Graphs without repeated cycle lengths

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Abstract

In 1975, Erdös proposed the problem of determining the maximum number $f(n)$ of edges in a simple graph of n vertices in which any two cycles are of different lengths. In this paper, it is proved that

$$f(n) \geq n + 36t$$

for $t = 1260r + 169$ $(r \geq 1)$ and $n \geq 540t^2 + \frac{175811}{2}t + \frac{7989}{2}$. Consequently, $\liminf_{n\to\infty} \dfrac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{2}{5}}$.

1 Introduction

Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, Erdös raised the problem of determining $f(n)$ (see [1], p. 247, Problem 11). Shi [2] proved that

$$f(n) \geq n + [(\sqrt{8n-23}+1)/2]$$

for $n \geq 3$. Lai [3,4,5,6,7] proved that for $n \geq \frac{6911}{16}t^2 + \frac{514441}{8}t - \frac{3309665}{16}$, $t = 27720r + 169$,

$$f(n) \geq n + 32t - 1,$$

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and for $n \geq e^{2m}(2m+3)/4$,

$$f(n) < n - 2 + \sqrt{nln(4n/(2m+3)) + 2n} + log_2(n+6).$$

Boros, Caro, Füredi and Yuster [8] proved that

$$f(n) \leq n + 1.98\sqrt{n}(1 + o(1)).$$

In this paper, we construct a simple graph G having no two cycles with the same length which leads to the following result.

Theorem. Let $t = 1260r + 169$ $(r \geq 1)$; then

$$f(n) \geq n + 36t$$

for $n \geq 540t^2 + \frac{175811}{2}t + \frac{7989}{2}$.

2 Proof of the theorem

Proof. Let $t = 1260r + 169, r \geq 1$, $n_t = 540t^2 + \frac{175811}{2}t + \frac{7989}{2}$, $n \geq n_t$. We shall show that there exists a graph G on n vertices with $n + 36t$ edges such that all cycles in G have distinct lengths.

Now we construct the graph G which consists of a number of subgraphs: B_i, $(0 \leq i \leq 21t-1, 27t \leq i \leq 28t+64, 29t-734 \leq i \leq 29t+267, 30t-531 \leq i \leq 30t+57, 31t-741 \leq i \leq 31t+58, 32t-740 \leq i \leq 32t+57, 33t-741 \leq i \leq 33t+57, 34t-741 \leq i \leq 34t+52, 35t-746 \leq i \leq 35t+60, 36t-738 \leq i \leq 36t+60, 37t-738 \leq i \leq 37t+799, i = 21t+2j+1 (0 \leq j \leq t-1), i = 21t+2j (0 \leq j \leq \frac{t-1}{2}), i = 23t+2j+1 (0 \leq j \leq \frac{t-1}{2})$, and $i = 26t)$.

Now we define these B_i. These subgraphs all have a common vertex x, otherwise their vertex sets are pairwise disjoint.

For $0 \leq i \leq t-1$, let the subgraph $B_{21t+2i+1}$ consist of a cycle

$$xu_i^1u_i^2 \ldots u_i^{25t+2i-1}x$$

and a path:

$$xu_{i,1}^1u_{i,1}^2 \ldots u_{i,1}^{(19t+2i-1)/2}u_i^{(23t+2i+1)/2}.$$

Based the construction, $B_{21t+2i+1}$ contains exactly three cycles of lengths:

$$21t + 2i + 1,\ 23t + 2i,\ 25t + 2i.$$

For $0 \leq i \leq \frac{t-3}{2}$, let the subgraph B_{21t+2i} consist of a cycle

$$xv_i^1v_i^2 \ldots v_i^{25t+2i}x$$

and a path:

$$xv_{i,1}^1v_{i,1}^2\dots v_{i,1}^{9t+i-1}v_i^{12t+i}.$$

Based the construction, B_{21t+2i} contains exactly three cycles of lengths:

