Embeddings and faithful enclosings of GDDs with a constant number of groups

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Abstract

We introduce, for block size 3, the problem of minimal enclosings of GDDs with u groups of size g into GDDs with u groups of size $g+1$. We prove the necessary conditions are sufficient for the embedding of any GDD with group size 1, i.e., a BIBD$(v, 3, \lambda)$, into a GDD with v groups of size 2. Minimal faithful enclosings of GDDs are given for group size 2 with any index into GDDs with group size 3, and we give some non-minimal faithful enclosings for group size 2 and any index. The minimality in several cases is shown by applying a new necessary condition for enclosing of GDDs.

1 Introduction

We have considered the general problem of enclosing of triple systems previously [9–11] and here extend the efforts to certain enclosings of GDDs. The general problem of enclosing or embedding of combinatorial structures has proven fruitful in the past [1–5,7,8] and we are further motivated by the comments in [6], pp. 155–156: "Enclosing of partial triple systems have not been seriously studied; in fact, as we see next, even enclosings of triple systems themselves have not been determined. . . . Both the enclosing and the faithful enclosing problems appear to be far from a solution at this point."

A design X is enclosed in a design Y if the points and blocks of X are among those of Y. If the index of Y is the same as that of X, the enclosing is called an embedding. An enclosing is minimal if, when X is augmented by some s new points to form Y, the increase in index is as small as possible. An enclosing is faithful if every new block has at least one new point. An embedding is trivially faithful since there is no increase in index, and thus even two old points will not appear in any new block. Faithful enclosing generalizes the concept of maximal independent subset [1, 2].

We refer the reader to [6] and [13] for well known facts about BIBDs, triple systems, and terms not defined here. A group divisible design is a set V with two types of subsets called blocks and groups. The v points of V are partitioned into disjoint sets called groups, and the blocks are k-element subsets such that all pairs of points not in a common group will appear together in exactly λ blocks. The number λ is the index of the design. Points within a group will have index zero with each other, i.e., they will not appear in a common block. The block size k is 3 throughout the paper, and in this note, the groups are uniform meaning that they all have the same size g. We use the superscript notation $\{3,\lambda\}$-GDD(g^u) to denote a uniform GDD with u groups of size g, index λ, and block size 3, with $v/g = u$. A design is resolvable if its blocks can be partitioned in classes such that each point occurs exactly once in each class. A resolvable GDD is referred to as a RGDD.

In this note we will seek to accomplish any enclosing by adding exactly one new point to each group so that the number of groups does not change. Put another way, we will give enclosings for group designs $X = \{3,\lambda\}$-GDD(g^u) into $Y = \{3,\lambda+x\}$-GDD$((g+1)^u)$ for a minimal x. If $x = 0$, the enclosing is an embedding. For accomplishing an enclosing by adding one more group but with a constant group size; see [12].

We prove in Section 2 that any GDD with group size 1 (a BIBD) may be embedded in a GDD with group size 2, and in Section 3 find minimal and faithful enclosings for designs with group size 2 into those with group size 3. In particular, we show that some $\{3,2\}$-GDD(2^u) may be enclosed into a $\{3,3\}$-GDD(3^u) but certain others cannot. This non-existence result motivates one to consider the question of enclosings in general for GDDs, and we point out that previously used techniques (see, for instance, [9] and [12]) are not useful for the enclosings considered here. Throughout the paper our enclosings are faithful and have minimal increase in index unless noted otherwise. We will make implicit use of Table 1 below [13, p.50]. It gives necessary and sufficient conditions for the existence of a λ-fold triple system of order v.

$\lambda \equiv 0 \pmod 6$	All $v \neq 2$
$\lambda \equiv 1,5 \pmod 6$	All $v \equiv 1,3 \pmod 6$
$\lambda \equiv 2,4 \pmod 6$	All $v \equiv 0,1 \pmod 6$
$\lambda \equiv 3 \pmod 6$	All odd v

Table 1: The Spectrum of Triple Systems

The necessary conditions [14] for a uniform $\{k, \lambda\}$-GDD(g^u) are:
(1) $u \geq k$, (2) $\lambda(u-1)g \equiv 0(\bmod k-1)$, (3) $\lambda u(u-1)g^2 \equiv 0(\bmod k(k-1))$.

