AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **28** (2003), Pages 3–24

Directed covering with block size 5 and v even

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Abstract

A directed covering design, $\mathrm{DC}(v,k,\lambda)$, is a $(v,k,2\lambda)$ covering design in which the blocks are regarded as ordered k-tuples and in which each ordered pair of elements occurs in at least λ blocks. Let $\mathrm{DE}(v,k,\lambda)$ denote the minimum number of blocks in a $\mathrm{DC}(v,k,\lambda)$. In this paper the values of the function $\mathrm{DE}(v,5,\lambda)$ are determined for all even integers $v \geq 5$ and λ odd.

1 Introduction

A transitively ordered k-tuple $(a_1,\ldots,a_k)$ is defined to be the set $\{(a_i,a_j) \mid 1 \leq i < j \leq k\}$. Let v, k and λ be positive integers. A directed covering (packing) design, denoted by $\mathrm{DC}(v,k,\lambda)$ ($\mathrm{DP}(v,k,\lambda)$), is a pair (X,A) where X is a set of points and A is a collection of transitively ordered k-tuples of X, called blocks, such that every ordered pair of X appears in at least (at most) λ blocks. Let $\mathrm{DE}(v,k,\lambda)$ ($\mathrm{DD}(v,k,\lambda)$) denote the minimum (maximum) number of blocks in a $\mathrm{DC}(v,k,\lambda)$ ($\mathrm{DP}(v,k,\lambda)$).

A $\mathrm{DC}(v,k,\lambda)$ with $|A| = \mathrm{DE}(v,k,\lambda)$ is called a minimum directed covering design and a $\mathrm{DP}(v,k,\lambda)$ with $|A| = \mathrm{DD}(v,k,\lambda)$ is called a maximum directed packing design. If we ignore the order of the blocks, a $\mathrm{DC}(v,k,\lambda)$ ($\mathrm{DP}(v,k,\lambda)$) is a standard $(v,k,2\lambda)$ covering (packing) design. Therefore, the following bounds, known as the Schönheim bounds, hold [24].

$$\mathrm{DE}(v,k,\lambda) \geq \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} 2\lambda \right\rceil \right\rceil = \mathrm{DL}(v,k,\lambda),$$

$$\mathrm{DD}(v,k,\lambda) \leq \left[\frac{v}{k} \left[\frac{v-1}{k-1} 2\lambda \right] \right] = \mathrm{DU}(v,k,\lambda).$$

Here $\lceil x \rceil$ is the smallest and $[x]$ is the largest integer satisfying $[x] \leq x \leq \lceil x \rceil$. The above bound has been sharpened by Hanani [20] in certain cases.

Theorem 1.1 *(i) If* $2\lambda(v-1) \equiv 0 \pmod{k-1}$ *and* $2\lambda v(v-1)/(k-1) \equiv -1 \pmod{k}$ *then* $DE(v,k,\lambda) \geq DL(v,k,\lambda) + 1$.
(ii) If $2\lambda(v-1) \equiv 0 \pmod{k-1}$ *and* $2\lambda v(v-1)/(k-1) \equiv 1 \pmod{k}$ *then* $DD(v,k,\lambda) \leq DU(v,k,\lambda) - 1$.

When $\mathrm{DE}(v,k,\lambda) = \mathrm{DL}(v,k,\lambda)$, the directed covering design is called minimal. Similarly, when $\mathrm{DD}(v,k,\lambda) = \mathrm{DU}(v,k,\lambda)$, the directed packing design is called optimal.

A directed balanced incomplete block design, $\mathrm{DB}[v,k,\lambda]$, is a $\mathrm{DC}(v,k,\lambda)$ where every ordered pair of points appears in exactly λ blocks. If a $\mathrm{DB}[v,k,\lambda]$ exists then it is clear that $\mathrm{DE}(v,k,\lambda) = 2\lambda v(v-1)/k(k-1) = \mathrm{DL}(v,k,\lambda) = \mathrm{DD}(v,k,\lambda)$. In the case $k = 5$, Street and Wilson [28] have shown the following:

Theorem 1.2 *Let* λ *and* $v \geq 5$ *be positive integers. The necessary and sufficient conditions for the existence of a* $DB[v,5,\lambda]$ *are that* $(v,\lambda) \neq (15,1)$ *and that* $\lambda(v-1) \equiv 0$ *(mod* 2*) and* $\lambda v(v-1) \equiv 0$ *(mod* 10*).*

In [25–27], Skillicorn discussed the function $\mathrm{DE}(v,4,1)$ and $\mathrm{DD}(v,4,1)$ and developed many other results including applications of directed designs to computer network and data flow machine architecture. The values of $\mathrm{DE}(v,5,\lambda)$ for all $v \geq 5$ and even λ have been determined by Assaf [9], and more recently, Assaf et al. [14] have determined the values of $\mathrm{DE}(v,4,\lambda)$ and $\mathrm{DD}(v,4,\lambda)$ for all positive integers v and λ. The values of $\mathrm{DE}(v,5,\lambda)$ for odd λ and v is considered by Alhalees [4]. It is our purpose here to discuss the function $\mathrm{DE}(v,5,\lambda)$ for every λ and even $v \geq 5$. Since λ even has been done in [9] we only need to treat the case λ is odd. We show the following.

Theorem 1.3 *Let* $v \geq 5$ *be an even integer and* λ *be an odd integer. Then* $DE(v,5,\lambda) = DL(v,5,\lambda) + e$ *where* $e = 1$ *if* $\lambda v(v-1)/2 \equiv -1$ (mod5)*; and* $e = 0$ *otherwise.*

2 Recursive Constructions

To describe our recursive constructions we need the notions of transversal designs, group divisible designs and covering (packing) designs with a hole of size h. For the definition of these designs see [6]. A (v, k, λ) covering design with a hole size h is said to be minimal if the total number of blocks β satisfies $|\beta| = \phi(v, k, \lambda) - \phi(h, k, \lambda)$ where $\phi(n, k, \lambda) = \lceil (n/k) \lceil \lambda(n-1)/(k-1) \rceil \rceil$. We shall adopt the following notation: a $T[k, \lambda, m]$ stands for a transversal design with block size k, index λ and group size m. A (K, λ)-GDD stands for a group divisible design with block sizes from K and index λ. When $K = \{k\}$ we simply write k for K. The group type of a (K, λ)-GDD is a listing of the group sizes using exponential notations, i.e. $1^a 2^b 3^c \ldots$ denotes a groups of size 1, b groups of size 2, etc.

The excess (complement) graph of a (v, k, λ) covering (packing) design is the graph on v vertices such that $\{a, b\}$ is an edge with multiplicity μ if $\{a, b\}$ appears in $\lambda + \mu$, $[(\lambda - \mu)]$ blocks. In a similar way one can define the directed complement and excess graphs of a DP(v, k, λ) and DC(v, k, λ) [13]. The directed graph is called symmetric if the number of edges entering a vertex is equal to the number of edges exiting the vertex.

We like to remark that the notions of transversal designs, group divisible designs, covering (packing) designs with a hole of size h can be easily extended to the directed case. In the sequel we write DT, DGDD with the appropriate parameters.

The following theorem will be used extensively in this paper. The proof of this result may be found in [1–3, 16–18, 20, 23, 29].

Theorem 2.1 *There exists a $T[6, 1, m]$ for all positive integers m, $m \notin \{2, 3, 4, 6\}$ with the possible exception of $m \in \{10, 14, 18, 22\}$.*

Theorem 2.2 *If there exists a $(6, \lambda)$- GDD of type 5^m and a minimal DC$(20+h, 5, \lambda)$ with a hole of size h and a minimal DC$(4u + h, 5, \lambda)$, $0 \le u \le 5$, then there exists a minimum DC$(20(m-1) + 4u + h, 5, \lambda)$.*

Proof: Take a $(6, \lambda)$-GDD of type 5^m and delete all but u points from last group. Inflate the resultant design by a factor of 4, i.e. replace each block of size 5 and 6 by the blocks of a $(5, 1)$-DGDD of type 4^5 and 4^6 respectively [20].

On the last group we construct a minimal DC$(4u + h, 5, \lambda)$ and on the remaining groups construct a minimal DC$(20 + h, 5, \lambda)$ with a hole of size h.

The application of the above theorem requires the existence of a $(6, \lambda)$-GDD of type 5^m . Our authority of the following Lemma is Hanani [20, p.286].

Lemma 2.1 *(i) There exists a $(6, \lambda)$-GDD of type 5^7 for $\lambda \ge 2$.*
(ii) There exists a $(6, \lambda)$-GDD of type 5^9 for λ even.

Another notion that is used in this paper is the notion of modified group divisible designs. Let k, λ, v and m be positive integers. A modified group divisible design (k, λ)-MGDD of type m^n is a quadruple $(V, \beta, \gamma, \delta)$ where V is a set of points with $|V| = mn$, $\gamma = \{G_1, G_2, \ldots, G_n\}$ is a partition of V into n sets, called groups, $\delta = \{R_1, R_2, \ldots, R_m\}$ is a partition of V into m sets, called rows, and β is a family of k-subsets of V, called blocks, with the following properties.

1) $|B \cap G_i| \le 1$ for all $B \in \beta$ and $G_i \in \gamma$.

2) $|B \cap R_i| \leq 1$ for all $B \in \beta$ and $R_i \in \delta$.

3) $|G_i| = m$ for all $G_i \in \gamma$.

4) Every 2-subset $\{x, y\}$ of V such that x and y are neither in the same group nor same row is contained in exactly λ blocks.

5) $|G_i \cap R_j| = 1$ for all $G_i \in \gamma$ and $R_j \in \delta$.

A resolvable MGDD (RMGDD) is one the blocks of which can be partitioned into parallel classes. It is clear that a $(5, 1)$-RMGDD of type 5^m is the same as RT$[5, 1, m]$ with one parallel class of blocks singled out, and since RT$[5, 1, m]$ is equivalent to $T[6, 1, m]$, we have the following existence theorem.

