On the basis number of the direct product of graphs

M. M. M. Jaradat

Department of Mathematics
Yarmouk University
Irbid-Jordan
mmjst4@yu.edu.jo

Abstract

The basis number $b(G)$ of a graph G is defined to be the least integer d such that G has a d-fold basis for its cycle space. In this paper we: give an upper bound of the basis number of the direct product of trees; classify the trees with respect to the basis number of the direct product of trees and paths of order greater than or equal to 5; give an upper bound of the basis number of the direct product of bipartite graphs; and investigate the basis number of the direct product of a bipartite graph and a cycle.

1 Introduction

Unless otherwise specified, all graphs considered here are finite, undirected and simple. Our terminology and notations will be standard except as indicated. For undefined terms, see [6]. For a given graph G, we denote the vertex set of G by $V(G)$ and the edge set by $E(G)$. Given a graph G, let $e_1, e_2, \ldots, e_{|E(G)|}$ be an ordering of its edges. Then a subset S of $E(G)$ corresponds to a $(0,1)$-vector $(b_1, b_2, \ldots, b_{|E(G)|})$ in the usual way with $b_i = 1$ if $e_i \in S$, and $b_i = 0$ if $e_i \notin S$. These vectors form an $|E(G)|$-dimensional vector space, denoted by $(Z_2)^{|E(G)|}$, over the field of integer numbers modulo 2. The vectors in $(Z_2)^{|E(G)|}$ which correspond to the cycles in G generate a subspace called the cycle space of G and denoted by $\mathcal{C}(G)$. We shall say that the cycles themselves, rather than the vectors corresponding to them, generate $\mathcal{C}(G)$. It is well-known that

$$\dim \mathcal{C}(G) = \gamma(G) = |E(G)| - |V(G)| + r \tag{1}$$

where $\gamma(G)$ is the cyclomatic number and r is the number of connected components.

A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called d-fold if each edge of G occurs in at most d of the cycles in the basis $\mathcal{B}$. The basis number $b(G)$ of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a d-fold basis. The fold of an edge e in a set $B \subset \mathcal{C}(G)$, denoted by $f_B(e)$, is the number of cycles in B containing e. The required basis of G is a basis $\mathcal{B}$ of $b(G)$-fold. Now, let $\varphi : G \to H$ be an isomorphism and $\mathcal{B}$ be a (required)

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basis of G; $\mathcal{B}' = \{\varphi(c) \mid c \in \mathcal{B}\}$ is called the corresponding (required) basis of $\mathcal{B}$ in H.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The direct product $G = G_1 \wedge G_2$ is the graph with the vertex set $V(G) = V_1 \times V_2$ and the edge set $E(G) = \{(u_1, u_2)(v_1, v_2) \mid u_1v_1 \in E_1 \text{ and } u_2v_2 \in E_2\}$. From the definition above, it is clear that (i) $d_{G_1 \wedge G_2}(x, y) = d_{G_1}(x)d_{G_2}(y)$ and (ii) $|E(G_1 \wedge G_2)| = 2|E_1||E_2|$ where $d_G(v)$ is the degree of the vertex v in the graph G. The largest degree of the vertices of G will be denoted by $\Delta(G)$. Also, we will denote a path by P or P^*, a cycle by C or C^*, a star by S or S^*, and a tree by T and T^*.

The first result concerning the basis number of a graph was obtained in 1937 by MacLane who proved the following theorem:

Theorem 1.1 (MacLane) *A graph G is planar if and only if $b(G) \leq 2$.*

Schmeichel [8] proved the existence of graphs that have arbitrary large basis number.

Theorem 1.2 (Schmeichel) *For any positive integer r, there exists a graph G with $b(G) \geq r$.*

Also, Schmeichel [8] proved that for $n \geq 5$, $b(K_n) = 3$ where K_n is the complete graph of n vertices, and for $m, n \geq 5$, $b(K_{n,m}) = 4$ except possibly for $K_{6,10}$, $K_{5,n}$ and $K_{6,n} (n = 5, 6, 7, 8)$ where $K_{n,m}$ is the complete bipartite graph of n and m vertices. Banks and Schmeichel [3] proved that $b(Q_n) = 4$ where Q_n is the n-cube.

Ali [1] investigated the basis number of the direct product of some special graphs. In fact he proved that for all $|V(C)| \geq 3$ and $|V(P)| \geq 2$, $b(C \wedge P) \leq 2$; for all $|V(P)| \geq 3$ and $|V(P^*)| \geq 2$, $b(P \wedge P^*) \leq 2$, and for all $|V(C)|$ and $|V(C^*)| \geq 3$, $b(C \wedge C^*) = 3$.

Al-Rhayyel and Jaradat [2] proved the following results concerning the basis number of the direct product of some special graphs: (i) $b(P \wedge S) = 2$, if $|V(S)| \geq 4$ and $|V(P)| \geq 3$ (ii) $b(C \wedge S) = 2$, if $|V(C)| \geq 4$ and $|V(S)| \geq 3$ (iii) $b(\theta \wedge S) = 3$, if $|V(\theta)| \geq 4$ and $|V(S)| \geq 4$ where θ is the theta graph, (iv) $b(S \wedge S^*) \leq 4$, and the equality holds for each $|V(S)| \geq 6$ and $|V(S^*)| \geq 6$ except possibly $|V(S)| = 6$ and $|V(S^*)| = 6, 7, 8, 9$ and $|V(S)| = 7$ and $|V(S^*)| = 6, 7, 8, 9, 11$.

Al-Rhayyel and Jaradat [2] proved the following result:

Lemma 1.1 (Al-Rhayyel and Jaradat) *Let G and H be two graphs. If $\Delta(G)$ and $\Delta(H) \geq 3$, then $b(G \wedge H) \geq 3$.*

We remark that knowing the number of components in a graph is very important for finding the dimension of the cycle space as in (1), so we need the following result from [6].

Theorem 1.3 ([6]) *Let G and H be two connected graphs. Then $G \wedge H$ is connected if and only if at least one of them contains an odd cycle.*

Our scope of investigations extend well beyond the special cases given in [1] and [2]. We give an appropriate upper bound for the basis number of the direct product of two bipartite graphs and classify the trees with respect to $b(T \wedge P)$ where $|V(P)| \geq 5$, which were previously unavailable even in relatively simple settings. We also investigate the basis number of the direct product of cycles and trees.

Our method in this paper not only allows the systematic treatment of the direct product of graphs, but also has found applications in some other graph products which will appear in subsequent papers.

2 The Upper Bound of the Basis Number of the Direct Product of Trees

It is worth pointing out that the question regarding the basis number of the direct product of two trees cannot be resolved directly using existing methods, simply because the trees do not have a uniform form. Therefore, we shall first make an appropriate decomposition for any tree. To achieve this, we introduce the following definition which will be of great use in our work.