2 Group Size 1, any index

Our first theorem shows that the *only* possible (faithful) embedding of GDDs occurs for group size 1, and our second theorem shows how to accomplish such an embedding. Recall that a GDD of type 1^v, with group size 1 and with v groups is a BIBD$(v, 3, \lambda)$.

Theorem 2.1 *Suppose there exists a (faithful) embedding of* $X = \{3, \lambda\}$*-GDD*(g^u) *into* $Y = \{3, \lambda\}$*-GDD*$((g+1)^u)$*. Then* $g = 1$*.*

Proof: Every embedding is faithful, trivially, since no new block will contain two old points if the index is fixed. Let $gu = v$, the number of points of X, and let r_x and r_y denote the replication numbers of X and Y, respectively, i.e. the number of blocks in which each point appears. By considering the number of new blocks two ways, it follows that

$$(r_y - r_x)gu \leq \lambda u\,(u-1)\,/2.$$

The left hand side gives the increase in appearance in blocks for points of X, a lower bound on the number of new blocks since we are hypothesizing an embedding. The right hand side is the number of pairs of new points which must appear λ times in possibly distinct blocks, an upper bound since 3 pairs can possibly combine to make one block. For any GDD with block size 3, one has that $\lambda(v-g) = 2r$ and $vr = 3b$, where r is the replication number or the number of blocks in which each point appears. Using these equations, one sees that

$$r_y - r_x = \lambda(u-1)/2.$$

On substituting into the above equation and simplifying, we get

$$\lambda(u-1)g/2 \leq \lambda(u-1)/2.$$

From this we get $g = 1$ since g is a positive integer.

In the next theorem we show how to enclose any BIBD into a GDD with no increase in index λ.

Theorem 2.2 *There is an embedding of every BIBD*$(v, 3, \lambda)$ *into a GDD with* $2v$ *points, index* λ*, and group size* 2*.*

Proof: Suppose first that X = BIBD$(v, 3, \lambda)$ = $\{3, \lambda\}$-GDD(1^v) with points $a_1, a_2, \ldots, a_v$. We add new points $1, 2, \ldots, v$ in order to construct $Y = \{3, \lambda\}$-GDD(2^v), a GDD with group size 2 and $2v$ points. The groups for Y will be the sets $\{a_i, i\}$ for $i = 1, 2, \ldots, v$. Let Z = BIBD$(v, 3, \lambda)$ be any design based on the v new points. For each block $\{x, y, z\}$ of Z, put blocks $\{a_x, y, z\}$, $\{x, a_y, z\}$, and $\{x, y, a_z\}$ in Y. The points of X do not appear in new blocks with each other, and new points will appear exactly λ times with each other. Each old point a_i appears exactly λ times with each new point since i appeared λ times with every other element from Z. The index is thus λ as required.

Corollary 2.3 *There exists a* $\{3,\lambda\}$*-GDD*(2^v) *whenever there exists a BIBD*$(v,3,\lambda)$.

3 Group size 2 and any index

In this section we consider GDDs with group size 2. The necessary conditions for existence require λ, u or $u-1$ to be a multiple of 3. These parameters determine the designs considered. However, we first give two examples with small parameters. Suppose $X = \{3,1\}$-GDD(2^3). We may take the groups to be $\{1,4\},\{2,5\},\{3,6\}$ and the blocks to be $\{1,2,3\},\{1,5,6\},\{2,4,6\}$, and $\{3,4,5\}$. We enclose X into $Y = \{3,2\}$-GDD(3^3) as follows. Add new points 7, 8, and 9, respectively, to the 3 groups. The new blocks of Y are:

{1, 8, 3}, {1, 8, 6}, {1, 9, 2}, {1, 9, 5}, {4, 8, 3},{4, 8, 6}, {4, 9, 2},

{4, 9, 5}, {7, 2, 3}, {7, 3, 5}, {7, 5, 6}, {7, 6, 2}, {7, 8, 9}, {7, 8, 9}.