Theorem 2.3 *There exists a $(5, 1)$-RMGDD of type 5^m for all positive integers m, $m \notin \{2, 3, 4, 6\}$, with the possible exception of $m \in \{10, 14, 18, 22\}$.*

The proof of the next theorem is the same as the proof of Theorem 2.3 of [5].

Theorem 2.4 *If there exists a $(5, 1)$-RMGDD of type 5^m and a $(5, \lambda)$-DGDD of type $4^m s^1$ and a $(5, \lambda)$-DGDD of type 4^5 and 4^6 and there exists a minimal DC$(20+h, 5, \lambda)$ with a hole of size h, then there exists a minimal DC$(20m + 4u + h + s, 5, \lambda)$ with a hole of size $4u + h + s$, where $0 \leq u \leq m - 1$.*

The application of the previous theorem requires the existence of a $(5, 1)$-DGDD of type $4^m s^1$. We shall use the following theorem.

Theorem 2.5 *(i) There exists a $(5, 1)$-DGDD of type $4^m s^1$ where $s = 0$ if $m \equiv 1 \pmod 5$, $s = 4$ if $m \equiv 0$ or $4 \pmod 5$ and $s = 4(m - 1)/3$ if $m \equiv 1 \pmod 3$* [5]. *(ii) There exists a $(5, 1)$-DGDD of type $4^m 8^1$ where $m \equiv 0$ or $2 \pmod 5$, $m \geq 7$, with the possible exception of $m = 10$* [19].

The following theorem is a generalization of Theorem 2.6 of [7].

Theorem 2.6 *If there exists a $(5, 1)$-RMGDD of type 5^m and a $(5, \lambda)$-DGDD of type $2^m s^1$ and a $(5, \lambda)$-DGDD of type 2^5 and 2^6 and there exists a minimal DC$(10+h, 5, \lambda)$ with a hole of size h, then there exists a minimal DC$(10m + 2u + h + s, 5, \lambda)$ with a hole of size $2u + h + s$, where $0 \leq u \leq m - 1$.*

We like to mention that for large v, instead of constructing a DC$(v, 5, \lambda)$, we will construct a DC$(v, 5, \lambda)$ with a hole of size h, $h > 5$, and then on the hole we construct a DC$(h, 5, \lambda)$.

Finally about the notation, a block of the form $\langle k\ k{+}m\ k{+}n\ k{+}j\ f(k)\rangle \pmod v$, where $f(k) = a$ if k is even and $f(k) = b$ if k is odd, is denoted by $\langle 0\ m\ n\ j\rangle \cup \{a, b\}$. Further, if a and b are to be inserted in the middle, then we write $\langle 0\ m{-}n\ j\rangle \cup \{a, b\}$.

3 Directed covering with index 1

Lemma 3.1 *i) There exists a minimal DC$(22, 5, 1)$ with a hole of size 2.*
ii) Let $v \equiv 2 \pmod{20}$ be a positive integer. Then DE$(v, 5, 1)$ = DL$(v, 5, 1)$.

Proof i) The construction of a minimal DC$(22, 5, 1)$ with a hole of size 2 is as follows:
1) Take a $B[21, 5, 1]$ in increasing order.
2) Take a $(23, 5, 1)$ minimal covering design, in decreasing order, with a hole of size three, say, $\{23, 22, 21\}$, [22]. Place the point 23 at the end of the blocks in which it is contained then replace it by 22. Then it is easy to check this construction yields the blocks of a minimal DC$(22, 5, 1)$ with a hole of size two.

ii) The construction of a minimal DC$(v, 5, 1)$ for all $v \equiv 2 \pmod{20}$ consists of the following two steps:

1) Take a $(v + 1, 5, 1)$ minimal covering design in decreasing order. This design has a block of size three, say, $\langle 3\ 2\ 1 \rangle$, [22]. Assume in this design we have the block $\langle v + 1\ v\ 10\ 9\ 8 \rangle$ where $\{8, 9, 10\}$ are arbitrary numbers. Replace this block by the block $\langle v\ 10\ 9\ 8\ 11 \rangle$. In all other blocks containing $v + 1$, place $v + 1$ at the end of the blocks and then replace it by v. Further, replace the block $\langle 3\ 2\ 1 \rangle$ by the block $\langle 3\ 2\ 1\ 11\ a \rangle$.

2) Take a $B[v-1, 5, 1]$ in increasing order. Assume we have the block $\langle 8\ 9\ 10\ 11\ a \rangle$ where a is an arbitrary number. Replace this block by the block $\langle 8\ 9\ 10\ v\ a \rangle$.

Lemma 3.2 *Let $v \equiv 4 \pmod{20}$ be a positive integer. Then $DE(v, 5, 1) = DL(v, 5, 1)$.*

Proof For all positive integers $v \equiv 4 \pmod{20}$, the construction consists of the following two steps:

(1) Take a $B[v + 1, 5, 1]$ in increasing order. Assume we have the block $\langle v - 3\ v - 2\ v - 1\ v\ v + 1 \rangle$ from which we delete the point $v + 1$. Further, place the point $v + 1$ at the beginning of the blocks in which it is contained and then replace it by v.

(2) Take a $(v - 1, 5, 1)$ minimal covering design in decreasing order [22]. This design has a block of size three, say, $\langle v - 1\ v - 2\ v - 3 \rangle$. Replace this block by $\langle v\ v - 3\ v - 2\ v - 1 \rangle$.

Remark: By deleting the two blocks of size four in Lemma 3.2, we obtain a directed covering with a hole of size 4 for all $v \equiv 4 \pmod{20}$.

Lemma 3.3 *Let $v \equiv 6 \pmod{20}$ be a positive integer. Then $DE(v, 5, 1) = DL(v, 5, 1)$.*

Proof. For all positive integers $v \equiv 6 \pmod{20}$ the construction is as follows:

1) Take a $(v+1, 5, 1)$ minimal covering design in increasing order, [21]. Assume we have the block $\langle v-3\ v-2\ v-1\ v\ v+1 \rangle$, which we replace by $\langle v-3\ v-2\ v-1\ v\ v-5 \rangle$. In all other blocks through $v + 1$, we place $v + 1$ at the beginning of each block and then replace it by v. Further, we may assume that the pair $(v - 5, v - 4)$ appears in at least two blocks. Take a block containing $(v - 5, v - 4)$ and place $v - 4$ before $v - 5$.

2) Take a $B[v - 1, 5, 1]$ in decreasing order. Assume we have the block $\langle v - 1\ v - 2\ v - 3\ v - 4\ v - 5 \rangle$ which we replace by $\langle v\ v - 1\ v - 2\ v - 3\ v - 4 \rangle$.

Lemma 3.4 *Let* $v \equiv 8 \ (mod\ 20)$ *be a positive integer. Then* $DE(v,5,1) = DL(v,5,1)$.

Proof. For $v = 8$ let $X = \{1, 2, \ldots, 8\}$; then the blocks are:

$\langle 1\ 2\ 5\ 4\ 3\rangle$, $\langle 4\ 3\ 5\ 2\ 1\rangle$, $\langle 8\ 7\ 6\ 2\ 1\rangle$, $\langle 3\ 6\ 7\ 8\ 4\rangle$, $\langle 5\ 6\ 7\ 8\ 3\rangle$, $\langle 1\ 2\ 6\ 7\ 8\rangle$, $\langle 4\ 6\ 7\ 8\ 5\rangle$.

For all other values the construction is as follows:

1) Take a $B[v-3, 5, 1]$ in increasing order.

2) Take a $(v+3, 5, 1)$ minimal covering design with a hole of size three, say, $\{v+1, v+2, v+3\}$ in decreasing order,[22], place the points of the hole at the end of each block containing them, then replace $v+3$ by v, $v+2$ by $v-1$ and $v+1$ by $v-2$.

Lemma 3.5 *Let* $v \equiv 0 \ (mod\ 10)$ *be a positive integer. Then* $DE(v,5,1) = DL(v,5,1)$.

Proof For $v \equiv 0 \pmod{20}$ the construction is as follows:

1) Take a $(v-1, 5, 1)$ minimal covering design in decreasing order [21], and assume that the pair $(5, 4)$ appears in at least two blocks.

2) Take a $B[v+1, 5, 1]$ in increasing order and assume we have the block $\langle 1\ 2\ 3\ v\ v+1\rangle$ where $\{1, 2, 3\}$ are arbitrary numbers. Replace this block by the block $\langle 4\ 1\ 2\ 3\ v\rangle$. In all other blocks through $v+1$, place $v+1$ at the beginning of each block, then replace it by v. Further, assume in (1) we have the block $\langle 5\ 4\ 3\ 2\ 1\rangle$ which we replace by $\langle 5\ 3\ 2\ 1\ v\rangle$.

For $v \equiv 10 \pmod{20}$, the values for $v = 10, 30, 50$ are given in the next table. In general, the construction in this table and all other tables is as follows: Let $X = Z_{v-n} \cup H_n$ or $X = (Z_2 \times Z_{(v-n)/2}) \cup H_n$ where $H_n = \{h_1, \ldots, h_n\}$ is the hole. The blocks are constructed by taking the orbits of the tabulated base blocks.

V	Point Scale	Base Blocks
10	Z_{10}	$\langle 2\ 0\ 5\ 7\ 1\rangle$
30	Z_{30}	$\langle 0\ 12\ 4\ 2\ 1\rangle$ $\langle 0\ 7\ 3\ 13\ 18\rangle$ $\langle 0\ 8\ 3\ 24\ 17\rangle$
50	Z_{50}	$\langle 0\ 1\ 27\ 9\ 20\rangle$ $\langle 0\ 34\ 39\ 36\ 49\rangle$ $\langle 0\ 46\ 3\ 44\ 24\rangle$ $\langle 0\ 12\ 23\ 4\ 37\rangle$ $\langle 0\ 18\ 40\ 6\ 35\rangle$

For all other values of v, take a $(5,1)$-DGDD of type 10^m, m is odd [15]; then on each group construct a minimal DC$(10, 5, 1)$.