Definition 2.1 *Let G be a connected graph of order greater than 2. A sequence $S(G) = \left\{P_3^{(1)}, P_3^{(2)}, \ldots, P_3^{(m)}\right\}$ is called a path-sequence of G if (i) $P_3^{(i)}$ is a path of length 2 for each $i = 1, 2, \ldots, m$, and (ii) $\bigcup_{i=1}^{m} E(P_3^{(i)}) = E(G)$.*

Proposition 2.1 *For each tree T of order ≥ 3, there is a path-sequence $S(T) = \left\{P_3^{(1)}, P_3^{(2)}, \ldots, P_3^{(m)}\right\}$ such that (i) every edge $uv \in E(T)$ appears in at most three paths of $S(T)$, (ii) each $P_3^{(j)}$ contains one edge which is not in $\bigcup_{i=1}^{j-1} P_3^{(i)}$, (iii) if uv appears in three paths of $S(T)$, then the paths have forms of either uva, uvb and cuv or auv, buv and uvc, (iv) for each end point v, the edge vv^* appears in at most two paths of $S(T)$, (v) $m = |V(T)| - 2 = |E(T)| - 1$.*

To understand the above proposition and for later use, let us consider the following examples:

Example 2.1 *Let T be a path of order n, i.e., $T = P = v_1v_2 \ldots v_n$. Then we can choose $S(T) = \left\{P_3^{(1)} = v_1v_2v_3, P_3^{(2)} = v_2v_3v_4, \ldots, P_3^{(n-2)} = v_{n-2}v_{n-1}v_n\right\}$.*

Example 2.2 *Let T be a star with $V(T) = \{v_1, v_2, \ldots, v_n\}$, and $d_T(v_1) = n - 1$. Then we can choose $S(T) = \left\{P_3^{(1)} = v_2v_1v_3, P_3^{(2)} = v_3v_1v_4, \ldots, P_3^{(n-2)} = v_{n-1}v_1v_n\right\}$.*

A tree T consisting of n equal order paths $\left\{P^{(1)}, P^{(2)}, \ldots, P^{(n)}\right\}$ is called an n-special star if there is a vertex, say v_1, such that v_1 is an end vertex for each path in $\left\{P^{(1)}, P^{(2)}, \ldots, P^{(n)}\right\}$ and $V(P^{(i)}) \cap V(P^{(j)}) = \{v_1\}$ for each $i \neq j$.

Example 2.3 *Let T be a 3-special star of order 7 in which $P^{(1)} = v_1v_2v_5$, $P^{(2)} = v_1v_3v_6$, $P^{(3)} = v_1v_4v_7$. Then we can choose* $S(T) = \Big\{P_3^{(1)} = v_5v_2v_1, P_3^{(2)} = v_2v_1v_4,$ $P_3^{(3)} = v_1v_4v_7, P_3^{(4)} = v_4v_1v_3, P_3^{(5)} = v_1v_3v_6\Big\}$.

Note that, in all the above examples, $S(T)$ satisfies the conditions of Proposition 2.1, and it is easy to see that $S(T)$ is not unique.

Now, let T be a tree of order ≥ 3. Let $A_v = \{v^{**} \mid v^*v, v^{**}v^* \in E(T)$ and $d_T(v^{**}) > 1\}$ and $EV = \{v \in V(T) \mid v$ is an end point of T and either of the following (i) or (ii) holds$\}$. (i) $d_T(v^*) = 2$ where $vv^* \in E(T)$. (ii) $d_T(v^*) \geq 3$ and $|A_v| \leq 1$ where $vv^* \in E(T)$. Then it is clear that $EV \neq \emptyset$.

Proof of Proposition 2.1. The proof is by mathematical induction on $|V(T)|$. If $|V(T)| = 3$, then T is a path of length 2. In this case we take $S(T) = \{P_3^{(1)} = T\}$. Now, assume T is a tree of order $n+1$. Let $v \in EV$ and v^* be a vertex such that $vv^* \in E(T)$. Set $T' = T - v$. By induction, there is a path-sequence $S(T')$ satisfying our proposition. To this end, we need to consider two cases:

Case 1. $d_T(v^*) = 2$. Then v^* is an end point of T'. Thus there is an edge $v^{**}v^*$ appearing at most twice in $S(T')$. If $v^{**}v^*$ appears in one path, then the path has the form $av^{**}v^*$; if it appears in two paths, then the paths have the following forms $av^{**}v^*$, and $bv^{**}v^*$. Therefore we take $S(T) = S(T') \cup \{P^{(n-1)} = v^{**}v^*v\}$.

Case 2. $d_T(v^*) \geq 3$. Then at most one vertex $v^{**} \in V(T)$ has the properties $v^*v^{**} \in E(T)$ and $d_T(v^{**}) > 1$. Let $A = \{v_{n_1}, v_{n_2}, \ldots, v_{n_r}\}$ be the set of all vertices adjacent to v^* other than v^{**}. Now, we need to consider three subcases:

Case 2a. v^*v^{**} appears in $S(T')$ once. Then we take $S(T)$ as in Case 1.

Case 2b. v^*v^{**} appears in $S(T')$ twice, say, $P_3^{(g_1)}$ and $P_3^{(g_2)}$. Then we have two subsubcases to consider:

Case 2b1. If either $P_3^{(g_1)} = v_{n_{i_0}}v^*v^{**}$ and $P_3^{(g_2)} = av^{**}v^*$, or $P_3^{(g_1)} = av^{**}v^*$ and $P_3^{(g_2)} = bv^{**}v^*$, then take $S(T)$ as in Case 1.

Case 2b2. If $P_3^{(g_1)} = v^{**}v^*v_{n_{i_0}}$ and $P_3^{(g_2)} = v^{**}v^*v_{n_{i_1}}$, then we claim that there exists $1 \leq i^* \leq r$, such that $v^*v_{n_{i^*}}$ appears in at most one path of $S(T')$. Thus, we take $S(T) = S(T') \cup \{P_3^{(n-1)} = vv^*v_{n_{i^*}}\}$. The proof of the claim goes as follows: Let $A = \{P_3^{(i_0)}, \ldots, P_3^{(i_r)}\} \subset S(T)$ be the set of all paths of the form $v_{n_i}v^*v_{n_j}$ where $i \neq j$. It is an easy matter to see that any path in $S(T')$ containing $v^*v_{n_i}$ is a path in A. Thus, if each edge of the form $v^*v_{n_i}$ appears two times ($1 \leq i \leq r$) in the paths of $S(T')$, then any ordering of $S(T')$ must contain a path in A, and both of its edges appear in the previous paths, which is a contradiction.

