Regular Hadamard matrix, maximum excess and SBIBD

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Abstract

When $k = q_1$, q_2, q_1q_2, q_1q_4, q_2q_3N, q_3q_4N, where q_1, q_2 and q_3 are prime powers, and where $q_1 \equiv 1 \pmod 4$, $q_2 \equiv 3 \pmod 8$, $q_3 \equiv 5 \pmod 8$, $q_4 = 7$ or 23, $N = 2^a 3^b t^2$, $a, b = 0$ or 1, $t \neq 0$ is an arbitrary integer, we prove that there exist regular Hadamard matrices of order $4k^2$, and also there exist $SBIBD(4k^2, 2k^2+k, k^2+k)$. We find new $SBIBD(4k^2, 2k^2+k, k^2+k)$ for 233 values of k.

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1 Preliminaries

An $n \times n$ matrix H is called a Hadamard matrix (or H-matrix) if every entry of the matrix is 1 or -1, and

$$HH^T = nI_n,$$

where I_n is an $n \times n$ identity matrix. In this paper we use H^T to denote the transpose of a matrix H.

We denote the excess of an *H-matrix* $H = [a_{ij}]$ by $\sigma(H)$, where

$$\sigma(H) = \sum_{1 \le i,j \le n} a_{ij}.$$

Let $\sigma(n) = \max\{\sigma(H)\}$. The weight of an H-matrix H, denoted by $W(H)$, is the number of ones in H. We define $W(n) = \max\{W(H)\}$. Note that the maxima are taken over all $n \times n$ H-matrices H. It is obvious that $\sigma(H) = 2W(H) - n^2$ and $\sigma(n) = 2W(n) - n^2$ (see [4], [5], [6], [7] for details).

Best [1] proved that

$$\sigma(n) \le n\sqrt{n}. \tag{1}$$

Definition 1 *(Regular Hadamard Matrix) A regular Hadamard matrix has the sum of each column of the matrix and the sum of each row of the matrix constant.*

Definition 2 *(SBIBD) A symmetric balanced incomplete block design, called an* $SBIBD(v, k, \lambda)$*, is defined by a* $v \times v$ *matrix* M*, which has every entry* 0 *or* 1*. The sum of each column and the sum of each row of the matrix is* k*. For any two columns* c_i*,* c_j *(and two rows* r_i*,* r_j*),* $1 \le i \ne j \le v$*, the inner product of* c_i *and* c_j *(*r_i *and* r_j*) is* λ *(see* [10]*).*

With the result of this paper and those of [4], [9], the status of the existence of $4k^2$-*Hadamard* matrices and $SBIBD(4k^2, 2k^2 + k, k^2 + k)$ is that they exist for $k \in$ {1, 3, 5, ···, 45, 49, ···, 69, 73, 75, 81, ···, 101, 105, 107, 109, ···, 125, 129, 131, 135, 137, 139, 143, ···, 149, 153, ···, 165, 169, ···, 175, ···, 189, 193, ···, 197, 201, ···, 207, 211, 215, 219, 221, 225, 227, 229, 233, 235, 241, ···, 251, 257, 259, 261, 267, 269, 273, 275, 277, 281, ···, 299, 303, 307, 313, ···, 327, 331, ···, 339, 343, ···, 353, 361, 363, 371, 373, 375, 379, 387, 389, 391, 393, 397, 401, 405, ···, 411, 415, 417, 419, 421, 427, 429, 433, 441, 443, 447, 449, 451, 457, 461, 467, 471, 475, 477, 489, 491, 495, 499, 507, 509, 511, 513, 519, 521, 523, 525, 529, 531, ···, 543, 547, 549, 551, 557, 559, 563, 567, 569, 571, 575, 577, 579, 583, 587, 591, 593, 601, 603, 605, 609, 613, 617, ···, 625, 633, 637, 641, 643, 645, 653, 655, 659, 661, 667, 671, 673, 675, 677, 679, 683, 687, 691, 695, 699, 701, 703, 707, 709, 723, 725, 729, 731, 733, 735, 739, 741, 747, 753, 757, 761, 763, 767, 769, 771, 773, 777, 779, 783, 787, 791, 797, 803, 807, 809, 811, 815, 819, 821, 827, 829, 831, 841, ···, 859, 865, 867, 871, 875, 877, 879, 881, 883, 885, 891, 895, 897, 907, 909, 921, 925, 929, 931, 937, 939, 941, ···, 947, 951, 953, 957, 959, 961, 963, 971, 975, 977, 979, 981, 993, 997, 999, q_1, q_2, q_1q_2, q_1q_4, q_2q_3N, q_3q_4N}, where q_1, q_2 and q_3 are prime powers, $q_1 \equiv 1 \pmod 4$,

$q_2 \equiv 3 \pmod 8$, $q_3 \equiv 5 \pmod 8$, $q_4 = 7$ or 23, $N = 2^a 3^b t^2$, $a, b = 0$ or 1, $t \neq 0$ is an arbitrary integer, $r \geq 0$. This means we find 233 new values less than 1000.

