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The enumeration of labelled spanning trees of $K_{m,n}$

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Abstract

Using a bijection to decompose a labelled rooted bipartite tree into several ones with smaller size and their exponential generating functions, this paper concerns the number of labelled spanning trees of the complete bipartite graph $K_{m,n}$.

1 Introduction

Regarding the number of labelled spanning trees of the complete graph K_n and complete bipartite graph $K_{m,n}$, various methods ([1, 3, 4, 9, 10] and [1, 2, 5, 6, 8]) have appeared. With the same purpose this paper establishes another bijection similar to that in [3], with the difference being that labelled rooted trees of height 2 are used, which is necessary for the bijection considered here. From the bijection we can derive an equation between two kinds of exponential generating functions for two classes of labelled rooted spanning trees in $K_{m,n}$: one class has no restriction; the other consists only of those spanning trees with height 1 or 2.

A labelled rooted spanning tree in the complete bipartite graph $K_{m,n}$ ($V(K_{m,n}) = A \cup B, |A| = m, |B| = n, m \geq 1, n \geq 1$) with root in B will be briefly called an $[m,n]$-tree in this paper, which may be called labelled rooted bipartite tree elsewhere. Vertices in A and B will be labelled from the set $\{1', 2', \ldots, m'\}$ and $\{1, 2, \ldots, n\}$ respectively.

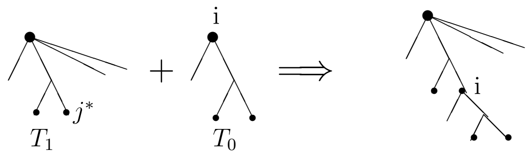

Figure 2.1

2 The number of labelled $[m, n]$-trees

Let T be an $[m, n]$-tree and v^* be its root. The height of a vertex $v (v \in V(T))$ and of T are defined by $W(v) = d(v^*, v)$ and $W(T) = \max\{W(v) \mid \forall\, v \in V(T)\}$ respectively, where $d(v^*, v)$ is the distance between v^* and v. Obviously there are m odd height vertices and n even height vertices in T. When $W(T)$ is 1 and 2, we call T a 1-rooted tree and 2-rooted tree, briefly as ORT and TRT.

Any even height vertex v_0 in T which is not a leaf will be called an IP. Let $OV(v_0) = \{v \mid v \in V(T)$ and the path from the root to v contains $v_0\}$. We denote by $OVG(v_0)$ the subgraph of T induced by the vertex set $\{v \mid v \in OV(v_0), d(v_0, v) \leq 2\}$, which is a labelled rooted subtree (with root v_0) of T. Let $OVW(v_0) = \max\{d(v_0, v) \mid v \in OV(v_0)\}$. When $OVW(v_0) = 1$, call v_0 a 1-IP, otherwise a 2-IP.

An $[m, n]$-tree in which there are $(k - r)$ 1-IPs and (r) 2-IPs is called an $[m, n, k, r]$-tree. From this definition, a TRT is an $[m, n, 1, 1]$-tree for some m and n, and an ORT is an $[m, 1, 1, 0]$-tree for some m.

Theorem 2.1 *There is a bijection between the set of all $[m, n, k, r]$-trees and the set of forests of r TRTs and $k-r$ ORTs in which the height 2 vertices are labelled from $\{1, 2, \ldots, n+k-1\}$ and the roots from $\{1, 2, \ldots, n\}$, and other height 1 vertices from $\{1', 2', \ldots, m'\}$.*

Proof. We first give the procedure to construct an $[m, n, k, r]$-tree from a forest F of r TRTs and $k - r$ ORTs.

1. In F mark the vertices $n + 1, n + 2, \ldots, n + r - 1$ by symbol *(star) and mark vertices $n + r, \ldots, n + k - 1$ by symbol #(double-cross).

2. Select out TRTs in F, let F_0 be this subset, and let the set of the left trees in F be F_1.

3. Find the tree T_0 in F_0 with the smallest root such that there is no starred vertex in T_0, let i be the root of T_0.