Case 2c. If $v^{**}v^*$ appears three times, say, $P_3^{(g_1)}$, $P_3^{(g_2)}$, and $P_3^{(g_3)}$, then either (i) $P_3^{(g_1)} = v^{**}v^*v_{n_{i_0}}$, $P_3^{(g_2)} = av^{**}v^*$, and $P_3^{(g_3)} = bv^{**}v^*$, or (ii) $P_3^{(g_1)} = v^{**}v^*v_{n_{i_0}}$, $P_3^{(g_2)} = v^{**}v^*v_{n_{i_1}}$ and $P_3^{(g_2)} = av^{**}v^*$. In both (i) and (ii), we can prove the same claim as in Case 2b2, and take the same $S(T)$ as in Case 2b2.

Remark 2.1 *From Proposition 2.1 and its proof it is an easy matter to see that there are at least two edges of T, each of which appears in only one path of $S(T)$.*

Now, let $P_3^{(i)} = a_ib_ic_i$ and $Q_3^{(j)} = d_je_jf_j$ be two paths. Let

$$\mathcal{B}_{i,j} = \{(a_i, e_j)(b_i, d_j)(c_i, e_j)(b_i, f_j)(a_i, e_j)\}.$$

Then $\mathcal{B}_{ij}$ is a basis of $P^{(i)} \wedge Q^{(j)}$ and also $|\mathcal{B}_{i,j}| = 1$. Therefore $\mathcal{B}_{i,j}$ is a 1-fold basis.

Lemma 2.1 *For every two trees T_1 and T_2 of order ≥ 3, and for every path-sequences $S(\ T_1\ ) = \left\{P_3^{(1)}, P_3^{(2)}, ..., P_3^{(|V(T_1)-2|)}\right\}$ and $S(\ T_2\ ) = \left\{Q_3^{(1)}, Q_3^{(2)}, ...,\ \ Q_3^{(|V(T_2)-2|)}\right\}$ of T_1 and T_2 as in Proposition 2.1, respectively, we have $\mathcal{B} = \bigcup_{j=1}^{(|V(T_2)|-2)} \bigcup_{i=1}^{(|V(T_1)|-2)} \mathcal{B}_{i,j}$ is linearly independent.*

Proof. Let $\mathcal{B}_i = \bigcup_{j=1}^{(|V(T_2)|-2)} \mathcal{B}_{i,j}$. Since each $Q_3^{(j)}$ contains an edge, say $e_j f_j$, which is not in $\cup_{k=1}^{(j-1)} E(Q_3^{(k)})$, the cycle of $\mathcal{B}_{i,j}$ contain $(a_i, e_j)(b_i, f_j)$ and $(b_i, f_j)(c_i, e_j)$ each of which is not in any cycle of $\bigcup_{k=1}^{(j-1)} \mathcal{B}_{i,k}$. Thus, for each $i = 1, 2, \ldots, |V(T_2)| - 2$, $\mathcal{B}_i$ is linearly independent. Similarly, each path $P_3^{(i)}$ contains an edge, say $b_i c_i$, which is not in $\cup_{k=1}^{(i-1)} E(P_3^{(k)})$ and each linear combination of cycles of $\mathcal{B}_i$ must contain at least one edge of $(b_i, d_j)(c_i, e_j)$ and $(c_i, e_j)(b_i, f_j)$ for some j which are not in any cycle of $\bigcup_{k=1}^{(i-1)} \mathcal{B}_k$. Therefore $\mathcal{B}$ is linearly independent.

As we mentioned before, one of the important steps in our work is to determine the number of components of the direct product of two connected graphs, so we give the following result.

Lemma 2.2 *Let T_1 and T_2 be any pair of trees of order greater than or equal to 2. Then $T_1 \wedge T_2$ consists of two components.*

Proof. Note that the size of $\mathcal{B}$ in Lemma 2.1 is

$$\begin{aligned} |\mathcal{B}| &= (|E(T_1)| - 1)(|E(T_2)| - 1) \\ &= |V(T_1)||V(T_2)| - 2|V(T_1)| - 2|V(T_2)| + 4. \end{aligned}$$

Since $\mathcal{B}$ is linearly independent,

$$\begin{aligned} |\mathcal{B}| &\leq \dim \mathcal{C}(T_1 \wedge T_2) \\ &= 2(|V(T_1)| - 1)(|V(T_2) - 1) - |V(T_1)||V(T_2)| + r \\ &= |V(T_1)||V(T_2)| - 2|V(T_1)| - 2|V(T_2)| + 2 + r \end{aligned}$$

where r is the number of components. Thus we have $2 \leq r$ (this inequality also holds from Theorem 1.3). Thus, it remains to prove $r \leq 2$. Choose $u_1u_2 \in E(T_1)$ and $v_1v_2 \in E(T_2)$ such that u_1 and v_1 are end points of T_1 and T_2, respectively. Then $u_1u_2 \wedge v_1v_2$ consists of two components: $G_1 = (u_1, v_1)(u_2, v_2)$ and $G_2 = (u_1, v_2)(u_2, v_1)$. Let $(u, v) \in V(T_1 \wedge T_2)$. We show that there is a path joining (u, v) with either (u_1, v_2) or (u_1, v_1). Since T_1 and T_2 are connected, there are two paths: $P_1 = u_1u_2u_3 \ldots u_n$ of T_1 where $u_n = u$,and $P_2 = v_1v_2v_3 \ldots v_m$ of T_2 where $v_m = v$. According to the relationship between m and n, there are three cases to consider:
Case 1. $m = n$. Then take

$$P = (u_1, v_1)(u_2, v_2) \dots (u_{n-1}, v_{n-1})(u_n, v_n).$$

Case 2. $n > m$. Then this case splits into two subcases:

Case 2(a). $s = n - m$ is odd. Then take

$$P = (u_n, v_m)(u_{n-1}, v_{m-1}) \cdots (u_{s+1}, v_1)(u_s, v_2)(u_{s-1}, v_1) \dots (u_3, v_2)(u_2, v_1)(u_1, v_2).$$

Case 2(b). $s = n - m$ is even. Then take

$$P = (u_n, v_m)(u_{n-1}, v_{m-1}) \cdots (u_{s+1}, v_1)(u_s, v_2)(u_{s-1}, v_1) \dots (u_3, v_1)(u_2, v_2)(u_1, v_1).$$

Case 3. $n < m$. Then this case splits into two subcases:

Case 3(a). $s = m - n$ is odd. Then take

$$P = (u_n, v_m)(u_{n-1}, v_{m-1}) \cdots (u_1, v_{s+1})(u_2, v_s)(u_1, v_{s-1}) \dots (u_1, v_4)(u_2, v_3)(u_1, v_2).$$

Case 3(b). $s = m - n$ is even. Then take

$$P = (u_n, v_m)(u_{n-1}, v_{m-1}) \cdots (u_1, v_{s+1})(u_2, v_s)(u_1, v_{s-1}) \dots (u_1, v_3)(u_2, v_2)(u_1, v_1).$$

Thus, we have $r = 2$.