Let G be an *abelian* group with the addition $\oplus$ and the subtraction $\ominus$. We denote by θ the zero element in G. Consider the polynomials in the elements of G over the field of rational numbers, $\sum_{g\in G} a(g)g$, where the integer $a(g)$ is the number of occurrences of g, and define the addition by

$$\sum_{g\in G} a(g)g + \sum_{g\in G} b(g)g = \sum_{g\in G}(a(g)+b(g))g.$$

We denote $\sum_{g\in A} g$ by A, $G = \sum_{g\in G} g$ and $G^* = G - \theta$. For any two subsets $A, B \subset G$, we define

$$A \ominus B = \sum_{a\in A, b\in B} (a \ominus b), \quad \triangle A = A \ominus A,$$
$$\triangle(A, B) = (A \ominus B) + (B \ominus A).$$

It is obvious that $\triangle(A, A) = 2\triangle A$. We define $\triangle\emptyset = 0$, $\triangle(\emptyset, A) = 0$ for any $A \subset G$.

Definition 3 *(DS) Let $D = \{a_1, \cdots, a_k\}$ be a subset of a group G of order v. If for every non-zero element $g \in G$ there are λ pairs (a_i, a_j), a_i, $a_j \in D$, such that*

$$a_i \oplus a_j = g,$$

we call D a (v, k, λ)-difference set (DS).

Definition 4 *(Incidence matrix) The incidence matrix $A = (a_{ij})$ of a $(v, k, \lambda)-$difference set D is defined by ordering the elements of the group $G = \{g_i\}$, $i = 1, \cdots, v$. and defining*

$$a_{ij} = \begin{cases} 1, & g_j \ominus g_i \in D, \\ 0, & \textit{otherwise} \end{cases}$$

Definition 5 *(SDS) Let $D_i \subset G$, $\mid D_i \mid = k_i$, $i = 1, \cdots, r$. If*

$$\sum_{i=1}^{r} \triangle D_i = (\sum_{i=1}^{r} k_i - \lambda)\theta + \lambda G,$$

$\lambda \geq 0$, then $D_1, \cdots, D_r$ are $r - \{v; k_1, \cdots, k_r; \lambda\}$ supplementary difference sets (SDS), where $v = \mid G \mid$.

If $k_1 = \cdots = k_r = k$, we simplify $D_1, \cdots, D_r$ to $r - \{v; k; \lambda\}$ SDS. When $r = 1$, the SDS become the difference set (DS).

We only consider $r = 4$. Then we define $\lambda = \sum_{i=1}^{4} k_i - v$ in this paper. In this case, we call D_1, D_2, D_3, D_4 type $H-$SDS.

Definition 6 *(Type H_1) Let D_1, D_2, D_3, $D_4 \subset G$ be SDS of order v, and $|D_i| = k_i$, $i = 1, 2, 3, 4$. Now $D_1, D_2, D_3, D_4 \in H_1$ if and only if*

$$\sum_{i=1}^{4} \triangle D_i = v\theta + \lambda G,$$

and

$$\triangle(D_1, D_2) + \triangle(D_3, D_4) = \lambda G,$$

where $\lambda = k_1 + k_2 + k_3 + k_4 - v$.

Definition 7 *($T-$matrix) Let T_1, T_2, T_3, T_4 be $n \times n$ matrices with entries $(0, \pm 1)$. Let I_n be an $n \times n$ identity matrix. Then we call T_1, T_2, T_3, T_4 $T-$matrices if*

(i) $T_i T_j = T_j T_i$, $1 \le i, j \le 4$, $i \ne j$,

(ii) there exists an $n \times n$ monomial matrix R with $R^T = R$, $R^2 = I_n$, such that $(T_i R)^T = T_i R$, $i = 1, 2, 3, 4$,

(iii) if $T_i = \left(t_{jk}^{(i)}\right)$, $1 \le j, k \le n$, $i = 1, 2, 3, 4$, then $\sum_{i=1}^{4} \mid t_{jk}^{(i)} \mid = 1$, $i \le j, k \le n$,

(iv) $\sum_{i=1}^{4} T_i T_i^T = nI_n$.

We use conditions (i) and (ii) to replace the condition of circulant $T-$matrices, and the matrix R may easily be found in *abelian* groups.

Definition 8 *($C-$partitions) A_1, A_2, $\cdots$, A_8 are called $C-$partitions of an abelian group G of order v, if the following three conditions are satisfied:*

(i) $A_i \cap A_j = \emptyset$, $i \ne j$;

(ii) $\bigcup_{i=1}^{8} A_i = G$;

(iii) $\sum_{i=1}^{8} \triangle A_i = v\theta + \sum_{i=1}^{4} \triangle(A_i, A_{i+4})$.

Lemma 1 (Seberry [7]) *The following conditions are equivalent:*

(i) There exists a Hadamard matrix of order $4k^2$ with maximum excess $8k^3$.

(ii) There exists a regular Hadamard matrix of order $4k^2$.