Lemma 3.6 *Let* $v \equiv 12 \ (mod\ 20)$ *be a positive integer. Then* $DE(v,5,1) = DL(v,5,1)$.

Proof For $v = 12$ let $X = \{1, 2, \ldots, 12\}$. Then the blocks are:

$\langle 8\ 3\ 12\ 1\ 2\rangle$ $\langle 4\ 7\ 2\ 11\ 1\rangle$ $\langle 1\ 10\ 6\ 3\ 4\rangle$ $\langle 2\ 4\ 9\ 5\ 3\rangle$
$\langle 12\ 7\ 5\ 4\ 6\rangle$ $\langle 3\ 8\ 6\ 11\ 5\rangle$ $\langle 5\ 10\ 2\ 7\ 8\rangle$ $\langle 6\ 1\ 9\ 8\ 7\rangle$
$\langle 11\ 4\ 9\ 8\ 10\rangle$ $\langle 7\ 3\ 12\ 10\ 9\rangle$ $\langle 9\ 11\ 6\ 2\ 12\rangle$ $\langle 10\ 1\ 5\ 12\ 11\rangle$
$\langle 5\ 8\ 9\ 4\ 1\rangle$ $\langle 2\ 6\ 4\ 10\ 12\rangle$ $\langle 12\ 11\ 3\ 7\ 8\rangle$.

For $v = 32$, the construction is as follows:

1) Take a $(31, 5, 1)$ minimal covering design in increasing order [22]. This design has a block of size three, say $\langle 29\ 30\ 31\rangle$, which we delete.

2) Take a $(33, 5, 1)$ covering design with $\phi(33, 5, 1) + 1$ blocks in decreasing order, [11]. Close observation of this design shows there is at least one triple, say, $\{31, 30, 29\}$, the pairs of which appear in two blocks. Assume in this design we have the block $\langle 33\ 32\ c\ b\ a\rangle$ which we replace by $\langle 32\ c\ b\ a\ e\rangle$. In all other blocks through 33, place 33 at the end of these blocks then replace it by 32. Assume in (1) we have the block $\langle a\ b\ c\ d\ e\rangle$ and that (d, e) appears in at least two blocks. Replace this block by the block $\langle a\ b\ c\ d\ 32\rangle$.

The above two steps give a design such that the pairs $(29, 30)$, $(29, 31)$, and $(30, 31)$ appear in zero blocks, $(30, 29)$, $(31, 29)$, $(31, 30)$ appear twice while each other ordered pair appears in at least one block. To fix this problem take three blocks containing the pairs $(30, 29)$, $(31, 29)$, and $(31, 30)$ and we switch the order of these pairs so that $(29, 30)$, $(29, 31)$, $(30, 31)$ each appears exactly once.

For $v = 52$, let $X = Z_{44} \cup H_8$. Then the required blocks are the following (mod 44)

$$\langle 22\ 2\ 8\ 18\ 0\rangle,\ \langle 2\ 0\ 4\ 1\ 9\rangle,\ \langle 26\ 15\ -\ 0\ 3\rangle \cup \{h_1, h_2\},$$

$$\langle 5\ 30\ -\ 0\ 17\rangle \cup \{h_3, h_4\},\ \langle 6\ 29\ -\ 0\ 19\rangle \cup \{h_5, h_6\},\ \langle 16\ 7\ -\ 0\ 27\rangle \cup \{h_7, h_8\}.$$

For $v = 92$ we first construct a $(93, 5, 1)$ covering design with $\phi(93, 5, 1)+1$ blocks such that there is a triple the pairs of which appear in at least two blocks. Such a design can be constructed by taking a $T[5, 1, 18]$, adjoin three new points to the groups and then on each group construct a $B[21, 5, 1]$. Now the construction of a minimal DC$(92, 5, 1)$ is exactly the same as DC$(32, 5, 1)$.

For $v = 132$, by adjoining 33 new points to the 33 parallel classes of RB$[100, 4, 1]$ we obtain a $(133, 5, 1)$ covering design with a hole of size 33, on which we construct a $(33, 5, 1)$ covering design with $\phi(93, 5, 1) + 1$ blocks. Now the construction of a minimal DC$(132, 5, 1)$ is the same as the DC$(32, 5, 1)$.

For a DC$(72, 5, 1)$, take a $T[5, 1, 7] - T[5, 1, 1]$ and inflate this design by a factor of two; that is, we replace each block by the blocks of a $(5, 1)$-DGDD of type 2^5, [15]. Adjoin two points $\{a, b\}$ to the groups of the resultant design and on each group we construct a minimal DC$(16, 5, 1)$ with a hole of size four such that the hole is on $\{a, b\}$ together with the two points of the hole of the directed $T[5, 1, 14] - T[5, 1, 2]$. Finally, on the hole of the directed $T[5, 1, 14] - T[5, 1, 2]$ with the two points $\{a, b\}$ we construct a minimal DC$(12, 5, 1)$.

For a minimal DC$(16, 5, 1)$ with a hole of the size 4, let $X = Z_{12} \cup \{a, b, c, d\}$. Then the blocks are:

$\langle 3\ 0\ a\ 7\ 1\rangle$	$\langle 2\ 5\ a\ 9\ 3\rangle$	$\langle 7\ 4\ a\ 5\ 11\rangle$	$\langle 10\ 1\ a\ 6\ 0\rangle$	$\langle 9\ 6\ a\ 8\ 2\rangle$	$\langle 8\ 11\ a\ 10\ 4\rangle$
$\langle 6\ 9\ b\ 1\ 7\rangle$	$\langle 11\ 3\ b\ 8\ 9\rangle$	$\langle 10\ 1\ b\ 5\ 11\rangle$	$\langle 8\ 0\ b\ 3\ 2\rangle$	$\langle 4\ 7\ b\ 6\ 0\rangle$	$\langle 5\ 2\ b\ 4\ 10\rangle$
$\langle 4\ 1\ c\ 8\ 2\rangle$	$\langle 6\ 10\ c\ 3\ 4\rangle$	$\langle 0\ 8\ c\ 5\ 6\rangle$	$\langle 9\ 5\ c\ 0\ 11\rangle$	$\langle 11\ 2\ c\ 7\ 1\rangle$	$\langle 7\ 3\ c\ 9\ 10\rangle$
$\langle 10\ 7\ d\ 2\ 8\rangle$	$\langle 0\ 4\ d\ 10\ 9\rangle$	$\langle 2\ 11\ d\ 0\ 6\rangle$	$\langle 1\ 9\ d\ 4\ 3\rangle$	$\langle 5\ 8\ d\ 7\ 1\rangle$	$\langle 3\ 6\ d\ 11\ 5\rangle$

For all other values of v, simple calculations show that v can be written in the form $v = 20m + 4u + h + s$ where m, u, h and s are chosen so that:

(1) There exists a $(5,1)$-RMGDD of type 5^m;
(2) There exists a $(5,1)$-DGDD of type $4^m s^1$;
(3) $4u+h+s = 12, 32, 52, 72, 92$;
(4) $0 \le u \le m-1$, $s \equiv 0 \pmod 4$ and $h = 0$.

Now apply Theorem 2.4 with $\lambda = 1$ to get the result.

Lemma 3.7 *Let $v \equiv 14 \ (mod\ 20)$ be a positive integer. Then $DE(v,5,1) = DL(v,5,1)$.*

Proof For $v = 14$, let $X = \{1,\ldots,14\}$. Then the required blocks are:

$\langle 1\ 8\ 12\ 2\ 3\rangle$	$\langle 11\ 7\ 2\ 4\ 1\rangle$	$\langle 6\ 1\ 3\ 10\ 4\rangle$	$\langle 9\ 5\ 4\ 3\ 2\rangle$	$\langle 5\ 12\ 4\ 6\ 7\rangle$
$\langle 11\ 3\ 6\ 8\ 5\rangle$	$\langle 2\ 7\ 5\ 10\ 8\rangle$	$\langle 9\ 8\ 1\ 7\ 6\rangle$	$\langle 3\ 1\ 2\ 14\ 13\rangle$	$\langle 6\ 14\ 13\ 4\ 5\rangle$
$\langle 8\ 4\ 11\ 9\ 10\rangle$	$\langle 10\ 7\ 3\ 12\ 9\rangle$	$\langle 10\ 12\ 1\ 5\ 11\rangle$	$\langle 4\ 14\ 13\ 12\ 8\rangle$	$\langle 13\ 14\ 3\ 7\ 11\rangle$
$\langle 5\ 13\ 14\ 1\ 9\rangle$	$\langle 9\ 8\ 7\ 13\ 14\rangle$	$\langle 11\ 12\ 10\ 13\ 14\rangle$	$\langle 6\ 2\ 9\ 12\ 11\rangle$	$\langle 14\ 13\ 10\ 2\ 6\rangle$.

For all other values of v, the construction is as follows:

1) Take a $(v-3,5,1)$ minimal covering design in increasing order [21]. Close observation of these designs show that they have a block of size three, say, $\langle v-5, v-4, v-3\rangle$. Delete this block.

2) Take a $(v+3,5,1)$ minimal covering design with a hole size of 9 in increasing order, [19]. Place the points $v+1$, $v+2$ and $v+3$ at the end of each block containing them, then replace $v+3$ by v, $v+2$ by $v-1$ and $v+1$ by $v-2$.

3) On $\{v-5, v-4, v-3, v-2, v-1, v\}$, construct a minimal DC$(6,5,1)$.

Lemma 3.8 *Let $v \equiv 16 \ (mod\ 20)$ be a positive integer. Then $DE(v,5,1) = DL(v,5,1)$.*

Proof For $v = 16$, see Lemma 3.6.

For all other values of v, the construction is as follows:

1) Take a $(v-1,5,1)$ minimal covering design in decreasing order [21].

2) Take a $(v+1,5,1)$ minimal covering design in increasing order with a hole of size 9 on $\{v-7,\ldots,v+1\}$, [19]. Place the point $v+1$ at the beginning of the blocks which contain it and then replace it by v.