The following result follows immediately from Lemmas 2.1 and 2.2 and Equation (1).

Corollary 2.1 *For every pair of trees T_1 and T_2, the set $\mathcal{B}$ in Lemma 2.1 is a basis for $\mathcal{C}(T_1 \wedge T_2)$.*

Theorem 2.1 *For every pair of connected graphs G and H, $G \wedge H$ is connected if and only if one of them contains an odd cycle. If both of them are bipartite graphs, then $G \wedge H$ consists of two components.*

Proof. The first part of this theorem directly follows from Theorem 1.3. To prove the second part of the theorem, first we prove it in the case $H = T$. Let T^* be a spanning tree of the bipartite graph G obtained by the usual way. Then by Lemma 2.1, we have that $T^* \wedge T$ consists of two components G_1 and G_2. To this end, it is sufficient to show that if $(u, v)(u^*, v^*) \in E(G \wedge T)$ with $uu^* \in E(G) \setminus E(T^*)$, then (u, v) and (u^*, v^*) belong to the same component. Since G is a bipartite graph, any cycle C is of even length and so the path $P = uu_2u_3 \dots u_s u^*$ of T^* is an odd length path. Hence $(u, v)(u_2, v^*)(u_3, v) \cdots (u_{s-1}, v)(u^*, v^*)$ is a path in one of the components of $T^* \wedge T$. Thus, $G \wedge T$ consists of two components. To prove the theorem for any bipartite graph H we apply the same argument as in the above on $G \wedge T^{**}$where T^{**} is a spanning tree for H.

Lemma 2.3 *For each graph H, $H \wedge K_2$ is a bipartite graph where $K_2 = vv'$ is a complete graph of order 2.*

Lemma 2.4 *Let H be a bipartite graph, and H_1 and H_2 be the two components of $H \wedge K_2$. If $(u, v) \in H_1$ (or H_2), then $(u, v') \in H_2$ (or H_1) where $vv' = K_2$.*

Proof. Assume that (u, v) and (u, v') belong to the same component, say, H_1. Let $P = (u, v)(u_2, v') \cdots (u_{n-1}, v)(u, v')$, a subgraph of $H \wedge K_2$, be a path which joins these two points. Now, we claim that $P^* = uu_2 \ldots u_{n-1}u$ is a cycle which contradicts the fact that H has no odd cycle. Suppose not. Then there are $i < j$ such that $u_i = u_j$, $u_l \neq u_k$ and the elements of both of $\{u_i u_{i+1} \ldots u_{j-1}\}$ and $\{u_{i+1}u_{i+2} \ldots u_j\}$ are distinct. To this end, either $(u_i, v)(u_{i+1}, v') \ldots (u_j, v')$ is a path subgraph of P or $(u_i, v')(u_{i+1}, v) \ldots (u_j, v)$ is a path subgraph of P. Thus, $P^{**} = u_i u_{i+1} \ldots u_j$ is an odd cycle in H. This is a contradiction.

Lemma 2.5 *Let H be a bipartite graph. Then each of the two components H_1 and H_2 of $H \wedge K_2$ is isomorphic to H.*

Proof. Note that $V(H \wedge K_2) = V(H_1) \cup V(H_2)$ and $\frac{|V(H \wedge K_2)|}{2} = |V(H_1)| = |V(H_2)|$. Now define $\varphi_i : V(H_i) \to V(H)$ by $\varphi_i((u, v)) = u$ and $\varphi_i((u, v')) = u$ for $i = 1, 2$. It is an easy matter to see that φ_1 and φ_2 are bijections and preserve the adjacency.

So far, we have furnished the necessary ground to deal with the basis number of the direct product of trees.

Theorem 2.2 *For every pair of trees T_1 and T_2, we have $b(T_1 \wedge T_2) \leq 5$.*

Proof. According to the order of trees, we need to consider two cases:
Case 1. One of them is of order less than or equal to 2. Then $T_1 \wedge T_2$ is either a null graph or by Lemma 2.5 consists of two vertex disjoint components each of which is a tree. In both cases $b(T_1 \wedge T_2) = 0$.
Case 2. Both of them are of order at least 3. Then it suffices to prove that the linearly independent set $\mathcal{B}$ is 5-fold where $\mathcal{B}$ is given as in Lemma 2.1. Let $S(T_1) = \{P_3^{(1)}, P_3^{(2)}, \ldots, P_3^{(|V(T_1)|-2)}\}$ and $S(T_2) = \{Q_3^{(1)}, Q_3^{(2)}, \ldots, Q_3^{(|V(T_2)|-2)}\}$ be two path-sequences of T_1 and T_2 as in Proposition 2.1. We handle the worst case where each of $S(T_1)$ and $S(T_2)$ contains at least one edge which appears in three paths. To this end, suppose that $ab \in E(T_1)$ appears in the following paths: $P_3^{(1)} = abc$, $P_3^{(2)} = bad$, and $P_3^{(3)} = eab$, and $fg \in E(T_2)$ appears in the following paths: $Q_3^{(1)} = fgh$, $Q_3^{(2)} = gfk$, and $Q_3^{(3)} = fgl$. It suffices to show that $f_{\mathcal{B}}((a, f)(b, g)) \leq 5$ and $f_{\mathcal{B}}((a, g)(b, f)) \leq 5$. To achieve that, we list all the possibilities of $\mathcal{B}_{i,j}$:

$$\begin{aligned}
\mathcal{B}_{1,1} &= (a, g)(b, f)(c, g)(b, h)(a, g), \\
\mathcal{B}_{1,2} &= (a, f)(b, g)(c, f)(b, k)(a, f), \\
\mathcal{B}_{1,3} &= (a, g)(b, f)(c, g)(b, l)(a, g), \\
\mathcal{B}_{2,1} &= (b, g)(a, f)(d, g)(a, h)(b, g), \\
\mathcal{B}_{2,2} &= (b, f)(a, g)(d, f)(a, k)(b, f), \\
\mathcal{B}_{2,3} &= (b, g)(a, f)(d, g)(a, l)(b, g), \\
\mathcal{B}_{3,1} &= (e, g)(a, f)(b, g)(a, h)(e, g), \\
\mathcal{B}_{3,2} &= (e, f)(a, g)(b, f)(a, k)(e, f), \\
\mathcal{B}_{3,3} &= (e, g)(a, f)(b, g)(a, l)(e, g).
\end{aligned}$$

Note that $(a, f)(b, g)$ appears in $\mathcal{B}_{1,2}$, $\mathcal{B}_{2,1}$, $\mathcal{B}_{2,3}$, $\mathcal{B}_{3,1}$ and $\mathcal{B}_{3,3}$, and $(a, g)(b, f)$ appears in $\mathcal{B}_{1,1}$, $\mathcal{B}_{1,3}$, $\mathcal{B}_{2,2}$, and $\mathcal{B}_{3,2}$. Thus $\mathcal{B}$ is a 5-fold basis.