(iii) There exists $SBIBD(4k^2, 2k^2 + k, k^2 + k)$.

Some very useful methods to construct *Hadamard* matrices with maximum excess from *Willamson* matrices and $T-$matrices are given in [7].

Lemma 2 (Xia and Liu [11]) *Let q be a prime power, if $q \equiv 1(mod\ 4)$, there exist $4 - \{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ supplementary difference sets.*

Lemma 3 (Xia and Liu [14]) *Let q be a power of a prime, $q \equiv 3 (mod\, 8)$, then there exist $4-\{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ supplementary difference sets.*

Lemma 4 (Chen [2], Xia [12]) *Let $q = 2^a 3^b N^2$, $a, b = 0$ or 1, and N be an arbitrary integer. Then there exist $(4q^2, 2q^2+q, q^2+q)$ difference sets and Williamson type matrices (type 1) A_1, A_2, A_3 and A_4 of order q^2 that satisfy*

$$\begin{array}{l} \sigma(A_1) = \sigma(A_2) = \sigma(A_3) = q^3, \;\; \sigma(A_4) = -q^3, \\ A_1^2 + A_2^2 + A_3^2 + A_4^2 = 4q^2 I_{q^2}, \\ A_i A_j + A_k A_l = 0, \;\; \{i,j,k,l\} = \{1,2,3,4\}. \end{array} \tag{2}$$

Lemma 5 (Xia and Xia [13]) *Let q_1 be a prime power, $q_1 \equiv 5 (mod\, 8)$, $q_2 = 2^a 3^b N^2$, a, $b = 0$ or 1, and N be an arbitrary integer. Then there exist $(1,-1)$ Williamson type matrices (type 1) A_1, A_2, A_3 and A_4 of order $(q_1 q_2)^2$ that satisfy:*

$$\begin{array}{l} \sigma(A_1) = \sigma(A_2) = \sigma(A_3) = (q_1 q_2)^3, \;\; \sigma(A_4) = -(q_1 q_2)^3, \\ A_1 A_1^T + A_2 A_2^T + A_3 A_3^T + A_4 A_4^T = 4(q_1 q_2)^2 I_{(q_1 q_2)^2}, \\ A_1 A_2^T + A_2 A_1^T + A_3 A_4^T + A_4 A_3^T = 0. \end{array} \tag{3}$$

Proposition 1 *Let $p \equiv 5 (mod\;\; 8)$ be a prime, $q \equiv 2^a 3^b N^2$, a, $b = 0$ or 1, N be an arbitrary integer, for any integer $r \geq 1$, there exist $(1,-1)$ matrices A_1, A_2, A_3 and A_4 of order $(p^r q)^2$ that satisfy*

$$\begin{array}{l} \sigma(A_1) = \sigma(A_2) = \sigma(A_3) = (p^r q)^3, \;\; \sigma(A_4) = -(p^r q)^3, \\ \sum_{i=1}^4 A_i A_i^T = 4(p^r q)^2 I_{(p^r q)^2}, \\ A_1 A_2^T + A_2 A_1^T + A_3 A_4^T + A_4 A_3^T = 0 \end{array} \tag{4}$$

Proof. When $q_1 = p^{2r+1}$, then $q_1 \equiv 5$ (mod 8). Then from Lemma 5, the result is true.
When $q_1 = p^{2r} = (p^r)^2$, from Lemma 4 we have the result. This completes the proof. □

Remark. By using Definition 6 we can say when $p \equiv 5$ (mod 8), $q = 2^a 3^b N^2$, a, $b = 0$ or 1, N is an arbitrary integer, for any integer $r \geq 1$, there exist *SDS* D_1, D_2, D_3 and D_4 of order $p^{2r}q^2$ and type H_1. We say $H_1(p^{2r}q^2) \neq \emptyset$, whenever such *SDS* exist for order $p^{2r}q^2$.

In Section 2 we use SDS to construct *SBIBD*. In Section 3 we use SDS and $T-$matrices to construct *SBIBD*. We find new results which give many new *SBIBDs*.

2 Construct SBIBD from SDS

Theorem 1 *If there exist $4-\{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ SDS on an abelian group G of order q^2, then there exist $SBIBD(4q^2, 2q^2+q, q^2+q)$.*

Proof. Let D_1, D_2, D_3, D_4 be $4-\{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ SDS on G, since we have

$$|D_1| = |D_2| = |D_3| = |D_4| = \frac{1}{2}q(q-1), \;\; \sum_{i=1}^{4} \triangle D_i = q^2\theta + q(q-2)G.$$

Let $g_1, \cdots, g_{q^2}$ be the arbitrary order of G, and set

$$A_i = \left(a_{jk}^{(i)}\right)_{1\le j,k\le q^2}, \quad a_{jk}^{(i)} = \begin{cases} -1 & \text{if } g_k \ominus g_j \in D_i, \\ 1 & \text{otherwise,} \end{cases} \quad i = 1,2,3,4, \tag{5}$$

$$R = (r_{jk})_{1\le j,k\le q^2}, \quad r_{jk} = \begin{cases} 1 & \text{if } g_j \oplus g_k = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

It is obvious that A_1, A_2, A_3, A_4 are matrices of type 1. In this case

(i) $A_iA_j = A_jA_i, i \neq j, i, j = 1,2,3,4,$

(ii) $(A_iR)^T = A_iR, i = 1,2,3,4,$

(iii) $\sum_{i=1}^4 A_iA_i^T = 4q^2I_{q^2}$.