4. Find the tree T_1 in F_0 that contains the smallest starred vertex. Let j^* be this starred vertex.

5. Merge T_0 and T_1 by identifying i and j^* and keeping i as the new vertex. (See Figure 2.1.)

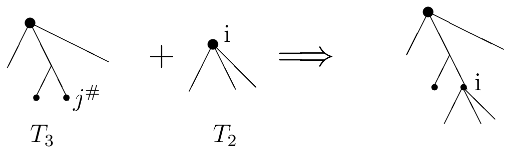

Figure 2.2

6. Repeat (3), (4) and (5) until F has no starred vertex.

7. Find the tree T_2 in F_1 with the smallest root; let i be this root.

8. Find the tree T_3 in F that contains the smallest double-crossed vertex. Let $j^{\#}$ be that vertex.

9. Replace $j^{\#}$ with T_2 in T_3 as (5). (See Figure 2.2.)

10. Repeat (7), (8) and (9) until F has no double-crossed vertex.

It is easy to see that we at last get an $[m, n, k, r]$-tree. The reverse procedure is as follows:

1. Select out all the (r) 2-IPs and $(k-r)$ 1-IPs; denote them by $V_2 = \{v_1, \ldots, v_r\}$ and $V_1 = \{v_{r+1}, \ldots, v_k\}$ respectively.

2. Find the smallest 1-IP i in V_1.

3. Remove $OVG(i)$ and relabel the original vertex i by $(n+r)^{\#}$. Then we get an ORT with root i.

4. Repeat (2), (3) and relabel the encountered 1-IPs subsequently by $(n + r + 1)^{\#}, \ldots, (n + k - 1)^{\#}$ until there is no vertex v in V_1 with $W(v) > 0$.

5. Find the smallest 2-IP i in V_2 with $OVW(i) = 2$.

6. Remove $OVG(i)$ and relabel the original vertex i by $(n + 1)^*$. Then we get a TRT with root i.

7. Repeat (5), (6) and relabel the encountered 2-IPs by $(n+2)^*, \ldots, (n+r-1)^*$ subsequently until there is no vertex v in V_2 with $W(v) > 0$.

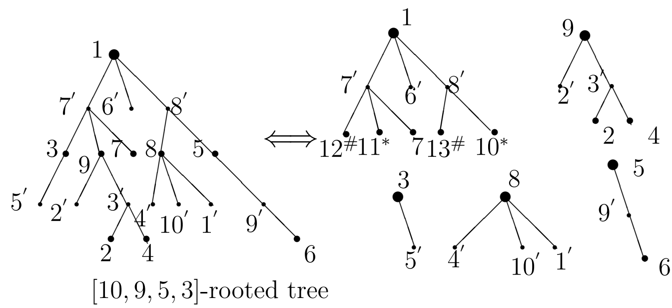

Figure 2.3

Example: Figure 2.3 shows a $[10, 9, 5, 3]$-tree together with its ORTs and TRTs.

Let $T_{1,m,1}$ be the number of ORTs of order $m + 1$ and consider the function

$$T_1(x, y) = \sum_{m=1}^{\infty} T_{1,m,1} \frac{x^m y}{m!}.$$

Obviously $T_1(x, y) = y(e^x - 1)$ and $\frac{T_1(x,y)}{y}$ is the exponential generating function for ORTs, where the label of the root is already defined. For convenience, let $T^*_{n-1}(x) = \sum_{m=1}^{\infty} T^*_{m,n-1} \frac{x^m}{m!}$ be the exponential generating function for $[m, n]$-trees of height 2, where the label of the root is already defined. Suppose the number of non-leaf height 1 vertices is t; then we have

$$T^*_{n-1}(x) = \sum_{m=1}^{\infty} \sum_{t=1}^{\min\{m,n-1\}} \binom{m}{t} t! S(n-1, t) \frac{x^m}{m!}, \tag{2.1}$$

where $S(n - 1, t)$ are the Stirling numbers of the second kind. If we let $(T^*(x))^n = T^*_n(x)$ then