The proof of the following corollary follows by the same lines as of the proof of Theorem 2.2 and by taking $P_3^{(2)}$ and $P_3^{(3)}$ from $S(T_1)$ and $Q_3^{(1)}$ and $Q_3^{(2)}$ from $S(T_2)$.

Corollary 2.2 *Let $S(T_1)$ and $S(T_2)$ be two path-sequences of T_1 and T_2, such that each edge of T_1 and T_2 appears in at most two paths of $S(T_1)$ and $S(T_2)$, respectively. Then $b(T_1 \wedge T_2) \leq 4$.*

We remark that by specializing the trees in the above corollary into stars we get the following result:

Corollary 2.3 (Al-Rhayyel and Jaradat) *For any pair of stars S and S^*, $b(S \wedge S^*) \leq 4$.*

Similarly, the proof of the next two results follows by taking $S(P)$ as in Example 2.1 and employing the same argument as in the proof of Theorem 2.1.

Corollary 2.4 *For every tree T and path P, we have $b(T \wedge P) \leq 3$.*

Corollary 2.5 *Let T be a tree and $S(T)$ be a path sequence of T as in Proposition 2.1 such that each edge of T appears in at most two paths of $S(T)$. Then $b(T \wedge P) \leq 2$.*

The following result shows that the upper bound in Corollary 2.1 is optimal.

Proposition 2.2 *Let T be the 3-special star of order 7 as in Example 2.3 and P be a path of order 5. Then $b(T \wedge P) = 3$.*

Proof. In order to prove this proposition it is enough to show that $T \wedge P$ is non-planar. Consider the subgraph H whose vertex set $A \cup B \cup \{(u_2, v_4)(u_5, v_3), (u_3, v_4), (u_6, v_2), (u_4, v_4), (u_7, v_3)\}$ where $A = \{(u_1, v_1), (u_1, v_3), (u_1, v_5)\}$ and $B = \{(u_2, v_2), (u_3, v_2), (u_4, v_2)\}$ and whose edge set consists of the following nine paths: $P_1 = (u_1, v_1)(u_2, v_2)$, $P_2 = (u_1, v_1)(u_3, v_2)$, $P_3 = (u_1, v_1)(u_4, v_2)$, $P_4 = (u_1, v_3)(u_2, v_2)$, $P_5 = (u_1, v_3)(u_3, v_2)$, $P_6 = (u_1, v_3)(u_4, v_2)$, $P_7 = (u_1, v_5)(u_2, v_4)(u_5, v_3)(u_2, v_2)$, $P_8 = (u_1, v_5)(u_3, v_4)(u_6, v_2)(u_3, v_2)$, $P_9 = (u_1, v_5)(u_4, v_4)(u_7, v_3)(u_4, v_2)$. By noting that each of A and B are independent sets of edges, it is an easy matter to see that H is homeomorphic to $K_{3,3}$. Thus, by Kuratowski's theorem, $T \wedge P_5$ is non planar.

3 Classification

In this section, we classify trees with respect to the basis number of $T \wedge P$ where $|V(P)| \geq 5$.

Theorem 3.1 *Let T be a tree of order at least 3. Then there is a path-sequence $S(T)$ satisfying Proposition 2.1 such that each edge appears in at most two paths of $S(T)$ if and only if T has no subgraph isomorphic to a 3-special star of order 7.*

Proof. If T has no subgraph isomorphic to a 3-special star of order 7, then either T is a path, in which case we can take $S(T)$ as in Example 2.1, or T is a path $P = v_1v_2\dots v_n$ in which v_i is adjacent to a set of end points, say $\{v_{i_1}, v_{i_2}\dots v_{i_{r_i}}\}$, where $1 \leq i \leq n$. To this end, we take

$$S(T) = \{P_3^{(1)} = v_{1_1}v_1v_{1_2}, P_3^{(2)} = v_{1_2}v_1v_{1_3}, P_3^{(3)} = v_{1_3}v_1v_{1_4}\dots P_3^{(r_1)} = v_{1_{r_1}}v_1v_2, P_3^{(r_1+1)} = v_1v_2v_{2_1}P_3^{(r_1+2)} = v_{2_1}v_2v_{2_2}\dots, P_3^{(r_1+r_2+1)} = v_{2_{r_2}}v_2v_3, \dots, P_3^{(r_1+r_2+r_3+r_4+\cdots+r_n+n-2)} = v_{n_{r_n-1}}v_nv_{n_{r_n}}\}.$$

Note that $r_1 + r_2 + r_3 + r_4 + \cdots + r_{n-1} + r_n + n - 2 = |E(T)| - 1$. The other direction is an easy consequence of Corollary 2.5 and Proposition 2.2.

Proposition 3.1 *Let G be a graph. Then $b(G) = 0$ if and only if G has no cycle.*

Proof. G has at least one cycle if and only if dim $\mathcal{C}(G) \geq 1$. And dim $\mathcal{C}(G) \geq 1$ if and only if $b(G) \geq 1$.

The following result is an immediate consequence of the above Proposition 3.1 and Lemma 2.5.

Corollary 3.1 *For any tree T and path P, we have $b(T \wedge P\ ) = 0$ if and only if at least one of $|V(T)|$ and $|V(P)|$ is less than or equal to 2.*

Proposition 3.2 *Let G be any graph. Then $b(G) = 1$ if and only if dim $\mathcal{C}(G) \geq 1$ and $\mathcal{C}(G)$ is generated from edge-disjoint cycles.*

Proof. If $\mathcal{C}(G)$ is generated from edge-disjoint cycles, then G is planar. By Euler's Theorem we have:

$$\text{The number of faces} = |E(G)| - |V(G)| + 2 = \dim \mathcal{C}(G) + 1.$$

Thus, dim $\mathcal{C}(G) =$ the number of bounded faces. Therefore, choose $\mathcal{B}$ to be the set of all bounded faces. Hence $\mathcal{B}$ is 1-fold. Now, to prove the other direction, assume $\mathcal{B}$ is a 1-fold basis. Then the cycles of $\mathcal{B}$ are edge-disjoint. Hence $\mathcal{C}(G)$ is generated from edge-disjoint cycles.

Corollary 3.2 *For every tree T and path P, we have that $b(T \wedge P) = 1$ if and only if T is a path and at least one of $|V(T)|$and $|V(P)|$ is of order 3 and the other is of order greater than or equal 3.*

Proof. If T is a path and at least one of $|V(T)|$and $|V(P)|$ is of order 3, then we have a direct product of two paths, one of which is of order 3. Thus, by taking $\mathcal{B} = \mathcal{B}_1$ as in the proof of Lemma 2.1, we have the result. To prove the other direction we assume the contrary. Thus, we need to consider two cases:
Case a. T is not a path. Then $|V(T)| \geq 4$. Now, if $|V(P)| \leq 2$, then $b(T \wedge P) = 0$. If $|V(P)| \geq 3$, then there is at least one edge of $T \wedge P$ belonging to at least two cycles. Thus $b(T \wedge P) \neq 1$.
Case b. T is a path. Then the orders of P and T are less than or equal to 2. Hence $b(T \wedge P) = 0$.