The next example encloses $X = \{3,1\}$-GDD(2^4) into $Y = \{3,2\}$-GDD(3^4). Take the groups of X as {1, 5}, {2, 6}, {3, 7}, and {4, 8} and add points $9,\dots,12$ to them, respectively, to get the groups for Y. The new blocks for Y consist of a BIBD$(4,3,2)$ on the new points and the following:

{9, 2, 3}, {9, 2, 4}, {9, 3, 4}, {9, 6, 7}, {9, 6, 8}, {9, 7, 8}, {10, 1, 4},

{10, 1, 7}, {10, 4, 7}, {10, 3, 5}, {10, 3, 8}, {10, 5, 8}, {11, 1, 2}, {11, 1, 8},

{11, 2, 8}, {11, 4, 6}, {11, 4, 5}, {11, 6, 5}, {12, 1, 6}, {12, 1, 3}, {12, 3, 6},

{12, 2, 5}, {12, 2, 7}, {12, 5, 7}.

Theorem 3.1 *Suppose* $u \equiv 0,1 \pmod 3$. *Then any* $X = \{3,1\}$*-GDD*(2^u) *can be minimally and faithfully enclosed into some* $Y = \{3,2\}$*-GDD*(3^u).

Proof: We form a $(2u-1)$ by u matrix of unordered pairs, say M, as follows.

Row 1: $\{1,2u\}$, and $\{1+j,1-j\}$ for $j=1,2,\dots,u-1$.

Row 2: $\{2,2u\}$, and $\{2+j,2-j\}$ for $j=1,2,\dots,u-1$.

...

Row $2u-1$: $\{2u-1,2u\}$ and $\{2u-1+j,2u-1-j\}$ for $j=1,2,\dots,u-1$.

Here the integers in pairs not involving $2u$ are reduced mod $(2u-1)$ leaving each of these entries in the range $1,2,\dots,2u-1$. Without loss of generality, the pairs in the first row of M are the groups of X. The new points are designated $a_0,a_1,\dots,a_{u-1}$. To assign the groups for Y, add new point a_i to the i-th pair in Row 1 of M, for $0 \le i \le u-1$. For each new point a_i with $i>0$, create new blocks with the pairs (after the first) in Column i of M except as follows. There will be two distinguished pairs in Column i one of which contains the group point $1+i$ and one of which contains the group point $1-i$, say pairs $\{x, 1+i\}$ and $\{1-i, y\}$. Put a_0 with these two pairs to make new blocks and put a_i with $\{x, 2u\}$ and $\{y, 2u\}$ to make new blocks. Notice, by the construction, a_i ($i = 1, 2, \ldots, u-1$) appears twice in some block or other with every point of X except its group-mates $1+i$ and $1-i$. Let us check the index for a_0. From column i, we have $x-(1+i) = 2i \pmod{2u-1}$ and $1-i-y = 2i \pmod{2u-1}$. Thus, $x = 1+3i \pmod{2u-1}$ and we claim x is unique. Also, $y = 1-3i \pmod{2u-1}$. For suppose on the contrary, that $1+3i = 1-3j \pmod{2u-1}$ for some j. Then $3(i+j) \equiv 0 (\bmod\, 2u-1)$. But $u \equiv 0, 1 \pmod 3$, by hypothesis. Thus, $2u \equiv 0, 2 \pmod 6$, or, $2u-1 \equiv 1, 5 \pmod 6$. In particular, as 3 is prime, 3 is prime to $2u-1$. Thus $i+j \equiv 0 \pmod{2u-1}$. But as i, j are positive integers bounded above by $u-1$, $i+j \le 2u-2$. That is, $i+j \equiv 0 \pmod{2u-1}$ is impossible — hence, the x-value and corresponding y-value determined by each column are distinct and comprise a partition of $2, 3, \ldots, 2u-1$. This shows a_0 is in a block with each point in the range $2, \ldots, 2u-1$ once from this source. But also, the set of groups (except the first) also partitions the points $2, \ldots, 2u-1$. So, the index of point a_0 is 2. The index for $2u$ is also 2 for the same reason. The index is now 2 for old points since each row of M may be recognized as a one-factor of the complete graph on $2u$ vertices, and the pairs in the matrix give a one-factorization of K_{2u}. Finally, let us add the blocks of a BIBD$(u, 3, 2)$ based on the new points, and the enclosing is faithful since each new block has at least one new point.

The construction of matrix M in the proof does not require $u \equiv 0, 1 \pmod 3$ of course, and in each column the x and y determined satisfy $x+y = 2u+1$. However, $x \bmod (2u-1)$ is not unique to one column when $u \equiv 2 \pmod 3$ so that the matching of points does not work.