3) On $\{v-7, v-6, \ldots, v\}$ take the following blocks:

$$\langle v-5\ v-4\ v-3\ v-2\ v\rangle \quad \langle v-6\ v-3\ v-2\ v-1\ v\rangle$$
$$\langle v-7\ v\ v-5\ v-4\ v-1\rangle \quad \langle v\ v-7\ v-6\ v-3\ v-2\rangle.$$

The above three steps guarantee that each ordered pair appears at least once except $(v-6, v-5)$ and $(v-6, v-4)$. Now consider the blocks of the $(v-1,5,1)$ minimal covering design in decreasing order. This design has v repeated pairs. Further, close observation of these designs shows that we may assume that $(v-5, v-6)$ and $(v-4, v-6)$ each appears at least twice. If $\{v-6, v-5, v-4\}$ appear in one block, say, $\langle y\ v-4\ v-5\ v-6\ x\rangle$, then replace this block by $\langle y\ v-6\ v-4\ v-5\ x\rangle$. Otherwise, there are two blocks; one contains $(v-5, v-6)$ and the other contains $(v-4, v-6)$. Then in the first block write $v-6$ in front of $v-5$, and in the second write $v-6$ in front of $v-4$. Then it is clear that the above construction yields the blocks of a minimal DC$(v, 5, 1)$ for $v \equiv 16 \pmod{20}$, $v \geq 36$.

Lemma 3.9 *Let $v \equiv 18 \ (mod\ 20)$ be a positive integer. Then $DE(v, 5, 1) = DL(v, 5, 1)$.*

Proof For $v = 18$ let $X = Z_{15} \cup \{a, b, c\}$. Then take the following blocks:

$$\begin{array}{lll} \langle 0\ 3\ 6\ 9\ 12\rangle + i,\ i \in Z_3 & \langle 3\ 9\ 14\ 7\ 1\rangle + i,\ i \in Z_8 & \langle 11\ 8\ 7\ 9\ 0\rangle + i,\ i \in Z_4 \\ \langle 12\ 0\ 11\ 13\ 4\rangle + i,\ i \in Z_3 & \langle 0\ 5\ 10\ a\ b\rangle + i,\ i \in Z_4 & \langle a\ c\ 0\ 5\ 10 >\rangle + i,\ i \in Z_4 \\ \langle b\ 10\ 5\ 0\ c\rangle + i,\ i \in Z_4 & \langle b\ 4\ 9\ 14\ a\rangle \ \ \langle 4\ 9\ 14\ c\ a\rangle & \langle c\ 14\ 9\ 4\ b\rangle. \end{array}$$

For $v \geq 38$ the construction is as follows:

1) Take a $B[v+3, 5, 1]$ in increasing order. Assume that $\{v+1\ v+2\ v+3\}$ are not contained in one block. So assume we have the following three blocks:

$$\langle a\ b\ c\ v\ v+3\rangle \quad \langle d\ e\ f\ v-1\ v+2\rangle \quad \langle g\ h\ i\ v-2\ v+1\rangle$$

which we replace by

$$\langle v-1\ a\ b\ c\ v\rangle \quad \langle v-2\ d\ e\ f\ v-1\rangle \quad \langle v\ g\ h\ i\ v-2\rangle$$

respectively. Further, place $v+3$, $v+2$, $v+1$ at the beginning of the blocks in which they are contained, then replace $v+3$ by v, $v+2$ by $v-1$ and $v+1$ by $v-2$.

2) Take a $(v-3, 5, 1)$ covering design in decreasing order. Assume that the pairs $(v-2, y)$, $(v-1, x)$, $(v, 2)$ are repeated in this design. Further, assume we have the blocks:

$$\langle v-1\ x\ c\ b\ a\rangle \quad \langle v-2\ y\ f\ e\ d\rangle \quad \langle v\ z\ i\ h\ g\rangle,$$

which we replace by

$$\langle v\ x\ c\ b\ a\rangle \quad \langle v-1\ y\ f\ e\ d\rangle \quad \langle v-2\ z\ i\ h\ g\rangle.$$

Then it is readily checked that the above steps yield the blocks of a minimal DC$(v, 5, 1)$ for $v \equiv 18 \pmod{20}$, $v \geq 38$.

4 Directed covering with index 3

We first observe that if $v \equiv 14 \pmod{20}$ then a minimal DC$(v, 5, 3)$ can be constructed by taking two copies of a minimal $(v, 5, 3)$ covering design in opposite order. We now turn our attention to the remaining cases.

Lemma 4.1 *Let $v \equiv 0$ or $6 \ (mod\ 10)$ and $\lambda \geq 3$ be positive integers; then $DE(v, 5, \lambda) = DL(v, 5, \lambda)$.*

Proof If λ is even then see [9]. If λ is odd, say, $2m+1$, then the blocks of a minimal DC$(v, 5, \lambda)$ are the blocks of a DB$[v, 5, 2m]$ together with blocks of a minimal DC$(v, 5, 1)$.

Lemma 4.2 *Let $v \equiv 2 \ (mod\ 20)$ be a positive integer. Then $DE(v, 5, 3) = DL(v, 5, 3)$.*

Proof For all $v \equiv 2 \pmod{20}$, $v \geq 22$, the construction is as follows:

1) Take the minimal DC$(v, 5, 2)$ in [9]. This design has a triple, say, $\{v-2, v-1, v\}$ the ordered pairs of which appear in three blocks.

2) Take a $B[v-1, 5, 1]$ in increasing order.

3) Take a $(v+1, 5, 1)$ minimal covering design in decreasing order [22]. This design has a block of size three, say, $\langle v-1, v, v+1\rangle$ which we delete. Further, take the blocks through $v+1$, place $v+1$ at the end of each block then replace it by v.

Now it is easily checked that the above three steps yield the blocks of a minimal DC$(v, 5, 3)$ for $v \equiv 2 \pmod{20}$.

Lemma 4.3 *Let $v \equiv 4$ (mod 20) be a positive integer. Then DE$(v, 5, 3)$ = DL$(v, 5, 3)$.*

Proof For all such v the construction is as follows:

1) Take the minimal DC$(v, 5, 2)$ in [9]. This design has a triple, say, $\{a, b, c\}$ the ordered pairs of which appear in three blocks.

2) Take a $(v-1, 5, 1)$ minimal covering design in increasing order [22]. This design has a block of size three, say, $\langle a\ b\ c\rangle$ which we delete. Further, assume we have the block $\langle 1\ 2\ 3\ 7\ 10\rangle$ where $\{1, 2, 3, 7, 10\}$ are arbitrary numbers such that $(7, 10)$ appears in two blocks. Replace this block by the block $\langle 1\ 2\ 3\ 7\ v\rangle$.

3) Take a $B[v+1, 5, 1]$ in decreasing order. Assume we have the block $\langle v+1\ v\ 3\ 2\ 1\rangle$ which we replace by $\langle v\ 3\ 2\ 1\ 10\rangle$. Further, take the remaining blocks through $v+1$, place $v+1$ at the end of each block then replace it by v.

It is readily checked that the above three steps yield the blocks of a minimal DC$(v, 5, 3)$, $v \equiv 4 \pmod{20}$.

Lemma 4.4 *Let $v \equiv 8$ (mod 20) be a positive integer. Then DE$(v, 5, 3)$ = DL$(v, 5, 3)$.*

Proof For $v = 8$, let $X = Z_8$. Then the required blocks are: $\langle 0\ 7\ 3\ 5\ 4\rangle \pmod 8$, $\langle 4\ 7\ 2\ 0\ 1\rangle + i$, $i \in Z_6$, $\langle 2\ 5\ 6\ 0\ 7\rangle + i$, $i \in Z_2$, $\langle 0\ 2\ 4\ 6\rangle + i$, $i \in Z_2$.

For $v = 28$, let $X = Z_{22} \cup H_6$. Then the required blocks are the following (mod 22), together with the blocks of a DC$(6, 5, 3)$ on $\{h_1, \ldots, h_6\}$.

$\langle 2\ 0\ 21\ 6\ 13\rangle$	$\langle 8\ 0\ h_1\ 2\ 12\rangle$	$\langle 0\ 14\ h_2\ 2\ 8\rangle$
$\langle 3\ 2\ h_3\ 0\ 1\rangle$	$\langle 7\ 3\ h_4\ 0\ 18\rangle$	$\langle 0\ 3\ h_5\ 8\ 13\rangle$
$\langle 4\ 0\ h_6\ 9\ 15\rangle$	$\langle 0\ 7\ -\ 1\ 2\rangle \cup \{h_1, h_2\}$	$\langle 10\ 13\ -\ 0\ 3\rangle \cup \{h_3, h_4\}$
$\langle 9\ 0\ -\ 17\ 4\rangle \cup \{h_5, h_6\}$.		

For $v = 48$, let $X = Z_{40} \cup H_8$. On $Z_{40} \cup H_7$ construct an optimal DP$(47, 5, 2)$ with a hole of size 7 on H_7. Further, take the following blocks (mod 40):

$\langle 0\ 2\ 6\ 24\ 14\rangle$	$\langle 5\ 2\ h_8\ 0\ 1\rangle$	$\langle 0\ 11\ -\ 26\ 3\rangle \cup \{h_1, h_2\}$
$\langle 0\ 5\ -\ 18\ 25\rangle \cup \{h_3, h_4\}$	$\langle 13\ 6\ -\ 0\ 29\rangle \cup \{h_5, h_6\}$	$\langle 9\ 0\ -\ 19\ 28\rangle \cup \{h_7, h_8\}$.

For $v = 68, 88$, take a $(6,3)$-GDD of type 5^n, $n = 7, 9$ and delete one point from the last group. Inflate the resulting design by a factor of two. Replace all its blocks by the blocks of a $(5,1)$-DGDD of type 2^5 and 2^6, [28]. Finally, on the first $(n-1)$

groups construct a minimal DC(10, 5, 3) and on the last group construct a minimal DC(8, 5, 3).