Lemma 3.1 *Let T and P be a tree and a path, respectively, such that $|V(T)| \geq 4$ and $|V(P)| \geq 5$. Then $b(T \wedge P) = 2$ if and only if there is $S(T)$ as in Proposition 2.1 and each edge appears in at most two paths of $S(T)$.*

Proof. The proof of the 'only if' direction follows from Corollary 2.5, Corollary 3.1 and Corollary 3.2. The 'if' direction follows from Proposition 2.2 and Theorem 3.4.

Corollary 3.3 (Ali) *For any pair of paths P and P^*, $b(\ P \wedge P^*) \leq 2$.*

Corollary 3.4 *For every tree T and a path P of order at least 5, we have $b(T \wedge P) = 3$ if and only if for each $S(T)$ as in Proposition 2.1, there is at least one edge appearing in at least three paths of $S(T)$.*

From the above results we give the following theorem which classifies trees with respect to $b(T \wedge P)$ where $|V(P)| \geq 5$.

Theorem 3.2 *Let T be a tree. Then (1) $b(T \wedge P) = 0$ if and only if $|V(T)| \leq 2$. (2) $b(T \wedge P) = 1$ if and only if $|V(T)| = 3$. (3) $b(T \wedge P) = 2$ if and only if $|V(T)| \geq 4$ and T has no subgraph isomorphic to a 3-special star of order 7. (4) $b(T \wedge P) = 3$ if and only if T has a subgraph isomorphic to a 3-special star of order 7.*

The following theorem provides us with necessary and sufficient conditions for $T_1 \wedge T_2$ to be non-planar.

Theorem 3.3 *For any two trees T_1 and T_2 such that $|V(T_2)|$, $|V(T_2)| \geq 5$, we have that $T_1 \wedge T_2$ is non-planer ($b(T_1 \wedge T_2) > 2$) if and only if one of the following holds: (i) $\Delta(T_1) \geq 3$ and $\Delta(T_2) \geq 3$. (ii) One of them is a path and the other contains a subgraph isomorphic to a 3-special star of order 7.*

Proof. Assume neither (i) nor (ii) holds. Then, either $\Delta(T_1) \leq 2$ and $\Delta(T_2) \leq 2$, and so by Corollary 2.5, $b(T_1 \wedge T_2) \leq 2$, which is a contradiction, or one of $\Delta(T_1)$ and $\Delta(T_2)$ is less than or equal to 2, say $\Delta(T_1)$, and the other greater than or equal to 3. Thus T_1 is a path and T_2 contains no subgraph isomorphic to a 3-special star of order 7. Therefore, by Theorem 3.1 and Corollary 2.5, we get a contradiction. On the other hand, if one of (i) and (ii) holds, then by Lemma 1.1 and Theorem 3.2, we get the result.

4 The Upper Bound of the Basis Number of the Direct Product of Two Bipartite Graphs

In this section, we give an upper bound of the basis number of the direct product of two bipartite graphs in terms of their basis numbers.

Remark 4.1 *Let G be a connected graph and $\mathcal{B}=\left\{c_1, c_1, ..., c_{\dim \mathcal{C}(G)}\right\}$ be a basis of $\mathcal{C}(G)$. If T is a spanning tree obtained by deleting $t_G=\left\{e_1, e_{2,...}, e_{\dim \mathcal{C}(G)}\right\} \in E(G)$, then: (i) If $c=\sum_{j=1}^{\alpha} c_{i_j}$ mod 2, then c contains at least one edge of t_G. (ii) For each $c_i, c_j \in \mathcal{B}$, there exists at least one edge in t_G, say, e_{i_0} in either c_i or c_j, but not in both.*

Theorem 4.1 *For every two bipartite graphs G and H, $b(G\wedge H) \leq b(G)+b(H)+5$.*

Proof. Let T and T^* be spanning trees of G and H. Let $\mathcal{B}^*$ be the basis of $\mathcal{C}(T\wedge T^*)$ as in Theorem 2.1. Let $\mathcal{B}_G$ and $\mathcal{B}_H$ be the required basis of G and H, respectively, Note that $G\wedge T^*$ is decomposed into $2(|V(T^*)|-1)$ edge-disjoint copies of G. Now, define $\mathcal{B}_{Ge}=\mathcal{B}^{(1)}_{Ge}\cup\mathcal{B}^{(2)}_{Ge}$, where $\mathcal{B}^{(1)}_{Ge}$ and $\mathcal{B}^{(2)}_{Ge}$ are the corresponding required basis of $\mathcal{B}_G$ in the two components of $G\wedge e$, where $e\in E(T^*)$. Let $\mathcal{B}^{**}=\bigcup_{e\in T^*}\mathcal{B}_{Ge}$. Since each of $\mathcal{B}^{(1)}_{Ge}$ and $\mathcal{B}^{(2)}_{Ge}$ is linearly independent and they are vertex-disjoint, $\mathcal{B}_{Ge}$ is linearly independent. Since $\mathcal{B}_{Ge}$ and $\mathcal{B}_{Ge'}$ are edge-disjoint for each $e\neq e'$, $\mathcal{B}^{**}$ is linearly independent. Define $\mathcal{B}_{He}=\mathcal{B}^{(1)}_{He}\cup\mathcal{B}^{(2)}_{He}$, where $\mathcal{B}^{(1)}_{He}$ and $\mathcal{B}^{(2)}_{He}$ are the corresponding required basis of $\mathcal{B}_H$ in the two components of $e\wedge H$, where $e\in E(G)$. Let $\mathcal{B}^{***}=\bigcup_{e\in E(G)}\mathcal{B}_{He}$. By the same argument as above, we have that $\mathcal{B}^{***}$ is linearly independent. Set $\mathcal{B}=\mathcal{B}^*\bigcup\mathcal{B}^{**}\bigcup\mathcal{B}^{***}$. Assume that $\sum_{i=1}^{\gamma}c_i+\sum_{e\in A\subset E(G)}\sum_{i=1}^{\delta_e}d_{e_i}=0 \pmod 2$ where $c_i\in\mathcal{B}^*$ and $d_{e_i}\in\mathcal{B}_{Ge}$. Then $\sum_{i=1}^{\gamma}c_i+\sum_{e\in A-e'}\sum_{i=1}^{\delta_e}d_{e_i}=\sum_{i=1}^{\delta_{e'}}d_{e'_i} \pmod 2$ where $A-e'=\{e_1,e_2,\dots,e_n\}$. Thus, $E(c_1\oplus c_2\oplus\dots\oplus c_\gamma\oplus d_{e1_1}\oplus d_{e1_2}\oplus\dots\oplus d_{e1_{\delta_{e1}}}\oplus d_{e2_1}\oplus\dots d_{en_{\delta_{en}}})=E(d_{e'_1}\oplus d_{e'_2}\oplus\dots\oplus d_{e'_{\delta_{e'}}})$ where the ring sum $c_1\oplus c_2\oplus\dots\oplus c_\gamma\oplus d_{e1_1}\oplus d_{e1_2}\oplus\dots\oplus d_{e1_{\delta_{e1}}}\oplus d_{e2_1}\oplus\dots d_{en_{\delta_{en}}}$ and $d_{e'_1}\oplus d_{e'_2}\oplus\dots\oplus d_{e'_{\delta_{e'}}}$ are cycles or edge-disjoint cycles. By Remark 4.1 we have that $E(d_{e'_1}\oplus d_{e'_2}\oplus\dots\oplus d_{e'_{\delta_{e'}}})$ contains at least one edge which is not in $E(c_1\oplus c_2\oplus\dots\oplus c_\gamma)$, and since $\{\mathcal{B}_{Ge}\}_{e\in T^*}$ are pairwise edge-disjoint, it is not in $E(d_{e1_1}\oplus d_{e1_2}\oplus\dots\oplus d_{e1_{\delta_{e1}}}\oplus d_{e2_1}\oplus\dots d_{en_{\delta_{en}}})$. Thus, $\mathcal{B}^*\cup\mathcal{B}^{**}$ is linearly independent. By the same argument as above we can show that $\mathcal{B}$ is linearly independent. Now,