Since $|D_i| = \frac{1}{2}q(q-1)$, there exist $\frac{1}{2}q(q+1)$ ones and $\frac{1}{2}q(q-1)$ negative ones in each row of A_i, $i = 1,2,3,4$, so $\sigma(A_i) = q^3, i = 1,2,3,4$. Set

$$H = \begin{pmatrix} -A_1 & A_2R & A_3R & A_4R \\ A_2R & A_1 & A_4^TR & -A_3^TR \\ A_3R & -A_4^TR & A_1 & A_2^TR \\ A_4R & A_3TR & -A_2^TR & A_1 \end{pmatrix}. \tag{7}$$

It is easy to verify that $HH^T = 4q^2I_{4q^2}$, $\sigma(A_i) = \sigma(A_iR) = \sigma(A_i^TR)$, $i = 1,2,3,4$. So we have

$$\sigma(H) = 2\{\sigma(A_1) + \sigma(A_2) + \sigma(A_3) + \sigma(A_4)\} = 8q^3.$$

From Lemma 1, $\frac{1}{2}(H+J)$ is a $SBIBD(4q^2, 2q^2+q, q^2+q)$. This completes the proof. □

Proposition 2 *Let q be a prime power, $q \equiv 1(mod\ 4)$ or $q \equiv 3(mod\ 8)$. There exists $SBIBD(4q^2, 2q^2+q, q^2+q)$.*

Proof. From Lemma 2, Lemma 3 and Theorem 1 the conclusion is true. □

Remark. When $q \equiv 1 \pmod 4$ is a prime power, there exist *Williamson* type matrices A_1, A_2, A_3 and A_4 of order q^2, that make the matrix H of (7) have maximum excess and the form

$$H = \begin{pmatrix} -A_1 & A_2 & A_3 & A_4 \\ A_2 & A_1 & A_4 & -A_3 \\ A_3 & -A_4 & A_1 & A_2 \\ A_4 & A_3 & -A_2 & A_1 \end{pmatrix}. \tag{8}$$

Lemma 6 *Let $q = 2^a3^bN^2$, $a, b = 0$ or 1, N be an arbitrary integer. There exists $SBIBD(4q^2, 2q^2+q, q^2+q)$.*

Proof. From Lemma 4, there exist DS of type $(4q^2, 2q^2+q, q^2+q)$, and the $(0,1)$ incidence matrix B of the DS is an $SBIBD(4q^2, 2q^2+q, q^2+q)$. The proof is completed. □

Remark. From Lemma 4 we know that there exist *Williamson* type matrices A_1, A_2, A_3, A_4 of order q^2 that satisfy (2). In this case, the matrix H of order $4q^2$ with maximum excess has the following form

$$H = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2 & A_1 & A_4 & A_3 \\ A_3 & A_4 & A_1 & A_2 \\ A_4 & A_3 & A_2 & A_1 \end{pmatrix}, \tag{9}$$

or

$$H = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2 & A_3 & A_4 & A_1 \\ A_3 & A_4 & A_1 & A_2 \\ A_4 & A_1 & A_2 & A_3 \end{pmatrix}. \tag{10}$$

Lemma 7 *Let $p \equiv 5(mod\ 8)$ be a prime, $q = 2^a 3^b p^c N^2$, $a, b, c = 0$ or 1, and N be an arbitrary integer. Then there exists $SBIBD(4q^2, 2q^2+q, q^2+q)$.*

Proof. When $c = 0$, from Lemma 6, the Lemma 7 is true. When $c = 1$, from Lemma 5 there exist $(1, -1)$ matrices (type 1) A_1, A_2, A_3 and A_4 of order q^2 that satisfy (3). Let

$$H = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2 & A_1 & A_4 & A_3 \\ -A_3^T & -A_4^T & A_1^T & A_2^T \\ -A_4^T & -A_3^T & A_2^T & A_1^T \end{pmatrix}; \tag{11}$$

then $HH^T = 4q^2 I_{4q^2}$, $\sigma(H) = 4(\sigma(A_1) + \sigma(A_2)) = 8q^3$. So the matrix H of (11) is an *Hadamard* matrix with maximum excess. In this case, from Lemma 1 there exists $SBIBD(4q^2, 2q^2+q, q^2+q)$. The proof is completed. □

Proposition 3 *Let $q = 2^{r_1} 3^{r_2} p^{r_3} N^2$, $p \equiv 5(mod\ 8)$ be a prime, r_1, r_2, r_3 be integers and $r_1, r_2, r_3 \geq 0$, N be an arbitrary integer. Then Lemma 7 still holds.*

Proof. Let $r_i = 2m_i + a_i$, $0 \leq a_i \leq 1$, $i = 1, 2, 3$; then $q = 2^{a_1} 3^{a_2} p^{a_3} (2^{m_1} 3^{m_2} p^{m_3} N)^2$. From Lemma 7 the result is true. □

3 Construct SBIBD from SDS and T-matrices

More details of *T-matrices* are discussed in [3]. In this paper we refer to the paper [15].