$$\begin{aligned}
e^{yT^*(x)} - 1 &= \sum_{n=2}^{\infty} \frac{T^*_{n-1}(x) y^{n-1}}{(n-1)!} \\
&= \sum_{m=1}^{\infty} \sum_{n=2}^{\infty} \sum_{t=1}^{\min\{m,n-1\}} \frac{S(n-1,t) x^m y^{n-1}}{(n-1)!(m-t)!} \\
&= \sum_{m=1}^{\infty} \sum_{t=1}^{m} \sum_{n=1}^{\infty} \frac{S(n,t) x^m y^n}{n!(m-t)!} \\
&= \sum_{m=1}^{\infty} \sum_{t=1}^{m} \frac{(e^y - 1)^t}{t!} \frac{m!}{(m-t)!} \frac{x^m}{m!} \\
&= \sum_{m=1}^{\infty} ((e^y)^m - 1) \frac{x^m}{m!} \\
&= \sum_{m=1}^{\infty} \frac{(xe^y)^m}{m!} - \sum_{m=1}^{\infty} \frac{x^m}{m!} \\
&= e^{xe^y} - e^x.
\end{aligned} \tag{2.2}$$

Let $T_{m,n}(m \geq 1, n \geq 1)$ be the number of $[m,n]$-trees and consider

$$T(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} T_{m,n}\frac{x^m y^n}{m!n!}.$$

Theorem 2.2 *The exponential generating function $T(x,y)$ satisfies:*

$$T(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} n^m m^{n-1}\frac{x^m y^n}{m!n!}. \tag{2.3}$$

Proof. When $n=1$, $T_1(x,y)$ is the exponential generating function for $[m,1]$-trees. Let $T_{2,m,n}(n>1)$ be the number of $[m,n]$-trees with height bigger than 1 and consider the function

$$T_2(x,y) = \sum_{n=2}^{\infty}\sum_{m=1}^{\infty} T_{2,m,n}\frac{x^m y^n}{m!n!}.$$

First we show that from the bijection in Theorem 2.1, $T_2(x,y)$ satisfies

$$T_2(x,y) = \sum_{n=2}^{\infty}\sum_{k=1}^{n}\sum_{r=1}^{N}\frac{S(n-1,r)}{(k-r)!(n-k)!}T^*_{n_1}(x)\cdots T^*_{n_r}(x)(e^x-1)^{k-r}y^n, \tag{2.4}$$

where $N=\min\{n-1,k\}$ and $n_1+\cdots+n_r=n-1$.

Let $T_2(x,y)=\sum_{n=2}^{\infty}T_n(x)\frac{y^n}{n!}$. Then the coefficient of $\frac{x^m}{m!}$ in $T_n(x)$ is the number of $[m,n]$-trees whose height bigger than 1. In fact the number k of IPs has $1\leq k\leq n$ and the number r of 2-IPs has $1\leq r\leq \min\{n-1,k\}$. By Theorem 2.1, we first define the roots of those TRTs and ORTs, there are $\binom{n}{r}\binom{n-r}{k-r}$ ways. Then we can consider a partition of the $n-1$ height 2 vertices into r blocks and put all vertices of each block as vertices of a TRT in $r!S(n-1,r)$ ways. From the definition of $T^*_{n-1}(x)$, we know that the exponential generating function for forests of r TRTs whose roots and height 2 vertices are already defined is $T^*_{n_1}(x)\cdots T^*_{n_r}(x)$. Similarly the function for forests of $k-r$ ORTs whose roots are already defined is $(\frac{T_1(x,y)}{y})^{k-r}$. Then we get

$$T_n(x) = \sum_{k=1}^{n}\sum_{r=1}^{N}\binom{n}{r}\binom{n-r}{k-r}r!S(n-1,r)T^*_{n_1}(x)\cdots T^*_{n_r}(x)\left(\frac{T_1(x,y)}{y}\right)^{k-r},$$

which completes the proof of (2.4).

Let $f(x,y)|_{\{y,n\}}$ be the term containing $\frac{y^n}{n!}$ in $f(x,y)$. Noting $S(m,l)\neq 0$ is equivalent to $0<l\leq m$, therefore from (2.2) and (2.4) we have