Theorem 3.2 *(a) Suppose $u \equiv 0, 4$ (mod 6). Then $X = \{3, 2\}$-GDD(2^u) can be faithfully enclosed into $Y = \{3, 4\}$-GDD(3^u), and the enclosing is minimal.*
(b) Suppose $u \equiv 1, 3$ (mod 6). Then $X = \{3, 2\}$-GDD(2^u) can be faithfully enclosed into $Y = \{3, 3\}$-GDD(3^u), and the enclosing is minimal.

Proof: Apply the construction in Theorem 3.1 twice. Applied once, the construction increases the index by 1 for points of X with each other and increases the index from 0 to 2 for points of X with new points. Add the blocks of 2 copies of a triple system BIBD$(u, 3, 2)$ based on the new points. In either case (a) or (b), X is enclosed into Y. However, when u is even, then $Y = \{3, 3\}$-GDD(3^u) does not exist. So the enclosing for part (a) is minimal. For part (b), it is possible to accomplish the enclosing with an increase of only 1 in the index. First, denote the new points by $y_1, y_2, \ldots, y_u$. Let Z denote a BIBD$(u, 3, 1)$ based on the new points. Z exists by Table 1. We add the blocks of 3 copies of Z to those of X. This means the index is 3 for new points with themselves. We next apply the construction from the proof of Theorem 3.1, and this raises the index to 3 for points of X with each other and to 2 for old points with new points. We need to add blocks which increase the index by 1 between old points and new points without increasing the index of either old points or new points with themselves. We do this in the following way. For $i = 1, 2, \ldots, u$, let $\{a_i, b_i\}$ denote a group of X. Use the blocks of 2 of the 3 copies of Z as follows. Consider the two copies of block $\{y_i, y_j, y_k\}$ from Z. We replace these two blocks by the following six blocks:

$$\{y_i, y_j, a_k\}, \{y_i, y_j, b_k\}, \{y_i, a_j, y_k\}, \{y_i, b_j, y_k\}, \{a_i, y_j, y_k\}, \{b_i, y_j, y_k\}.$$

We do this for all the corresponding pairs of blocks in the 2 copies of Z. This completes the enclosing. Note the index for new points with each other remains the same, but since the index is 1 for Z, each old point now appears one more time with each new point (except within its group). As a check, we count the new blocks created in the proof. There are $2u(2u-2)/2$ pairs of old points in distinct blocks from the construction in Theorem 3.1. There are $u(u-1)/6$ blocks in one copy of Z. The last part tripled the $2u(u-1)/6$ blocks in the remaining two copies of Z. Thus there are $2u(2u-2)/2 + u(u-1)/6 + u(u-1) = 19u(u-1)/6$ blocks constructed. An easy calculation shows this is precisely the quantity of blocks of Y less the quantity of blocks of X.

Corollary 3.3 *Suppose $u \equiv 0 (\bmod 3)$. Then any $\{3, \lambda\}$-GDD(2^u) can be faithfully enclosed into some $\{3, 2\lambda\}$-GDD(3^u).*

Proof: Apply the construction in Theorem 3.1 λ times. Of course the enclosing is not in general a minimal one.

If the index $\lambda = 3$, it turns out that there are no congruential restrictions on X or Y. That is, $X = \{3, 3\}$-GDD(2^u) should presumably minimally enclose into $Y = \{3, 4\}$-GDD(3^u) for all $u \geq 3$. As will be seen, a new idea is necessary because $u = 3s + 2$ must be considered. Here is an example for $v = 6$ and $u = 3$. The groups for Y are $\{1, 4, a\}$, $\{2, 5, b\}$, and $\{3, 6, c\}$, where we have added new points a, b, and c, respectively, to the groups $\{1, 4\}, \{2, 5\}, \{3, 6\}$ for X. The new blocks are:

$$\{a, 2, 3\}, \{a, 2, 6\}, \{a, 2, c\}, \{a, 2, c\}, \{a, 3, b\}, \{a, 3, b\}, \{a, 6, b\}, \{a, 6, b\},$$
$$\{a, 5, 3\}, \{a, 5, 6\}, \{a, 5, c\}, \{a, 5, c\}, \{b, 1, 3\}, \{b, 1, 6\}, \{b, 1, c\}, \{b, 1, c\},$$
$$\{b, 4, 3\}, \{b, 4, 6\}, \{b, 4, c\}, \{b, 4, c\}, \{c, 2, 1\}, \{c, 2, 4\}, \{c, 5, 1\}, \{c, 5, 4\}.$$

Of course u in this example is 0 mod 3.