Theorem 2 *If there exist* $4-\{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ *SDS* D_1, D_2, D_3, D_4 *of order* q^2 *in an abelian group* G, *and every entry of* G *appears an even number of times in* D_1, D_2, D_3, D_4, *then there exist T-matrices* T_1, T_2, T_3, T_4 *that satisfy*

$$\sigma(T_1) = q^3, \quad \sigma(T_2) = \sigma(T_3) = \sigma(T_4) = 0.$$

Proof. Let

$$\begin{array}{ll} E_1 = G \backslash (D_1 \cup D_2 \cup D_3 \cup D_4), & E_2 = (D_1 \cap D_2) \backslash E_5, \\ E_3 = (D_1 \cap D_3) \backslash E_5, & E_4 = (D_1 \cap D_4) \backslash E_5, \\ E_5 = D_1 \cap D_2 \cap D_3 \cap D_4, & E_6 = (D_3 \cap D_4) \backslash E_5, \\ E_7 = (D_2 \cap D_4) \backslash E_5, & E_8 = (D_2 \cap D_3) \backslash E_5. \end{array}$$

From [15] we know

$$\begin{array}{l} E_i \cap E_j = \emptyset, \ i \neq j, \ 1 \leq i, j \leq 8, \\ G = \cup_{i=1}^{8} E_i, \\ \sum_{i=1}^{8} \triangle E_i = q^2\theta + \sum_{i=1}^{4} \triangle (E_i, E_{i+4}), \end{array}$$

and

$$\begin{array}{ll} D_1 = E_5 \cup E_2 \cup E_3 \cup E_4, & D_2 = E_5 \cup E_2 \cup E_7 \cup E_8 \\ D_3 = E_5 \cup E_3 \cup E_6 \cup E_8, & D_4 = E_5 \cup E_4 \cup E_6 \cup E_7. \end{array}$$

Set $|E_i| = e_i$, $i = 1, \cdots, 8$. We have

$$\begin{array}{ll} |D_1| = e_2 + e_3 + e_4 + e_5, & |D_2| = e_2 + e_5 + e_7 + e_8, \\ |D_3| = e_3 + e_5 + e_6 + e_8, & |D_4| = e_4 + e_5 + e_6 + e_7. \end{array}$$

Since $|D_1| = |D_2| = |D_3| = |D_4| = \frac{1}{2}q(q-1)$, then

$$e_2 - e_6 = e_3 - e_7 = e_4 - e_8 = 0.$$

Since

$$\begin{aligned} q^2 &= |G| = |\cup_{i=1}^{8} E_i| = \sum_{i=1}^{8} e_i = e_1 + e_5 + 2(e_2 + e_3 + e_4) \\ &= e_1 - e_5 + 2(e_2 + e_3 + e_4 + e_5) = e_1 - e_5 + q(q-1), \end{aligned}$$

then $e_1 - e_5 = q$. Let $g_1, \cdots, g_{q^2}$ be an arbitrary ordering of elements of G, and

$$T_i = \left(t_{jk}^{(i)}\right)_{1 \leq j,k \leq q^2}, \ t_{jk}^{(i)} = \begin{cases} 1 & \text{if } g_k \ominus g_j \in E_i, \\ -1 & \text{if } g_k \ominus g_j \in E_{i+4}, \quad i = 1, 2, 3, 4. \\ 0 & \text{otherwise,} \end{cases}$$

T_1, T_2, T_3, T_4 are T-matrices of order q^2 and

$$\sigma(T_1) = q^3, \quad \sigma(T_2) = \sigma(T_3) = \sigma(T_4) = 0.$$

This completes the proof. □

Proposition 4 *Let q be a prime power and $q \equiv 3\,(mod\ 8)$; then there exist T-matrices T_1, T_2, T_3 and T_4 of order q^2 that satisfy Theorem 2.*

Theorem 3 *If there exist T-matrices T_1, T_2, T_3 and T_4 of order t^2, and $\sigma(T_1) = t^3$, $\sigma(T_2) = \sigma(T_3) = \sigma(T_4) = 0$, then there exists $SBIBD(4k^2, 2k^2 + k, k^2 + k)$, $k = tq$, where $q \equiv 1(mod\ 4)$ is any prime power.*

Proof. When $q \equiv 1 \pmod 4$ is a prime power, from Lemma 2 we know there exist $4 - \{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ SDS. In this case from Theorem 1 we have *Williamson* type (type 1) matrices A_1, A_2, A_3 and A_4 of order q^2 which satisfy

(i) $A_i = A_i^T$, $A_iA_j = A_jA_i$, $1 \le i, j \le 4$, $i \ne j$,

(ii) $\sum_{i=1}^4 A_i^2 = 4q^2 I_{q^2}$,

(iii) $\sigma(A_1) = \sigma(A_2) = \sigma(A_3) = \sigma(A_4) = q^3$.