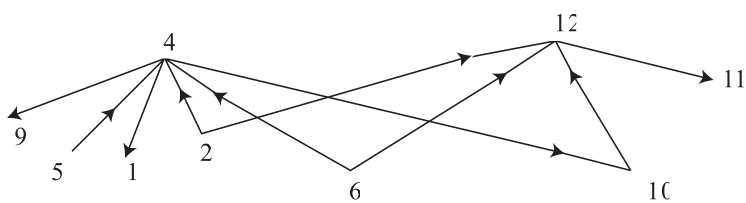
For $v = 128$, apply Theorem 2.2 with $m = 7$, $u = 2$, $h = 0$ and $\lambda = 3$.

For all other values of v, simple calculations show that v can be written in the form $v = 20m + 4u + h + s$ where m, u, h and s are chosen as in Lemma 3.6 with the difference that $4u + h + s = 8, 28, 48, 68, 88$. Now apply Theorem 2.4 with $\lambda = 3$ to get the result.

Lemma 4.5 *Let $v \equiv 12 \pmod{20}$ be a positive integer. Then $DE(v, 5, 3) = DL(v, 5, 3)$.*

Proof For $v = 12$ the construction is as follows:

1) Take the blocks of the minimal DC(12, 5, 1) (Lemma 3.6). The excess graph of this design contains the following digraph.



From this design, delete the block ⟨2 6 4 10 12⟩. Further, replace the block ⟨12 11 3 7 8⟩ by the block ⟨4 12 3 7 8⟩.

2) Take the minimal DC(12, 5, 2) in [9]. Apply the permutation (2 a), (b 6) and replace c by 10; we obtain a minimal DC(12, 5, 2) such that the ordered pairs of $\{2, 6, 10\}$ appear in three blocks. Apply the permutation (3 b) to the blocks of this design; then replace 0 by 9, a by 11, and b by 12. We obtain a minimal DC(12, 5, 2) on $\{1, \ldots, 12\}$ such that we have the following blocks: ⟨8 7 4 3 1⟩ ⟨1 4 7 2 10⟩ ⟨4 9 8 2 6⟩, which we replace by ⟨11 8 7 3 1⟩ ⟨1 7 4 2 10⟩ ⟨9 8 4 2 6⟩.

Then it is easy to check that the above two steps yield the blocks of a minimal DC(12, 5, 3).

For $v = 32, 52, 72, 92$ the construction is as follows:

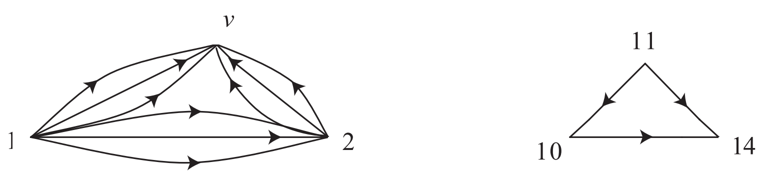
1) Take a minimal DC(v, 5, 1) with a hole of size 8 such that $\{1, 2, 10, 11, 14, v\}$ are points of the hole.

2) Take a $B[v - 1, 5, 2]$ in increasing order and assume we have the two blocks ⟨3 4 5 10 14⟩ and ⟨6 7 8 11 14⟩, which we replace by ⟨3 4 5 v 14⟩ ⟨6 7 8 v 14⟩.

3) Take a ($v+1$, 5, 2) optimal packing design in decreasing order, [7]. This design has a triple, say, $\{1, 2, v+1\}$, the pairs of which appear in zero blocks. Further, assume we have the following blocks: ⟨$v+1$ v 5 4 3⟩ ⟨$v+1$ v 8 7 6⟩, which we

replace by $\langle v\ 5\ 4\ 3\ 10\rangle$ $\langle v\ 8\ 7\ 6\ 11\rangle$. Further, place the point $v+1$ at the end of the blocks in which it is contained, and then replace it by v.

4) On the hole of size eight, take the minimal DC(8, 5, 1) of Lemma 3.4 and notice that the excess graph contains the following digraph, say, on {1, 2, v} and {10, 11, 14}.



It is easy to check that the above four steps yield the blocks of a minimal DC$(v, 5, 3)$ for $v = 32, 52, 72, 92$.

To complete the proof of our Lemma we need to construct a minimal DC$(v, 5, 1)$ with a hole of size 8 for $v = 32, 52, 72, 92$.

For $v = 32$, let $X = Z_2 \times Z_{12} \cup H_8$. Then the required blocks are the following mod $(-, 12)$:

$\langle(0,0)\ (0,3)\ -\ (0,1)\ (0,8)\rangle \cup \{h_1, h_2\}$ $\langle(1,3)\ (1,0)\ -\ (1,1)\ (1,8)\rangle \cup \{h_1, h_2\}$
$\langle(1,1)\ (0,1)\ h_3\ (0,0)\ (1,0)\rangle$ $\langle(0,0)\ (1,7)\ h_4\ (1,1)\ (0,2)\rangle$
$\langle(0,3)\ (1,6)\ h_5\ (0,0)\ (1,8)\rangle$ $\langle(1,7)\ (0,0)\ h_6\ (0,4)\ (1,10)\rangle$
$\langle(1,2)\ (0,0)\ h_7\ (0,5)\ (1,9)\rangle$ $\langle(1,10)\ (0,0)\ h_8\ (1,2)\ (0,6)\rangle$.

For $v = 52$, let $X = Z_{44} \cup H_8$. Then take the following blocks (mod 44):

$\langle 12\ 8\ 0\ 2\ 26\rangle$ $\langle 1\ 2\ 5\ 0\ 13\rangle$ $\langle 3\ 0\ -\ 25\ 18\rangle \cup \{h_1, h_2\}$
$\langle 0\ 19\ -\ 28\ 5\rangle \cup \{h_3, h_4\}$ $\langle 0\ 17\ -\ 27\ 6\rangle \cup \{h_5, h_6\}$ $\langle 0\ 31\ -\ 16\ 7\rangle \cup \{h_7, h_8\}$.

For $v = 72$, let $X = Z_{64} \cup H_8$. Then take the following blocks (mod 64):

$\langle 30\ 14\ 6\ 2\ 0\rangle$ $\langle 1\ 0\ 16\ 42\ 6\rangle$ $\langle 37\ 46\ 3\ 0\ 17\rangle$
$\langle 0\ 7\ 31\ 20\ 39\rangle$ $\langle 1\ 0\ -\ 11\ 4\rangle \cup \{h_1, h_2\}$ $\langle 17\ 0\ -\ 29\ 2\rangle \cup \{h_3, h_4\}$
$\langle 5\ 18\ -\ 43\ 0\rangle \cup \{h_5, h_6\}$ $\langle 42\ 9\ -\ 0\ 23\rangle \cup \{h_7, h_8\}$.

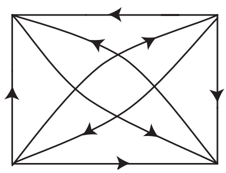
For $v = 92$, let $X = Z_{84} \cup H_8$. Then take the following blocks (mod 84):

$\langle 4\ 12\ 42\ 0\ 26\rangle$ $\langle 22\ 1\ 3\ 7\ 0\rangle$ $\langle 0\ 56\ 31\ 48\ 5\rangle$
$\langle 54\ 9\ 20\ 44\ 0\rangle$ $\langle 68\ 0\ 27\ 45\ 13\rangle$ $\langle 0\ 13\ 15\ 49\ 1\rangle$
$\langle 5\ 24\ -\ 33\ 0\rangle \cup \{h_1, h_2\}$ $\langle 6\ 31\ -\ 13\ 0\rangle \cup \{h_3, h_4\}$ $\langle 27\ 0\ -\ 10\ 47\rangle \cup \{h_5, h_6\}$
$\langle 11\ 0\ -\ 32\ 55\rangle \cup \{h_7, h_8\}$.

For a minimal DC$(132, 5, 3)$, apply Theorem 2.2 with $m = 7$, $h = 0$, $u = 3$ and $\lambda = 3$.

For all other values of v, write $v = 20m + 4u + h + s$ where m, u, h and s are chosen as in Lemma 3.6; then apply Theorem 2.4 with $\lambda = 3$ to get the result.

Lemma 4.6 *There exists a minimal DC*$(v,5,2)$ *for all* $v \equiv 18 \pmod{20}$, $v \geq 38$, $v \neq 178$, *such that the excess digraph is the following digraph:*



Proof We first show that the excess digraph of a minimal DC$(8,5,2)$ and DC$(13,5,2)$ is the above digraph.

For $v = 8$, take the blocks of the minimal DC$(8,5,2)$ from [9], and replace the blocks $\langle 0\ 3\ 2\ 5\ 6\rangle$ $\langle 4\ 1\ 6\ 3\ 0\rangle$ $\langle 5\ 2\ 3\ 1\ 4\rangle$ by $\langle 0\ 2\ 3\ 5\ 6\rangle$ $\langle 1\ 4\ 6\ 3\ 0\rangle$ $\langle 5\ 4\ 2\ 3\ 1\rangle$.

For $v = 13$ take the blocks of the minimal DC$(13,5,2)$ from [9] and replace the blocks $\langle 2\ 3\ 7\ 13\ 5\rangle$ $\langle 12\ 7\ 8\ 9\ 3\rangle$ $\langle 11\ 7\ 4\ 12\ 8\rangle$ by the blocks $\langle 2\ 7\ 3\ 13\ 5\rangle$ $\langle 12\ 8\ 7\ 9\ 3\rangle$ $\langle 7\ 11\ 4\ 12\ 8\rangle$.

Now to prove our lemma we show that for such v there exists a minimal DC$(v,5,2)$ with a hole of size 8 or 13. But this can be done exactly the same as Lemma 5.9 of [6].

Lemma 4.7 *Let* $v \equiv 18 \pmod{20}$ *be a positive integer. Then* $DE(v,5,3) = DL(v,5,3)$.

Proof For $v = 18$ the construction is as follows:

1) Take an optimal DP$(18,5,2)$ [8]. This design has every ordered pair appearing in two blocks except the pairs of a triple, say, $\{a,b,c\}$, which appear in zero blocks.