$$\begin{aligned}
T_2(x,y) &= \sum_{n=2}^{\infty}\sum_{k=1}^{n}\sum_{r=0}^{\min\{n-1,k\}}\frac{S(n-1,r)}{(k-r)!(n-k)!}T^*_{n_1}(x)\cdots T^*_{n_r}(x)(e^x-1)^{k-r}y^n\\
&= \sum_{n=2}^{\infty}\sum_{k=1}^{n}\sum_{r=0}^{k}\frac{S(n-1,r)(yT^*(x))^{n-1}}{(n-1)!}\frac{(n-1)!(e^x-1)^{k-r}y}{(k-r)!(n-k)!}\\
&= \sum_{n=2}^{\infty}\sum_{k=1}^{n}\sum_{r=0}^{k}\left.\frac{(e^{yT^*(x)}-1)^r}{r!}\right|_{\{y,n-1\}}\frac{(e^x-1)^{k-r}(n-1)!y}{(k-r)!(n-k)!}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=2}^{\infty}\sum_{k=1}^{n}\frac{(e^{xe^y}-1)^k}{k!}\left.\frac{n!}{(n-k)!}\right|_{\{y,n-1\}}\frac{y}{n} \\
&= \sum_{n=2}^{\infty}(e^{nxe^y}-1)\Big|_{\{y,n-1\}}\frac{y}{n} \\
&= \sum_{n=2}^{\infty}\sum_{m=1}^{\infty}\left.\frac{n^m x^m e^{my}}{m!}\right|_{\{y,n-1\}}\frac{y}{n} \\
&= \sum_{n=2}^{\infty}\sum_{m=1}^{\infty}n^m m^{n-1}\frac{x^m y^n}{m!n!}. \qquad (2.5)
\end{aligned}
$$

Therefore

$$T(x,y)=T_1(x,y)+T_2(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}n^m m^{n-1}\frac{x^m y^n}{m!n!}. \quad \square$$

For the first few terms of $T(x,y)$ we have

$$
\begin{aligned}
T(x,y) \;=\; & \frac{xy}{1!1!}+\frac{2xy^2}{1!2!}+\frac{x^2y}{2!1!}+\frac{3xy^3}{1!3!}+\frac{8x^2y^2}{2!2!}+\frac{x^3y}{3!1!} \\
&+\frac{4xy^4}{1!4!}+\frac{36x^2y^3}{2!3!}+\frac{24x^3y^2}{3!2!}+\frac{x^4y}{4!1!}+\cdots.
\end{aligned}
$$

From Theorem 2.2 we now can get the well known result which gives the number of rooted spanning trees of the labelled complete graph $K_n (n>1)$; let $\tau(K_n)$ be that number.

Theorem 2.3

$$\tau(K_n)=n^{n-1}. \qquad (2.6)$$

Proof. To a labelled rooted spanning tree of K_n, suppose the number of vertices whose height is even is r; then $1\le r\le n-1$. In fact if we first select r labels out of $\{1,\ldots,n\}$ for the r vertices and the remaining $n-r$ labels for the $n-r$ odd height vertices, then from the definition of $T_{m,n}$ we have that the number of labelled rooted spanning trees of K_n with r even height vertices is $\binom{n}{r}T_{n-r,r}$. Recall the Abel identity ([7])

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}(y-k)^{n-k}(x+k)^{k-1}x,$$

and replace x,y,k by $-n,0,r$ respectively. Then from Theorem 2.2 we get

$$\tau(K_n)=\sum_{r=1}^{n-1}\binom{n}{r}T_{n-r,r}=\sum_{r=1}^{n-1}\binom{n}{r}r^{n-r}(n-r)^{r-1}=n^{n-1}. \quad \square$$

Let $\tau(K_{m,n})$ be the number of labelled spanning trees of $K_{m,n}$. Then from Theorem 2.2 we have

Theorem 2.4

$$\tau(K_{m,n}) = m^{n-1}n^{m-1}. \tag{2.7}$$

Proof. We first calculate the number of labelled rooted spanning trees of $K_{m,n}$. From the definition of $T_{m,n}$ we know that the number of labelled rooted spanning trees of $K_{m,n}$ with root in A and B is $T_{n,m}$ and $T_{m,n}$ respectively, which tells us that the number of labelled rooted spanning trees of $K_{m,n}$ is

$$T_{n,m} + T_{m,n} = m^n n^{m-1} + n^m m^{n-1}.$$

Because there are $m+n$ vertices in all and any vertex can be selected out as the root, the number of labelled spanning trees of $K_{m,n}$ is

$$\tau(K_{m,n}) = \frac{m^n n^{m-1} + n^m m^{n-1}}{m+n},$$

which completes the proof.

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