$$\begin{aligned}
|\mathcal{B}| &= |\mathcal{B}^*|+|\mathcal{B}^{**}|+|\mathcal{B}^{***}|\\
&= 2|V(T^*)||E(T)|-|V(T^*)||V(T)|+2+\sum_{|E(T^*)|}|\mathcal{B}_{Ge}|+\sum_{|E(G)|}|\mathcal{B}_{He}|\\
&= 2|V(T^*)||E(T)|-|V(H)||V(G)|+2|E(T^*)|\dim\mathcal{C}(G)+2|E(G)|\dim\mathcal{C}(H)\\
&= 2|V(T^*)|(|E(T)|+\dim\mathcal{C}(G))+2|E(G)|\dim\mathcal{C}(H)-|V(G)||V(P)|+2\\
&= 2|V(T^*)||E(G)|+2|E(G)|\dim\mathcal{C}(H)-|V(G)||V(P)|+2\\
&= 2|E(G)|(|V(T^*)|+\dim\mathcal{C}(H))-|V(G)||V(P)|+2\\
&= 2|E(G)||E(H)|-|V(G)||V(P)|+2=\dim\mathcal{C}(G\wedge H).
\end{aligned}$$

Therefore, $\mathcal{B}$ is a basis of $\mathcal{C}(G\wedge H)$. To this end, we need to show that $\mathcal{B}$ is a $(5+b(H)+b(G)))$ - fold basis. Let $e\in\mathcal{B}$. Then (i) If $e\in E(G\wedge H)-E(T\wedge T^*)$, then $f_{\mathcal{B}^*}=0$, $f_{\mathcal{B}^{**}}\leq b(G)$, and $f_{\mathcal{B}^{***}}(e)\leq b(H)$.
(ii) If $e\in E(T\wedge T^*)$, then $f_{\mathcal{B}^*}(e)\leq 5$, $f_{\mathcal{B}^{**}}(e)\leq b(G)$ and $f_{\mathcal{B}^{***}}(e)\leq b(H)$. Therefore,

we have the result.

The following corollaries are straightforward consequences from the proof of the previous results.

Corollary 4.1 *Let H and G be two bipartite graphs. If there exist two spanning trees T and T^* for H and G, respectively, and $S(T)$ and $S(T^*)$ as in Proposition 2.1 such that each edge of T and T^* appears in at most two paths of $S(T)$ and $S(T^*)$, respectively, then $b(G \wedge H) \leq 4 + b(G) + b(H)$.*

Corollary 4.2 *For every path P and bipartite graph G, $b(G \wedge P) \leq 3 + b(G)$. Moreover, if there is a spanning tree T of G and $S(T)$ as in Proposition 2.1 such that each edge of T appears in at most two paths, then $b(G \wedge P) \leq 2 + b(G)$.*

By specializing G in the above corollary into a cycle of even order we have the following result.

Corollary 4.3 (Ali) *If C and C^* are even cycles and P is a path, then $b(C \wedge P) \leq 2$ and $b(C \wedge C^*) \leq 3$.*

5 The Basis Number of the Direct Product of a Bipartite Graph and a Cycle

In this section, we investigate the upper bound of the basis number of the direct product of a bipartite graph with a cycle.

Theorem 5.1 *For any tree T and cycle C, we have $b(T \wedge C) \leq 3$. Moreover, the equality holds if T contains a subgraph isomorphic to a 3-special star of order 7.*

Proof. Let $S(T) = \{P_3^{(1)} = a_1b_1c_1, P_3^{(2)} = a_2b_2c_2, \ldots, P_3^{(|V(T)|-2)} = a_{|V(T)|-2}\, b_{|V(T)|-2}\, c_{|V(T)|-2}\}$ be a path sequence as in Proposition 2.1 and $C = u_1u_2 \ldots u_{|V(C)|}u_1$. For each $i = 1, 2, \ldots, |V(T)| - 2$, set

$$\begin{aligned}\mathcal{B}_i = \{&(b_i, u_j)(c_i, u_{j+1})(b_i, u_{j+2})(a_i, u_{j+1})(b_i, u_j) | j = 1, 2, \ldots, |V(C)| - 2\}\\ &\cup\{(b_i, u_{|V(C)|})(c_i, u_1)(b_i, u_2)(a_i, u_1)(b_i, u_{|V(C)|})\}\\ &\cup\{(b_i, u_1)(c_i, u_{|V(C)|})(b_i, u_{|V(C)|-1})(a_i, u_{|V(C)|})(b_i, u_1)\}.\end{aligned}$$