2) Take the following blocks of a minimal DC$(18,5,1)$ on $X = Z_{15} \cup \{a,b,c\}$: $\langle 4\ 8\ 0\ 2\ 1\rangle$ (mod 15) $\langle 12+k\ 9+k\ 6+k\ 3+k\ k\rangle$, $k = 0,1,2$, $\langle 0\ 5\ 10\ b\ a\rangle$, orbit length 3, $\langle c\ b\ 10\ 5\ 0\rangle$, orbit length 2, $\langle b\ c\ 12\ 7\ 2\rangle$, orbit length 1, $\langle a\ 0\ 5\ 10\ c\rangle$, orbit length 3, $\langle a\ 3\ 8\ 13\ b\rangle$, orbit length 2, $\langle b\ 13\ 8\ 3\ c\rangle$, orbit length 2, $\langle c\ 3\ 8\ 13\ a\rangle$, orbit length 2.

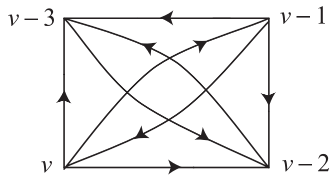
3) Adjoin the block $\langle c\ a\ b\rangle$.

For $v = 178$, apply Theorem 2.6 with $m = 16$, $h = s = 0$, $\lambda = 3$ and $u = 9$, and see [15] for a $(5,3)$-DGDD of type 2^{16}.

For all other values of $v \equiv 18 \pmod{20}$, the construction is as follows:

1) Take a minimal DC$(v,5,2)$ such that the excess graph is the following digraph (overpage) on $\{v-3, v-2, v-1, v\}$.

2) Take a $(v-3,5,1)$ minimal covering design in decreasing order [21]. Assume that the ordered pair (d,x) appears in at least two blocks, and that we have the block $\langle d\ c\ b\ a\ x\rangle$ where x is to the right of d. Replace this block by $\langle c\ b\ a\ x\ v-2\rangle$.



3) Take a $B[v+3,5,1]$ in increasing order. Delete from this design the block $\langle v-1\ v\ v+1\ v+2\ v+3\rangle$. Further, assume we have the block $\langle a\ b\ c\ v-2\ v+1\rangle$, which we replace by $\langle d\ a\ b\ c\ v-2\rangle$. In all other blocks through $v+1$, $v+2$, $v+3$, we place these points at the beginning of these blocks and then replace $v+1$ by $v-2$, $v+2$ by $v-1$, and $v+3$ by v.

Then it is easy to check that the above steps yield the blocks of a minimal DC$(v,5,3)$, $v \equiv 18 \pmod{20}$, $v \neq 18, 178$.

5 Directed covering with index 5

Notice that when $v \equiv 2$ or $4 \pmod{10}$ then a minimal DC$(v,5,5)$ can be constructed by taking a minimal DC$(v,5,2)$ and a minimal DC$(v,5,3)$. Furthermore, the case $v \equiv 0$ or $6 \pmod{10}$ follows from Lemma 4.1. The only case left is $v \equiv 8 \pmod{10}$. The following lemma is most useful for us.

Lemma 5.1 *(i) There exists a minimal DC*$(v,5,1)$ *with a hole of size* 2 *for* $v = 8, 18, 28, 38, 48, 58, 68, 78, 88, 98$.
(ii) There exists a minimal DC$(22,5,5)$ *with a hole of size* 2.

Proof (i) For $v = 88, 98$, take a $(6,1)$-GDD of type 8^6 and delete 5 and 0 points from the last group respectively and inflate the resulting design by a factor of two. Replace the blocks of the resulting design which are of size 5 and 6 by the blocks of a $(5,1)$-DGDD of type 2^5 and 2^6 respectively [28]. Adjoin two new points to the groups, and on the first five groups construct a minimal DC$(18,5,1)$ with a hole of size 2 and on the last group construct a minimal DC$(8,5,1)$ with a hole of size 2 in the case $v = 88$, and a minimal DC$(18,5,1)$ with a hole of size 2 when $v = 98$.

For all other values see the next table (overpage).

For a minimal DC$(22,5,5)$ with a hole of size 2, take one copy of a minimal DC$(22,5,4)$ with a hole of size 2 [9], and one copy of a minimal DC$(22,5,1)$ with a hole of size 2 (Lemma 3.1).

Lemma 5.2 *Let* $v \equiv 8 \pmod{10}$ *be a positive integer. Then DE*$(v,5,5) =$ *DL*$(v,5,5)$.

Proof For $v = 8, 18, 28, \ldots, 98$, the construction is as follows:

1) Take the minimal DC$(v,5,4)$ given in [9, p. 39]. This design has a triple, say, $\{a,b,c\}$, the ordered pairs of which appear in five blocks.

2) Take a minimal DC$(v,5,1)$ with a hole of size two, say, $\{b,c\}$.

Then it is clear that the above steps yield the blocks of a minimal DC$(v,5,5)$ for $v = 8, 18, \ldots, 98$.

v	Point Set	Base Blocks
8	$Z_6 \cup H_2$	$\langle 0\ 1\ -\ 4\ 3\rangle \cup \{h_1, h_2$
18	$Z_{16} \cup H_2$	$\langle 0\ 3\ 1\ 9\ 5\rangle$ $\langle 0\ 13\ -\ 8\ 7\rangle \cup \{h_1, h_2\}$
28	$Z_{26} \cup H_2$	$\langle 0\ 4\ 2\ 1\ 0\rangle$ $\langle 15\ 7\ 3\ 0\ 20\rangle$ $\langle 0\ 3\ -\ 15\ 10\rangle \cup \{h_1, h_2\}$
38	$Z_{36} \cup H_2$	$\langle 1\ 0\ 4\ 11\ 9\rangle$ $\langle 9\ 3\ 0\ 21\ 1\rangle$ $\langle 13\ 0\ 6\ 2\ 26\rangle$ $\langle 19\ 0\ -\ 5\ 14\rangle \cup \{h_1, h_2\}$
48	$Z_{46} \cup H_2$	$\langle 0\ 1\ 3\ 17\ 8\rangle$ $\langle 8\ 23\ 3\ 1\ 0\rangle$ $\langle 4\ 0\ 35\ 22\ 10\rangle$ $\langle 0\ 20\ 4\ 32\ 13\rangle$ $\langle 22\ 6\ -\ 17\ 0\rangle \cup \{h_1, h_2\}$
58	$Z_{56} \cup H_2$	$\langle 3\ 0\ 9\ 1\ 31\rangle$ $\langle 31\ 4\ 15\ 0\ 36\rangle$ $\langle 30\ 17\ 7\ 44\ 0\rangle$ $\langle 1\ 0\ 17\ 35\ 3\rangle$ $\langle 11\ 41\ 5\ 49\ 0\rangle$ $\langle 0\ 13\ -\ 4\ 23\rangle \cup \{h_1, h_2\}$
68	$Z_{66} \cup H_2$	$\langle 1\ 3\ 21\ 0\ 7\rangle$ $\langle 0\ 5\ 40\ 49\ 15\rangle$ $\langle 0\ 36\ 8\ 24\ 47\rangle$ $\langle 7\ 0\ 3\ 1\ 28\rangle$ $\langle 23\ 13\ 5\ 35\ 0\rangle$ $\langle 49\ 20\ 33\ 9\ 0\rangle$ $\langle 0\ 5\ -\ 34\ 9\rangle \cup \{h_1, h_2\}$
78	$Z_{76} \cup H_2$	$\langle 3\ 0\ 16\ 1\ 39\rangle$ $\langle 66\ 0\ 12\ 4\ 47\rangle$ $\langle 0\ 27\ 6\ 34\ 52\rangle$ $\langle 0\ 9\ 29\ 53\ 40\rangle$ $\langle 7\ 1\ 0\ 49\ 3\rangle$ $\langle 14\ 0\ 64\ 5\ 59\rangle$ $\langle 18\ 44\ 0\ 33\ 8\rangle$ $\langle 55\ 16\ -\ 35\ 0\rangle \cup \{h_1, h_2\}$

For $v = 128$, apply Theorem 2.2 with $m = 7$, $h = 0$, $u = 2$ and $\lambda = 5$.

For $v = 138$, apply Theorem 2.2 with $m = 7$, $h = 2$, $u = 4$ and $\lambda = 5$.

For all other values of v, $v \neq 178$, write $v = 20m + 4u + h + s$, where m, u, h and s are chosen as in Lemma 3.6 with the difference that $h = 0, 2$ and $4u + h + s = 8, 18, \ldots, 88, 98$.

Now apply Theorem 2.4 with $\lambda = 5$ to get the result.

For $v = 178$, apply Theorem 2.6 with $m = 16$, $h = s = 0$, $\lambda = 5$, $u = 9$, and see [15] for a $(5, 5)$-DGDD of type 2^{16}

6 Directed covering with index 7

When $v \equiv 18 \pmod{20}$, then the blocks of a minimal DC$(v, 5, 7)$ can be constructed by taking two copies of a $(v, 5, 7)$ minimal covering design [10], one in some order and the other in opposite order.

Lemma 6.1 *Let $v \equiv 4$ (mod 20) be a positive integer. Then DE$(v, 5, 7)$ = DL$(v, 5, 7)$.*

Proof For all integers $v \equiv 4 \pmod{20}$, $v \geq 24$, the construction is as follows:

1) Take an optimal DP$(v, 5, 2)$ [8]. In this design there is a 2-subset, say, $\{v-2, v-1\}$, the ordered pairs of which appear in zero blocks while each other ordered pair appears in two blocks.

2) Take two copies of a $(v, 5, 4)$ minimal covering design one in increasing order, the other in decreasing order [12]. This design has a triple the pairs of which appear in six blocks. Assume in both copies the triple is $\{v-3, v-2, v-1\}$. Further, assume in this step we have the blocks $\langle 9\ 10\ 11\ v-3\ v-1\rangle$ $\langle 1\ 2\ 3\ v-2\ v-3\rangle$, which we replace by $\langle 9\ 10\ 11\ v-3\ v\rangle$ $\langle v-1\ 1\ 2\ 3\ v-2\rangle$.