Let

$$\begin{aligned} B_1 &= T_1 \times A_1 + T_2 \times A_2 + T_3 \times A_3 + T_4 \times A_4, \\ B_2 &= T_1 \times A_2 - T_2 \times A_1 + T_3 \times A_4 - T_4 \times A_3, \\ B_3 &= T_1 \times A_3 + T_2 \times A_4 - T_3 \times A_1 - T_4 \times A_2, \\ B_4 &= T_1 \times A_4 - T_2 \times A_3 - T_3 \times A_2 + T_4 \times A_3, \end{aligned} \tag{12}$$

where $\times$ is the Kronecker product. It is obvious that $B_iB_j = B_jB_i$, $i \ne j$, $i, j = 1, 2, 3, 4$, and

$$\sum_{i=1}^{4} B_iB_i^T = (\sum_{i=1}^{4} T_iT_i^T) \times (\sum_{i=1}^{4} A_i^2) = 4(tq)^2 I_{(tq)^2}.$$

Since $\sigma(T_i \times A_i) = \sigma(T_i)\sigma(A_i)$, $i = 1, 2, 3, 4$,

$$\sigma(B_1) = \sigma(B_2) = \sigma(B_3) = \sigma(B_4) = (tq)^3.$$

Let

$$Q = R \times I_{q^2}, \tag{13}$$

where R is a monomial matrix of order t^2 that satisfies $R = R^T$, $R^2 = I$, and $(T_iR)^T = T_iR$, $i = 1, 2, 3, 4$. It is easy to show that Q is a permutation matrix and $(B_iQ)^T = B_iQ$, $i = 1, 2, 3, 4$. Let

$$H = \begin{pmatrix} B_1 & B_2Q & -B_3Q & B_4Q \\ B_2Q & -B_1 & B_4^TQ & B_3^TQ \\ B_3Q & B_4^TQ & B_1 & -B_2^TQ \\ -B_4Q & B_3^TQ & B_2^TQ & B_1 \end{pmatrix}. \tag{14}$$

Then $HH^T = 4k^2 I_{4k^2}$, and $\sigma(H) = 2(\sum_{i=1}^4 \sigma(B_i)) = 8k^3$. In this case, the matrix H of order $4k^2$ defined from (12), (13), (14) has the maximum excess. There exist $SBIBD(4k^2, 2k^2 + k, k^2 + k)$, $k = tq$. The proof is complete. □

Proposition 5 *When $k = q_1 q_2$, where $q_1 = 1(mod\ 4)$, $q_2 = 3(mod\ 8)$ are prime powers, there exist $SBIBD(4k^2, 2k^2 + k, k^2 + k)$.*

Proof. From [15] we know that there exist T-matrices T_1, T_2, T_3, T_4 satisfying Theorem 3. This completes the proof. □

Theorem 4 *Suppose*

1. *there exist T-matrices T_1, T_2, T_3, T_4 of order t^2 that satisfy*

$$\sigma(T_1) = t^3,\ \sigma(T_2) = \sigma(T_3) = \sigma(T_4) = 0,$$

and

2. *there exist $(1, -1)$ matrices (type 1) A_1, A_2, A_3 and A_4 of order q^2 that satisfy*
   (i) $\sum_{i=1}^{4} A_i A_i^T = 4q^2 I_{q^2}$,
   (ii) $A_1 A_2^T + A_2 A_1^T + A_3 A_4^T + A_4 A_3^T = 0$,
   (iii) $\sigma(A_1) = \sigma(A_2) = \sigma(A_3) = q^3 = -\sigma(A_4)$.

Then there exist $SBIBD(4k^2, 2k^2 + k, k^2 + k)$, $k = tq$.

Proof. Let

$$\begin{aligned} B_1 &= T_1 \times A_1 + T_2 \times A_2 + T_3 \times A_3 + T_4 \times A_4, \\ B_2 &= T_1 \times A_2 + T_2 \times A_1 + T_3 \times A_4 + T_4 \times A_3, \\ B_3 &= T_1 \times A_3^T + T_2 \times A_4^T - T_3 \times A_1 T - T_4 \times A_2^T, \\ B_4 &= -T_1 \times A_4^T - T_2 \times A_3^T + T_3 \times A_2^T + T_4 \times A_1^T, \end{aligned}$$