Suppose $u = 6$. Then $X = \{3, 3\}$-GDD(2^6) is enclosed into $Y = \{3, 4\}$-GDD(3^6) as follows. Use matrix M as in Theorem 3.1 and use two copies of $Z = $ BIBD$(6, 3, 2)$ as in Theorem 3.2. This method, however, cannot be used when $u \equiv 2 \pmod 3$.

Theorem 3.4 *For any $u \geq 3$, any $X = \{3, 3\}$-GDD(2^u) can be minimally and faithfully enclosed into $Y = \{3, 4\}$-GDD(3^u).*

Proof: In view of the examples just above, we may assume $u \neq 3, 6$. First, we determine the number of new blocks necessary to complete the design. Let b_x and b_y denote the quantity of blocks in X and Y, respectively. Since for any GDD, $vr = bk$ and $\lambda(v - g) = r(k - 1)$, we get

$$b_y - b_x = 6u(u-1) - 2u(u-1) = 4u(u-1).$$

Every pair of points of X not in the same group must appear exactly once in some new block. There are $2u(2u-2)/2 = 2u(u-1)$ such pairs. Assume for the moment we put each such pair in one new block with one of the new points. There are $u(u-1)/2$ pairs of new points which must appear $\lambda+1 = 4$ times in blocks together. If we put one old point with each such pair, we would have $4u(u-1)/2 = 2u(u-1)$ new blocks of this type. The proposed enclosing would thus have $2u(u-1)+2u(u-1) = 4u(u-1)$ new blocks, exactly the number required. Since the index for old points would increase by 1, the enclosing would be minimal. A scheme for arranging for the appropriate new blocks uses a matrix T, which is obtained from an idempotent Self-Orthogonal Latin Square (SOLS), that is, a Latin Square which is orthogonal to its transpose. SOLS exist for $u \neq 2, 3, 6$; see [15]. The rows and columns of$T$ are indexed by the new points, say $y_1, y_2, \ldots, y_u$. The entry in the (i, j)-cell, corresponding to the pair y_i, y_j, is an unordered but indexed pair of groups of X, say $\{G_n, G_m\}$, where $n \neq i, j$ and $m \neq i, j$. Let $G_n = \{a, b\}$ and $G_m = \{c, d\}$. (We ignore the entries on or above the diagonal of T.) For the (i, j)-cell (with $i < j$) containing $\{G_n, G_m\}$ we add the following new blocks to create Y: $\{a, c, y_i\}, \{b, d, y_i\}, \{a, d, y_j\}, \{b, c, y_j\}$, and $\{y_i, y_j, a\}, \{y_i, y_j, b\}, \{y_i, y_j, c\}, \{y_i, y_j, d\}$. The points y_i, and y_j will appear with each other 4 times in blocks and 2 times in blocks with each point from the two groups. The properties of an SOLS insure symmetry, idempotence, and that in each row and column, each indexed group will occur in two cells. Consequently the new index is 4 for all points of the design.

We next establish a powerful new necessary general condition for enclosings of GDDs. We use it here as a fundamental counting result which is just what we need in order to show the minimality of several enclosings in this section.

Lemma 3.5 *Suppose $X = \{3, \lambda\}$-GDD(2^u) is enclosed in $Y = \{3, \lambda + x\}$-GDD(3^u). Then $\lambda \leq 3x$.*

Proof: We establish the desired result by counting the new pairs of new points necessary to increase the index from 0 to $\lambda + x$ for the u new points. This number is

$$(\lambda + x)u(u - 1)/2.$$

There are two kinds of blocks which we count that contain pairs of "old-new" points, say Type A which are blocks with 2 old points and one new point and Type B which have 2 new points and one old point. Because the index for old points with each other is to increase by x, it follows that the total of blocks of Type A is at most

$$x[2u(2u - 2)/2] = 2xu(u - 1).$$

These Type A blocks contribute $4xu(u - 1)$ "old-new" pairs. There are altogether $(\lambda+x)(2u)(u-1)$ old-new pairs required for Y. So, the remaining number of old-new pairs is at least

$$(\lambda + x)(2u)(u - 1) - 4xu(u - 1) = 2u(\lambda - x)(u - 1).$$

These should occur in $(\lambda - x)u(u - 1)$ blocks of Type B, two pairs per block. Note each of these Type B blocks gives only one new-new pair. Thus, the difference between the total of new-new pairs necessary and those from Type B blocks must be non-negative. Hence

$$\begin{aligned}(\lambda + x)u(u-1)/2 - (\lambda - x)u(u-1) &\geq 0,\\ (\lambda + x)/2 &\geq \lambda - x,\\ 3x &\geq \lambda.\end{aligned}$$