3) Take a $(v-1, 5, 1)$ minimal covering design in increasing order [22]. This design has a block of size three, say, $\langle v-3\ v-2\ v-1\rangle$, which we delete.

4) Take a $B[v+1, 5, 1]$ in decreasing order. Assume in this design we have the two blocks $\langle v+1\ v\ 11\ 10\ 9\rangle$ $\langle v\ v-1\ 3\ 2\ 1\rangle$ which we replace by $\langle v\ 11\ 9\ 10\ v-1\rangle$ $\langle v\ 3\ 2\ 1\ v-3\rangle$. In all other blocks through $v+1$, we place $v+1$ at the end of the blocks then replace it by v. Furthermore, take the block through $(v-1, v-2)$, say $\langle v-1\ v-2\ c\ b\ a\rangle$ and replace by $\langle v-2\ v-1\ c\ b\ a\rangle$.

Now it is easy to check that the above four steps yield the blocks of a minimal DC$(v, 5, 7)$ for all $v \equiv 4 \pmod{20}$.

Lemma 6.2 *Let $v \equiv 8\ (mod\ 20)$ be a positive integer. Then $DE(v, 5, 7) = DL(v, 5, 7)$.*

Proof For $v = 8, 28$, see the next table, and notice that "$\bullet m$" following the block means take m copies of this block.

For $v = 48, 88$, take a $(5, 7)$-GDD of type 4^6 and 4^{11} respectively [20]. Inflate this design by a factor of two and replace each block of size 5 by the blocks of a $(5, 1)$-DGDD of type 2^5. Finally, on the groups construct a minimal DC$(8, 5, 7)$.

For $v = 128$, apply Theorem 2.2 with $m = 7$, $h = 0$, $u = 2$ and $\lambda = 7$.

For $v = 68$ take a $(6, 7)$-GDD of type 5^7 and delete one point from last group [20]. Inflate this design by a factor of two, that is, replace each block by the blocks of a $(5, 1)$-DGDD of types 2^5 and 2^6 respectively [28]. Finally on the first six groups construct a minimal DC$(10, 5, 7)$ and on the last group construct a minimal DC$(8, 5, 7)$.

For all other values of v, write $v = 20m + 4u + h + s$, where m, u, h and s are chosen as in Lemma 4.4, and then apply Theorem 2.4 with $\lambda = 5$ to get the result.

v	Point Set	Base Blocks
8	Z_8	$\langle 0\ 1\ 2\ 4\ 5\rangle \bullet 2$ $\langle 4\ 3\ 2\ 1\ 0\rangle$ $\langle 1\ 4\ 2\ 0\ 6\rangle$ $\langle 1\ 4\ 3\ 0\ 6\rangle$
28	Z_{28}	$\langle 0\ 1\ 2\ 3\ 8\rangle \bullet 2$ $\langle 8\ 3\ 2\ 1\ 0\rangle \bullet 2$ $\langle 0\ 6\ 14\ 19\ 2\rangle \bullet 3$ $\langle 19\ 14\ 10\ 3\ 0\rangle \bullet 3$ $\langle 0\ 16\ 3\ 10\ 20\rangle \bullet 2$ $\langle 0\ 3\ 10\ 16\ 20\rangle$ $\langle 2\ 0\ 1\ 4\ 12\rangle$ $\langle 5\ 2\ 0\ 18\ 14\rangle$ $\langle 21\ 2\ 0\ 15\ 8\rangle$ $\langle 16\ 0\ 9\ 20\ 3\rangle$ $\langle 20\ 3\ 10\ 15\ 0\rangle$ $\langle 0\ 15\ 4\ 19\ 9\rangle$

Lemma 6.3 *Let $v \equiv 12\ (mod\ 20)$ be a positive integer. Then $DE(v, 5, 7) = DL(v, 5, 7)$.*

Proof For $v = 12$, let $X = Z_{10} \cup \{a, b\}$. Then the blocks are the following: take two copies of a minimal DP$(12, 5, 2)$ with a hole of size two $\{a, b\}$. Further, take the following blocks (mod 10) $\langle a\ b\ 4\ 3\ 0\rangle$ $\langle 0\ 4\ 5\ b\ a\rangle$ $\langle 8\ 4\ 3\ 1\ 0\rangle$ $\langle 0\ 1\ 5\ 3\ 2\rangle$ and $\langle 8\ 6\ 4\ 2\ 0\rangle$, orbit length two.

For $v = 32, 52, 72, 92$ the construction is as follows:

1) Take the minimal DC$(v, 5, 2)$ in [9]. This design has a triple, say, $\{1, 2, 3\}$, the ordered pairs of which appear in three blocks.

2) Take an optimal DP$(v, 5, 2)$ with a hole of size two, say, $\{1, 2\}$, [8].

3) Again take an optimal DP$(v, 5, 2)$ with a hole of size two, say, $\{1, 3\}$.

4) Take a minimal DC$(v, 5, 1)$ with a hole of size 8 (Lemma 4.5); then on the hole take the following minimal DC$(8, 5, 1)$ on $\{1, \ldots, 8\}$: $\langle 6\ 7\ 8\ 4\ 5\rangle$ $\langle 3\ 2\ 1\ 7\ 6\rangle$ $\langle 4\ 8\ 1\ 2\ 3\rangle$ $\langle 1\ 3\ 2\ 4\ 5\rangle$ $\langle 5\ 1\ 2\ 3\ 8\rangle$ $\langle 5\ 4\ 8\ 7\ 6\rangle$ $\langle 6\ 7\ 3\ 2\ 1\rangle$.

For $v = 132$, apply Theorem 2.2 with $m = 7$, $h = 0$, $u = 3$ and $\lambda = 7$.

For all other values of v write $v = 20m + 4u + h + s$ where m, u, h and s are chosen as in the proof of Lemma 3.6. Now apply Theorem 2.4 with $\lambda = 7$ to get the result.

Lemma 6.4 *(i) There exists a minimal DC*$(22, 5, 7)$ *with a hole of size* 2.
(ii) Let $v \equiv 2 \pmod{20}$ *be a positive integer. Then DE*$(v, 5, 7) =$ *DL*$(v, 5, 7)$.

Proof We first construct a minimal DC$(v, 5, 1)$ with a hole of size two by taking a $B[v - 1, 5, 1]$ in increasing order and a minimal $(v + 1, 5, 1)$ covering design with a hole of size 3 in decreasing order, [22]. Assume the hole is $\{v - 1, v, v + 1\}$. Place the point $v + 1$ at the end of the blocks in which it is contained and then replace it by v.

(i) To construct a minimal DC$(22, 5, 7)$ with the hole of the size 2, take three copies of minimal DC$(22, 5, 2)$ with a hole the size 2, which is equivalent to an optimal DP$(22, 5, 2)$, [8]. Further, take a minimal DC$(22, 5, 1)$ with a hole of size 2.

(ii) We now construct a minimal DC$(v, 5, 7)$ as follows:

1) Take an optimal DP$(v, 5, 2)$ with a hole of size two, say, $\{v - 1, v\}$, [8].

2) Take two copies of minimal DC$(v, 5, 2)$, [9]. This design has a triple the ordered pairs of which appear in three blocks. Assume in both copies the triple is $\{v - 2, v - 1, v\}$.

3) Take a minimal DC$(v, 5, 1)$ with a hole of size 2, say, $\{v - 2, v - 1\}$.

Then it is readily checked that the above three steps yield the blocks of a minimal DC$(v, 5, 7)$.

Lemma 6.5 *DE*$(v, 5, 7) =$ *DL*$(v, 5, 7)$ *for* $v = 14, 34, 54, 74, 94$.

Proof For $v = 14$ the construction is as follows:

1) Take the minimal DC$(14, 5, 2)$ in [9]. This design has a triple, say, $\{12, 13, 14\}$, the ordered pairs of which appear in three blocks.

2) Take two copies of an optimal DP$(14, 5, 2)$ with a hole of size 2 [8]. Assume in both copies that the hole is $\{13, 14\}$.

3) Take the minimal DC$(14, 5, 1)$ from Lemma 3.7. Close observation of this design shows that the ordered pairs $(13, 14)$ and $(14, 13)$ appear in four blocks.

It is clear now that the above three steps yield the blocks of a minimal DC$(14, 5, 7)$.

For $v = 94$, the construction is the same as for $v = 14$ with the difference that in third step we take a minimal DC$(94, 5, 1)$ with a hole of size 14, then on the hole we take a copy of a minimal DC$(14, 5, 1)$ from Lemma 3.7. To complete this construction we need to construct a minimal DC$(94, 5, 1)$ with a hole size 14. Take a $T[6, 1, 8]$, delete two points from last group then inflate the resultant design by a factor of two and replace any blocks which are of size 5 and 6, by the blocks of a $(5, 2)$-DGDD type 2^5 and 2^6 respectively [28]. Adjoin two points to the groups and on the first five groups construct a minimal DC$(18, 5, 1)$ with a hole of size two (Lemma 5.1), and take these two points with last group as the hole of size 14.

For $v = 34, 54, 74$, the construction is as follows:

1) Take the minimal DC$(v, 5, 2)$ in [9]. This design has a triple, say, $\{1, 2, 3\}$, the ordered pairs of which appear in three blocks.

2) Take two copies of an optimal DP$(v, 5, 2)$ [8]. This design has a hole of size two, say, $\{1, 2\}$, in the first copy and $\{1, 3\}$ in the second copy. Further, assume we have $\langle 1\ 6\ 8\ 9\ 10\rangle$ $\langle 10\ 9\ 8\ 3\ 2\rangle$, which we replace by $\langle 1\ 8\ 9\ 10\ 3\rangle$ $\langle 6\ 10\ 9\ 8\ 2\rangle$.