It is easy to verify that

$$\sum_{i=1}^{4} B_i B_i^T = (\sum_{i=1}^{4} T_i T_i^T) \times (\sum_{i=1}^{4} A_i A_i^T) = 4k^2 I_{k^2},$$

and

$$\sigma(B_1) = \sigma(B_2) = \sigma(B_3) = \sigma(B_4) = k^3.$$

Set

$$H = \begin{pmatrix} B_1 & B_2 R & -B_3 R & B_4 R \\ B_2 R & -B_1 & B_4^T R & B_3^T R \\ B_3 R & B_4^T R & B_1 & -B_2^T R \\ -B_4 R & B_3^T R & B_2^T R & B_1 \end{pmatrix},$$

where $R = R_1 \times R_2$, R_1, R_2 are monomial matrices of order t^2 and q^2, and $(T_i R_1)^T = T_i R_1$, $(A_i R_2)^T = A_i R_2$, $i = 1, 2, 3, 4$. In this case $HH^T = 4k^2 I_{4k^2}$, and $\sigma(H) = 2\sum_{i=1}^{4} \sigma(B_i) = 8k^3$. Then H is a *Hadamard* matrix with maximum excess, and $\frac{1}{2}(H + J)$ is a $SBIBD(4k^2, 2k^2 + k, k^2 + k)$. □

Proposition 6 *Let* $k = 2^{a_1}3^{a_2}p_1^{a_3}p_2^{a_4}N^2$*,* a_1*,* a_2*,* a_3*,* $a_4 = 0$ *or* 1*,* $p_1 = 5$ *(mod 8),* $p_2 = 3$ *(mod 8) be primes and* N *be an arbitrary integer. Then there exists* $SBIBD(4k^2, 2k^2 + k, k^2 + k)$*.*

Proof. When $a_4 = 0$, from Lemma 7, the result is true. When $a_4 = 1$, set $t = p_2$, $q = 2^{a_1}3^{a_2}p_1^{a_3}N^2$. From Lemma 5, Proposition 4 and Theorem 3, we can prove the result is correct. □

Remark. Let $k = 2^{a_1}3^{a_2}p_1^{a_3}p_2^{a_4}N^2$, where a_1, a_2, a_3, $a_4 \geq 0$, $p_1 \equiv 5 \pmod 8$, $p_2 \equiv 3 \pmod 8$; then Proposition 6 is still true. Let $a_i = 2s_i + r_i$, where $s_i \geq 0$, $0 \leq r_i \leq 1$, $i = 1, 2, 3, 4$. Then $k = 2^{r_1}3^{r_2}p_1^{r_3}p_2^{r_4}(2^{s_1}3^{s_2}p_1^{s_3}p_2^{s_4}N)^2$ satisfies the condition of Proposition 6.

Proposition 7 *If* $q \equiv 1 (mod\ 4)$ *is a prime power, there exist* $SBIBD(4(7q)^2$*,* $2(7q)^2 + 7q$*,* $(7q)^2 + 7q)$*.*

Proposition 8 *When* $p_2^{a_4}$ *in Proposition 6 is replaced by* 7*, the conclusion of Proposition 6 is still true.*

Proof. Let $g = x \oplus 2$ be a generator of $GF(7^2)$. Set

$$F_i = \{g^{16j+i}(mod\ x^2 \oplus 1, mod\ 7) : j = 0, 1, 2\}, \quad i = 0, 1, \cdots, 15.$$

$$\begin{array}{llll} E_1 = \{0\} \cup F_{11} \cup F_{12} \cup F_{15}, & E_2 = F_0 \cup F_{13}, & E_3 = F_3 \cup F_6, & E_4 = F_4 \cup F_{14}, \\ E_5 = F_{10}, & E_6 = F_1 \cup F_2, & E_7 = F_7 \cup F_8, & E_8 = F_5 \cup F_9. \end{array}$$

It is easy to verify that

$$\sum_{i=1}^{8} \triangle E_i = 49\theta + \sum_{i=1}^{4} \triangle(E_i, E_{i+4}).$$

Without loss of generality, let $g_1, \cdots, g_{49}$ be an arbitrary order on the elements of $GF(7^2)$. Set

$$T_i = \left(t_{jk}^{(i)}\right)_{1 \leq j,k \leq 49}, \quad t_{jk}^{(i)} = \begin{cases} 1, & \text{if } g_k \ominus g_j \in E_i, \\ -1, & \text{if } g_k \ominus g_j \in E_{i+4}, \quad i = 1, 2, 3, 4. \\ 0, & \text{otherwise,} \end{cases}$$

The matrices T_1, T_2, T_3, T_4 are T-matrices of order 49, and

$$\sigma(T_1) = 7^3, \ \sigma(T_2) = \sigma(T_3) = \sigma(T_4) = 0.$$

From Theorem 3 and Theorem 4, we know that Propositions 7 and 8 are both true. This completes the proof. □

From Proposition 8 we know, for any integer $r \geq 1$, there exist $SBIBD(4 \cdot 7^{2r}, 2 \cdot 7^{2r} + 7^r, 7^{2r} + 7^r)$. When r is even, $q = 7^r = 1 \pmod 4$, from Proposition 7 we know the conclusion is true. When r is odd, then $7^{r-1} = 1 \pmod 4$. In this case let $q = 7^{r-1}$, and then from Proposition 7, the conclusion is true. For any integer $a, b \geq 1$, $p \equiv 5 \pmod 8$ a prime, from Proposition 8 we know there exist $SBIBD(4(7^a p^b)^2, 2(7^a p^b)^2 + 7^a p^b, (7^a p^b)^2 + 7^a p^b)$. From Proposition 8 we conclude that for $a, b, c \geq 0$, $p \equiv 5 \pmod 8$ a prime, there exist $SBIBD(4(3^a 7^b p^c)^2$, $2(3^a 7^b p^c)^2 + 3^a 7^b p^c$, $(3^a 7^b p^c)^2 + 3^a 7^b p^c)$.