Theorem 3.6 *Suppose $u \equiv 0, 1 \pmod 3$. Then any $X = \{3,4\}$-GDD(2^u) can be minimally and faithfully enclosed into $Y = \{3,6\}$-GDD(3^u).*

Proof: Minimality comes from Lemma 3.5. Use the construction with matrix M twice, and add the blocks of 3 copies of $Z = \text{BIBD}(u,3,2)$ based on the new points. Use two copies of Z as in the proof of Theorem 3.2. This encloses X into Y.

Theorem 3.7 *(a) Suppose $u \equiv 1, 3 \pmod 6$. Then $X = \{3,5\}$-GDD(2^u) can be minimally and faithfully enclosed into $Y = \{3,7\}$-GDD(3^u).*
(b) Suppose $u \equiv 0, 4 \pmod 6$. Then $X = \{3,5\}$-GDD(2^u) can be minimally and faithfully enclosed into $Y = \{3,8\}$-GDD(3^u).

Proof: For part (a), suppose $u \equiv 1, 3 \pmod 6$. We use the construction with matrix M twice, and add the blocks of two copies of $Z_1 = \text{BIBD}(u,3,3)$ based on the new points. Use two copies of Z_1 as in the proof of Theorem 3.2. Lastly, add the blocks of $Z_2 = \text{BIBD}(u,3,1)$ based on the new points. This encloses X into Y. For part (b), we note the two Zs in the proof of part (a) do not exist. Use the method with matrix M twice and add 2 copies of $Z = \text{BIBD}(u,3,4)$ and decompose them as in Theorem 3.2. The minimality of this enclosing follows since $\lambda(u-1)g$ must be even for the existence of Y. Since u is even and g is 3, the index must be 8 not 7 (i.e., even).

Theorem 3.8 *Any $X = \{3,6\}$-GDD(2^u) can be minimally and faithfully enclosed into $Y = \{3,8\}$-GDD(3^u).*

Proof: The enclosing is minimal by the lemma. Use the enclosing of Theorem 3.4 twice.

Let us describe an enclosing for $X = \{3,6t\}$-GDD(2^u) for $t \geq 1$. Apply the method of Theorem 3.4 $2t$ times. The new index for Y is $8t$ ($x = 2t$). By Lemma 3.5, $2t$ is the minimal increase possible; thus the enclosing is minimal. In similar fashion, for index $6t + \lambda$ with $1 \leq \lambda \leq 6$, we may use the methods of this section to construct an enclosing of X into some Y with minimal increase in index — just apply Theorem 3.4 $2t$ times and then use the method corresponding to λ. We conclude with this idea in the following main theorem.

Theorem 3.9 *Suppose $X = \{3, 6t + \lambda\}$-GDD(2^u) for $0 \leq t$ and $1 \leq \lambda \leq 6$. Then X can be minimally and faithfully enclosed into $Y = \{3, 8t + \lambda + y\}$-GDD($3^u$) where y is given by one of the following $(\lambda, \lambda + y)$ pairs:*
$(1,2); (2,3)$ *when* $u \equiv 1, 3 \pmod 6$*;* $(2,4)$ *when* $u \equiv 0, 4$ *(mod 6);*
$(3,4); (4,6); (5,7)$ *when* $u \equiv 1, 3$ *(mod 6);* $(5,8)$ *when* $u \equiv 0, 4$ *(mod 6);* $(6,8)$*.*

There is a very interesting character to certain of the enclosings. We borrow a term from analysis and define a faithful enclosing to be *dense* if every new block contains at least one old point; see, for example, Theorem 3.8. A dense and faithful enclosing is evidently minimal but not conversely (as illustrated by Theorem 3.6).

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(Received 2/1/2002)