3) Take a minimal DC$(v, 5, 1)$ with a hole of size six, say, $\{1, 2, \ldots, 6\}$, and on the hole construct the following minimal DC$(6, 5, 1)$:

$$\langle 1\ 2\ 4\ 3\ 5\rangle\ \langle 6\ 3\ 4\ 1\ 2\rangle\ \langle 2\ 3\ 1\ 6\ 5\rangle\ \langle 5\ 4\ 2\ 1\ 6\rangle.$$

Now it is readily checked that the above three steps yield the blocks of a minimal DC$(v, 5, 7)$ for $v = 34, 54, 74$. To complete this construction we need to construct a minimal DC$(v, 5, 1)$ with a hole of size six. For this purpose see the next table.

v	Point Set	Base Blocks
34	$Z_2 \times Z_{14} \cup H_6$	$\langle(1,3)\ (0,6)\ (0,0)\ (1,0)\ (1,1)\rangle$ $\langle(1,8)\ (0,0)\ (1,2)\ (0,6)\ (0,2)\rangle$ $\langle(0,0)\ (1,13)\ h_1\ (1,6)\ (0,1)\rangle$ $\langle(0,0)\ (1,12)\ h_2\ (0,3)\ (1,7)\rangle$ $\langle(1,4)\ (0,0)\ h_3\ (0,5)\ (1,10)\rangle$ $\langle(0,1)\ (0,10)\ -\ (0,3)\ (0,0)\rangle \cup \{h_5, h_6\}$ $\langle(1,10)\ (1,1)\ -\ (1,0)\ (1,3)\rangle \cup \{h_5, h_6\}$ $\langle(1,7)\ (0,0)\ h_4\ (0,6)\ (1,3)\rangle$.
54	$Z_{48} \cup H_6$	$\langle 0\ 7\ 15\ 3\ 29\rangle$ $\langle 0\ 11\ 31\ 1\ 17\rangle$ $\langle 1\ 3\ 13\ 0\ 43\rangle$ $\langle 2\ 21\ -\ 0\ 25\rangle \cup \{h_1, h_2\}$ $\langle 0\ 21\ -\ 5\ 14\rangle \cup \{h_3, h_4\}$ $\langle 20\ 9\ -\ 0\ 33\rangle \cup \{h_5, h_6\}$
74	$Z_{68} \cup H_6$	$\langle 3\ 21\ 15\ 0\ 35\rangle$ $\langle 1\ 0\ 7\ 5\ 29\rangle$ $\langle 35\ 0\ 21\ 8\ 51\rangle$ $\langle 19\ 0\ 9\ 57\ 45\rangle$ $\langle 8\ 0\ 42\ 52\ 1\rangle$ $\langle 0\ 31\ -\ 2\ 3\rangle \cup \{h_1, h_2\}$ $\langle 0\ 25\ -\ 3\ 40\rangle \cup \{h_3, h_4\}$ $\langle 9\ 4\ -\ 0\ 23\rangle \cup \{h_5, h_6\}$

Lemma 6.6 *Let $v \equiv 14 \pmod{20}$ be a positive integer. Then DE$(v, 5, 7)$ = DL$(v, 5, 7)$.*

Proof. For $v = 14, 34, 54, 74, 94$, see the previous lemma.

For $v = 134$, apply Theorem 2.2 with $m = 7$, $h = 2$, $u = 3$ and $\lambda = 7$.

For all other values write $v = 20m + 4u + h + s$ where m, u, h and s are chosen as in the proof of Lemma 3.6 with the difference that $4u + h + s = 14, 34, 54, 74, 94$. Now apply 2.4 with $\lambda = 7$ to get the result.

7 Directed covering with index 9

Again, in this section we notice that a minimal DC$(v, 5, 9)$ for all $v \equiv 12 \pmod{20}$ can be constructed by taking two copies of a minimal $(v, 5, 9)$ covering design in opposite directions. Further, a minimal DC$(v, 5, 9)$ for all $v \equiv 8 \pmod{10}$ can be constructed by taking a minimal DC$(v, 5, 2)$ and minimal DC$(v, 5, 7)$.

Lemma 7.1 *Let $v \equiv 4 \pmod{10}$ be a positive integer. Then DE$(v, 5, 9)$ = DL$(v, 5, 9)$.*

Proof For all such v the construction is as follows:

1) Take two copies of a minimal DC$(v, 5, 2)$. This design, as presented in [9], has a triple, say, $\{v-2, v-1, v\}$, the ordered pairs of which appear in three blocks.

2) Take two copies of an optimal DP$(v, 5, 2)$ with a hole of size two, [8]. Assume the hole is $\{v-1, v\}$ in the first copy and $\{v-2, v-1\}$ in the second copy.

3) Take a minimal DC$(v, 5, 1)$.

Then it is readily checked that the above construction yields a minimal DC$(v, 5, 9)$ for all $v \equiv 4 \pmod{20}$.

Lemma 7.2 *Let $v \equiv 2 \pmod{20}$ be a positive integer. Then DE$(v, 5, 9)$ = DL$(v, 5, 9)$.*

Proof For $v = 22, 42, 62, 82$ the construction is given in the next table.

v	Point Set	Base Blocks
22	Z_{22}	$\langle 1\ 2\ 0\ 3\ 8\rangle \bullet 3$ $\langle 5\ 2\ 0\ 15\ 11\rangle \bullet 3$ $\langle 0\ 3\ 17\ 7\ 11\rangle \bullet 3$ $\langle 2\ 4\ 0\ 1\ 13\rangle \bullet 2$ $\langle 1\ 0\ 13\ 6\ 16\rangle \bullet 2$ $\langle 0\ 5\ 2\ 18\ 10\rangle \bullet 2$ $\langle 1\ 0\ 2\ 10\ 4\rangle$ $\langle 1\ 0\ 15\ 11\ 7\rangle$ $\langle 7\ 2\ 0\ 16\ 11\rangle$ $\langle 3\ 15\ 0\ 12\ 7\rangle$
42	Z_{42}	Take 14 copies of a $(42, 5, 1)$ optimal packing design [30] such that 7 of them in some order and the other 7 are in opposite order. Further, take the following blocks: $\langle 0\ 1\ 3\ 21\ 5\rangle$ $\langle 5\ 19\ 2\ 1\ 0\rangle$ $\langle 4\ 0\ 11\ 21\ 32\rangle$ $\langle 0\ 30\ 21\ 13\ 6\rangle$ $\langle 0\ 3\ 22\ 1\ 10\rangle$ $\langle 0\ 15\ 4\ 27\ 21\rangle$ $\langle 0\ 13\ 21\ 5\ 29\rangle$ $\langle 10\ 3\ 0\ 31\ 32\rangle$ $\langle 27\ 5\ 0\ 14\ 21\rangle$
62	Z_{62}	Take 14 copies of a $(62, 5, 1)$ optimal packing design [30] such that 7 of them are in some order and the other 7 are in opposite order. Further, take the following blocks: $\langle 0\ 1\ 8\ 3\ 31\rangle$ $\langle 0\ 13\ 4\ 50\ 35\rangle$ $\langle 0\ 16\ 5\ 26\ 45\rangle$ $\langle 0\ 6\ 17\ 30\ 48\rangle$ $\langle 0\ 1\ 8\ 3\ 34\rangle$ $\langle 35\ 16\ 44\ 0\ 4\rangle$ $\langle 47\ 30\ 6\ 0\ 20\rangle$ $\langle 19\ 8\ 0\ 44\ 31\rangle$ $\langle 7\ 4\ 2\ 1\ 0\rangle$ $\langle 30\ 4\ 10\ 0\ 41\rangle$ $\langle 33\ 18\ 42\ 0\ 5\rangle$ $\langle 31\ 7\ 46\ 0\ 17\rangle$ $\langle 31\ 43\ 8\ 22\ 0\rangle$
82	Z_{82}	Take 16 copies of a $(82, 5, 1)$ optimal packing design [30] such that 8 of them are in some order and the other 8 are in opposite order. Further, take the following blocks: $\langle 1\ 3\ 0\ 7\ 41\rangle$ $\langle 46\ 0\ 5\ 63\ 14\rangle$ $\langle 59\ 45\ 0\ 7\ 18\rangle$ $\langle 8\ 21\ 47\ 62\ 0\rangle$ $\langle 0\ 10\ 22\ 51\ 67\rangle$ $\langle 0\ 42\ 8\ 3\ 1\rangle$ $\langle 30\ 54\ 13\ 4\ 0\rangle$ $\langle 61\ 41\ 6\ 0\ 25\rangle$ $\langle 51\ 33\ 22\ 10\ 0\rangle$

For $v = 142$, apply Theorem 2.2 with $m = 7$, $h = 2$, $u = 5$ and $\lambda = 9$. The application of this theorem requires a minimal DC$(22, 5, 9)$ with a hole of size 2. Such design can be constructed by taking 4 copies of minimal DC$(22, 5, 2)$ with a hole of size 2 [9, p. 31] and one copy of a minimal DC$(22, 5, 1)$ with a hole of size 2, Lemma 3.1.

For all other values of v, simple calculations show that v can be written in the form $v = 20m + 4u + h + s$ where m, u, h and s are chosen as in the proof of Lemma 3.6 with the difference that $4u+h+s = 22, 42, 62, 82$ and $h = 2$. Now apply Theorem 2.4 with $\lambda = 9$.

8 Conclusion.

We have shown that if v is an even integer, $v \geq 5$, and $1 \leq \lambda \leq 9$ is an odd integer then DE$(v, 5, \lambda) =$ DL$(v, 5, \lambda)$. In [9] we also have shown that if λ is even and $v \geq 5$ is an odd integer then DE$(v, 5, \lambda) =$ DL$(v, 5, \lambda) + e$ where $e = 1$ if $\lambda v(v-1)/2 \equiv -1 \pmod 5$; and $e = 0$ otherwise. On the other hand, for $\lambda = 10$ and $v \geq 5$, there exists a DB$[v, 5, 10]$, and since DE$(v, 5, \lambda') =$ DE$(v, 5, 10) +$ DE$(v, 5, \lambda)$ where $\lambda' = \lambda + 10$, it follows that DE$(v, 5, \lambda) =$ DL$(v, 5, \lambda) + e$ for all even $v \geq 5$ and λ odd.

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(Received 13 Nov 2000; revised 29 July 2002)