Lemma 8 *There exist* $4-\{23^2; 23\cdot 11, 23\cdot 21\}$ *SDS of order* 23^2.

Proof. Let $g = x+2$ be a generator oF $GF(23)^2$. Set

$$E_i = \{g^{48j+i}(mod\ x^2+1, mod\ 23) : j = 0, 1, \cdots, 10\},\ \ i = 0, \cdots, 47.$$

Put

$$\begin{array}{l} A_1 = \{0\} \cup E_9 \cup E_{12} \cup E_{13} \cup E_{28} \cup E_{41} \cup E_{44} \cup E_{45}, \\ A_2 = E_0 \cup E_{16} \cup E_{17} \cup E_{29} \cup E_{32} \cup E_{33}, \\ A_3 = E_2 \cup E_4 \cup E_{18} \cup E_{20} \cup E_{34} \cup E_{36}, \\ A_4 = E_3 \cup E_8 \cup E_{19} \cup E_{24} \cup E_{35} \cup E_{40}, \\ A_5 = E_1 \cup E_5 \cup E_6 \cup E_{22} \cup E_{38}, \\ A_6 = E_{10} \cup E_{21} \cup E_{25} \cup E_{26} \cup E_{37} \cup E_{42}, \\ A_7 = E_7 \cup E_{11} \cup E_{23} \cup E_{27} \cup E_{39} \cup E_{43}, \\ A_8 = E_{14} \cup E_{15} \cup E_{30} \cup E_{31} \cup E_{46} \cup E_{47}. \end{array} \quad (15)$$

Let $g_1, \cdots, g_{23^2}$ be an arbitrary order on the elements of $GF(23)^2$. Set matrix

$$T_i = \left(t_{jk}^{(i)}\right)_{1\le j,k\le 23^2},\ \ t_{jk}^{(i)} = \begin{cases} 1, & \text{if } g_k - g_j \in A_i, \\ -1, & \text{if } g_k - g_j \in A_{i+4}, \\ 0, & \text{otherwise.} \end{cases} \quad i = 1, 2, 3, 4. \quad (16)$$

Then T_1, T_2, T_3 and T_4 defined in (16) are $T-$matrices of order 23^2 and

$$\sigma(T_1) = 23^3,\ \ \sigma(T_2) = \sigma(T_3) = \sigma(T_4) = 0.$$

In this case the set $\{A_i\}_{i=1}^8$ defined in (15) is the C-Partition (see [15] for details). The set

$$\begin{array}{ll} D_1 = A_5 \cup A_2 \cup A_3 \cup A_4, & D_2 = A_5 \cup A_2 \cup A_7 \cup A_8, \\ D_3 = A_5 \cup A_3 \cup A_6 \cup A_8, & D_4 = A_5 \cup A_4 \cup A_6 \cup A_7, \end{array} \quad (17)$$

is the $4-\{23^2; 23\cdot 11, 23\cdot 21\}$ SDS. This completes the proof. □

Proposition 9 *There exist* $SBIBD(4\cdot 23^2; 2\cdot 23^2+23, 23^2+23)$.

Proposition 9 follows easily from Theorem 1.

Proposition 10 *When* 7 *in Proposition 7 is replaced by* 23, *there exist* $SBIBD(4\cdot (23q)^2, 2\cdot (23q)^2+23q, (23q)^2+23q)$.

Proposition 11 *There exist* $SBIBD(4\cdot 23^{2r}, 2\cdot 23^{2r}+23^r, 23^{2r}+23^r)$.

Proof. For any integer $r\ge 1$, when r is even, $q = 23^r \equiv 1 \pmod 4$, from Proposition 7, there exist $SBIBD(4\cdot 23^{2r}, 2\cdot 23^{2r}+23^r, 23^{2r}+23^r)$. When r is odd, then $q = 23^{r-1} \equiv 1 \pmod 4$, and in this case the conclusion is again true. □

Remark. For any integers $a, b \ge 1$, and $p \equiv 5 \pmod 8$ a prime, there exist $SBIBD(4\cdot (23^a p^b)^2, 2\cdot (23^a p^b)^2 + 23^a p^b, (23^a p^b)^2 + 23^a p^b)$.
For any $a, b, c \ge 0$, $p \equiv 5 \pmod 8$ a prime, there exist $SBIBD(4\cdot (3^a 23^b p^c)^2, 2\cdot (3^a 23^b p^c)^2 + 3^a 23^b p^c, (3^a 23^b p^c)^2 + 3^a 23^b p^c)$.

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