-overfull subgraph; then by Lemma 1.3 it is of the form $G' - x$. For every $v \in V(G')$ with $d(v, G') = \delta(G')$ the graph $G' - v$ is also a $\Delta(G')$-overfull subgraph. As $\tau_\delta > 2$, we get a contradiction to Theorem 1.2. Thus G' meets the criteria of Theorem 1.6 and is Class 1. Consequently, G is Class 1.
Case 2. Let $\tau_\delta > \Delta(G)$. If G is $\Delta(G)$-full, then $n\Delta(G) - \Delta(G) = 2e(G) \leq \tau_\delta\delta(G) + (n - \tau_\delta)\Delta(G)$ and thus $1 \geq \Delta(G)/\tau_\delta \geq \Delta(G) - \delta(G)$, in contradiction to $\Delta - \delta \geq 2$. Thus G is neither $\Delta(G)$-overfull nor $\Delta(G)$-full. Let H be the subgraph induced by the vertices of minimum degree in G. If H is complete, then

$$e_G(V(H), V(H)^c) = \delta(G)\tau_\delta - \tau_\delta(\tau_\delta - 1) = \tau_\delta(\delta(G) - \tau_\delta + 1) < 0,$$

giving us a contradiction. So we find two vertices x, y of minimum degree, which are not connected. We add the edge xy to G and denote the resulting graph by G^1. Obviously, $\Delta(G^1) = \Delta(G)$ and $\delta(G^1) = \delta(G)$. However, G^1 has two less vertices of minimum degree and is not $\Delta(G)$-overfull, as a short calculation shows. Let τ_δ^1 denote the number of vertices of minimum degree in G^1. If $\tau_\delta^1 > \Delta(G^1)$, then G^1 cannot be $\Delta(G^1)$-full, using the same argument for G^1 as for G. Thus we can recursively add i edges to G in such a way that $\tau_\delta^i \leq \Delta(G^i)$, $\Delta(G^i) = \Delta(G)$ and $\delta(G^i) = \delta(G)$. Note that G^i cannot be $\Delta(G^i)$-overfull. Then we are in Case 1 and know that G^i is Class 1 and consequently, that G is Class 1. □

In the case $\tau_\delta < \delta$, we need a further condition on the minimum degree, besides the degree-boundedness, for the Overfull Conjecture to hold.

Theorem 2.7 *Let G be a graph of odd order n which is degree-bounded such that $\tau_\delta < \delta$. Denote by d the smallest degree in G greater than $\delta(G)$. If there exists an odd integer s such that $d - \delta(G) \geq s \geq \frac{3\delta(G) - 3\tau_\delta + 3}{3 - 2c^*}$, then: G is Class* 2 *if and only if G is $\Delta(G)$-overfull.*

Proof. The necessity is clear; let us show the sufficiency. Suppose that G is not $\Delta(G)$-overfull. We will show that then G is Class 1. For the integer s as in the hypothesis, it holds that $s \geq 3$ as otherwise $c^* < 0$. Add the complete graph K_s to G and connect every vertex of K_s with every vertex of minimum degree in G. We call the resulting graph G_s. This graph has even order $n + s$ and

- for every $v \in V(K_s)$ it holds that $d(v, G_s) = \tau_\delta + s - 1$;
- for every $u \in V(G)$ with $d(u, G) = \delta$ it holds that $d(u, G_s) = \delta + s$;
- for every vertex $x \in V(G)$ with $d(x, G) > \delta$ it holds that $d(x, G_s) = d(x, G) \geq \delta + s$.

As a consequence we get $\delta(G_s) = \tau_\delta + s - 1$ and $\Delta(G_s) = \Delta(G)$. We are now going to show that the graph G_s is degree-bounded. Now

$$\begin{aligned} \frac{3\delta(G_s) - \Delta(G_s)}{2} &= \frac{3(\tau_\delta + s - 1) - \Delta(G)}{2} \\ &\geq \frac{3}{2}(\tau_\delta + s - 1 - \delta) + c^* n, \end{aligned}$$

as G is degree-bounded. With the choice of s we get $\frac{3}{2}(\tau_\delta + s - 1 - \delta) \geq c^* s$ and it follows that

$$\frac{3\delta(G_s) - \Delta(G_s)}{2} \geq c^*(n + s) = c^* n(G_s).$$

Thus G_s is degree-bounded. Furthermore, G_s cannot have a $\Delta(G_s)$-overfull subgraph. Assume that the opposite holds. With Lemma 1.3 this subgraph is of the form $G_s - x_0$. But then $G_s - x$ is a $\Delta(G_s)$-overfull subgraph of G_s for every $x \in V(K_s)$. Since $s \geq 3$, we get a contradiction to Theorem 1.2. Thus G_s does not have a $\Delta(G_s)$-overfull subgraph and we can apply Theorem 1.6. Hence G_s is Class 1 and, as $\Delta(G_s) = \Delta(G)$, the graph G is Class 1, too. □

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