It is an easy matter to see that $\mathcal{B}_i$ is linearly independent and is 1-fold. Let $\mathcal{B}' = \bigcup_{i=1}^{|V(T)|-2} \mathcal{B}_i$. By induction on $|V(T)|$, we get that $\bigcup_{i=1}^{|V(T)|-3} \mathcal{B}_i$ is linearly independent. Since $P_3^{(|V(T)|-2)}$ contains an edge, say $a_{|V(T)|-2}b_{|V(T)|-2}$, which is not in any other path of $S(T)$, each cycle of $\mathcal{B}_{|V(T)|-2}$ contains at least one edge of the form $(a_{|V(T)|-2}, u_j)(b_{|V(T)|-2}, u_{j+1})$ and $(a_{|V(T)|-2}, u_{j+1})(b_{|V(T)|-2}, u_j)$ which are not in any cycle of $\bigcup_{i=1}^{|V(T)|-3} \mathcal{B}_i$. Thus, $\mathcal{B}'$ is linearly independent. To this end, we have essentially two cases to consider:
Case a. $|V(C)|$ is odd. Then $T \wedge C$ is connected. Let

$$R = (a_{|V(T)|-2}, v_1)(b_{|V(T)|-2}, v_2)(a_{|V(T)|-2}, v_3) \dots (a_{|V(T)|-2}, v_{|V(C)|})$$
$$(b_{|V(T)|-2}, v_1)(a_{|V(T)|-2}, v_2) \dots (b_{|V(T)|-2}, v_{|V(C)|})(a_{|V(T)|-2}, v_1)$$

and

$$\mathcal{B} = \mathcal{B}' \cup \{R\}.$$

We now prove that R is independent from the cycles of $\mathcal{B}$. Let $E_e = E(e \wedge C)$. Then it is easy to verify that $\{E_e\}_{e\in E(T)}$ is a partition of $E(T \wedge C)$. Moreover, $E_{a_{|V(T)|-2}b_{|V(T)|-2}} = E(R)$ and $E_{a_{|V(T)|-2}b_{|V(T)|-2}} \cup E_{b_{|V(T)|-2}c_{|V(T)|-2}} = E(\mathcal{B}_{|V(T)|-2})$. Thus, if R is a sum modulo 2 of some cycles of $\mathcal{B}'$, say $\{k_1, k_2, \dots, k_r\}$, then $\mathcal{B}_{|V(T)|-2} \subset \{k_1, k_2, \dots, k_r\}$. Since $E_{b_{|V(T)|-2}c_{|V(T)|-2}} \subset E(\mathcal{B}_{|V(T)|-2})$, there is an i_0 and a non-empty set $B_{i_0} \subset \mathcal{B}_{i_0}$ such that $P_3^{(i_0)} \in S(T)$, $B_{i_0} \subset \{k_1, k_2, \dots, k_r\}$ and either $b_{|V(T)|-2}c_{|V(T)|-2} = a_{i_0}b_{i_0}$ or $b_{i_0}c_{i_0}$. Suppose $a_{i_0}b_{|V(T)|-2}c_{|V(T)|-2} = P_3^{(i_0)} \in S(T)$. Since $E(B_{i_0}) \subset E_{a_{i_0}b_{|V(T)|-2}} \cup E_{b_{|V(T)|-2}c_{|V(T)|-2}} = E(\mathcal{B}_{i_0})$, there is i_1 and a non-empty set $B_{i1} \subset \mathcal{B}_{i_1}$ such that $P_3^{(i_1)} \in S(T)$, $B_{i_1} \subset \{k_1, k_2, \dots, k_r\}$ and either $a_{i_0}b_{i_0} = a_{i_1}b_{i_1}$ or $b_{i_1}c_{i_1}$ and so on, it implies that there is an integer i_t and a non-empty set $B_{i_t} \subset \mathcal{B}_{i_t}$ such that $P_3^{(i_t)} \in S(T)$, one of its edges appears only in $P_3^{(i_t)}$, say $a_{i_t}b_{i_t}$, $b_{|V(T)|-2}c_{|V(T)|-2} \in P_3^{(i_t)}$(if not, then T has a cycle) and $B_{i_t} \subset \{k_1, k_2, \dots, k_r\}$. Therefore, $E(B_{i_t}) \cap E_{a_{i_t}b_{i_t}} \subseteq E(R)$ which is a contradiction. So $\mathcal{B}$ is linearly independent.

Case b. $|V(C)|$ is even. Then $T \wedge C$ consists of two isomorphic components H_1 and H_2. Take

$$f_1 = (a_{|V(T)|-2}, v_1)(b_{|V(T)|-2}, v_2)(a_{|V(T)|-2}, v_3) \dots (a_{|V(T)|-2}, v_{|V(C)|-1})$$
$$(b_{|V(T)|-2}, v_{|V(C)|})(a_{|V(T)|-2}, v_1) \quad \text{and}$$
$$f_2 = (b_{|V(T)|-2}, v_1)(a_{|V(T)|-2}, v_2)(b_{|V(T)|-2}, v_3) \dots (b_{|V(T)|-2}, v_{|V(C)|-1})$$
$$(a_{|V(T)|-2}, v_{|V(C)|})(b_{|V(T)|-2}, v_1).$$

Now, let $B_i^{(l)}$ be the set of cycles of B_i which are cycles in H_l for $l = 1, 2$. Note that one of f_1 and f_2 is a cycle of H_1, say f_1, and the other of H_2. By employing the same argument as in Case a to each component, we obtain that each of $B^{(1)} \cup \{f_1\}$ and $B^{(2)} \cup \{f_2\}$ is an independent set. Hence $\mathcal{B} = B^{(1)} \cup B^{(2)} \cup \{f_1\} \cup \{f_2\}$ is linearly independent. Now,

$$\begin{aligned} |\mathcal{B}| &= \sum_{i=1}^{|V(T)|-2} |\mathcal{B}_i| + \delta \\ &= (|V(T)| - 2)(|V(C)|) + \delta = \dim \mathcal{C}(T \wedge C). \end{aligned}$$

where

$$\delta = \begin{cases} 1, & \text{if } |V(C)| \text{ is odd ,} \\ 2, & \text{if } |V(C)| \text{ is even.} \end{cases}$$

Thus, $\mathcal{B}$ is a basis of $T \wedge C$ and it is easy to see that it is 3-fold.

By taking $\mathcal{B} = \mathcal{B}^* \cup \mathcal{B}^{**}$ where $\mathcal{B}^* = \mathcal{B}$ as in Theorem 5.1 and $\mathcal{B}^{**} = \cup_{e\in E(C)}\mathcal{B}_{G_e}$ as in Theorem 4.1, and following their proofs, we get the following result:

Corollary 5.1 *For every bipartite graph G and a cycle C, we have $b(G \wedge C) \leq b(G) + 3$.*

Corollary 5.2 *If G has a spanning tree T such that has no subgraph isomorphic to a 3-special star of order 7, then $b(G \wedge C) \leq b(G) + 2$.*

Proof. It follows from the proof of Corollary 5.1 and the fact that each edge of T appears in at most 2 paths of $S(T)$.

By specializing the bipartite graph G in the above corollary into an even cycle, we obtain the following result:

Corollary 5.3 (Ali) *If C^* is an even cycle and C is a cycle, then $b(C^* \wedge C) \leq 3$.*

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