$$21t+2i,\ 22t+2i+1,\ 25t+2i+1.$$

For $0\le i\le \frac{t-3}{2}$, let the subgraph $B_{23t+2i+1}$ consist of a cycle

$$xw_i^1w_i^2\dots w_i^{26t+2i+1}x$$

and a path:

$$xw_{i,1}^1w_{i,1}^2\dots w_{i,1}^{(21t+2i-1)/2}w_i^{(25t+2i+1)/2}.$$

Based the construction, $B_{23t+2i+1}$ contains exactly three cycles of lengths:

$$23t+2i+1,\ 24t+2i+2,\ 26t+2i+2.$$

For $58\le i\le t-742$, let the subgraph $B_{27t+i-57}$ consist of a cycle

$$C_{27t+i-57}=xy_i^1y_i^2\dots y_i^{132t+11i+893}x$$

and ten paths sharing a common vertex x; the other end vertices are on the cycle $C_{27t+i-57}$:

$$xy_{i,1}^1y_{i,1}^2\dots y_{i,1}^{(17t-1)/2}y_i^{(37t-115)/2+i}$$
$$xy_{i,2}^1y_{i,2}^2\dots y_{i,2}^{(19t-1)/2}y_i^{(57t-103)/2+2i}$$
$$xy_{i,3}^1y_{i,3}^2\dots y_{i,3}^{(19t-1)/2}y_i^{(77t+315)/2+3i}$$
$$xy_{i,4}^1y_{i,4}^2\dots y_{i,4}^{(21t-1)/2}y_i^{(97t+313)/2+4i}$$
$$xy_{i,5}^1y_{i,5}^2\dots y_{i,5}^{(21t-1)/2}y_i^{(117t+313)/2+5i}$$
$$xy_{i,6}^1y_{i,6}^2\dots y_{i,6}^{(23t-1)/2}y_i^{(137t+311)/2+6i}$$
$$xy_{i,7}^1y_{i,7}^2\dots y_{i,7}^{(23t-1)/2}y_i^{(157t+309)/2+7i}$$
$$xy_{i,8}^1y_{i,8}^2\dots y_{i,8}^{(25t-1)/2}y_i^{(177t+297)/2+8i}$$
$$xy_{i,9}^1y_{i,9}^2\dots y_{i,9}^{(25t-1)/2}y_i^{(197t+301)/2+9i}$$
$$xy_{i,10}^1y_{i,10}^2\dots y_{i,10}^{(27t-1)/2}y_i^{(217t+305)/2+10i}.$$

Since a cycle with d chords contains $\binom{d+2}{2}$ distinct cycles, $B_{27t+i-57}$ contains exactly 66 cycles of lengths:

$27t+i-57,$	$28t+i+7,$	$29t+i+210,$	$30t+i,$
$31t+i+1,$	$32t+i,$	$33t+i,$	$34t+i-5,$
$35t+i+3,$	$36t+i+3,$	$37t+i+742,$	$38t+2i-51,$
$38t+2i+216,$	$40t+2i+209,$	$40t+2i,$	$42t+2i,$
$42t+2i-1,$	$44t+2i-6,$	$44t+2i-3,$	$46t+2i+5,$
$46t+2i+744,$	$48t+3i+158,$	$49t+3i+215,$	$50t+3i+209,$
$51t+3i-1,$	$52t+3i-1,$	$53t+3i-7,$	$54t+3i-4,$
$55t+3i-1,$	$56t+3i+746,$	$59t+4i+157,$	$59t+4i+215,$
$61t+4i+208,$	$61t+4i-2,$	$63t+4i-7,$	$63t+4i-5,$
$65t+4i-2,$	$65t+4i+740,$	$69t+5i+157,$	$70t+5i+214,$
$71t+5i+207,$	$72t+5i-8,$	$73t+5i-5,$	$74t+5i-3,$
$75t+5i+739,$	$80t+6i+156,$	$80t+6i+213,$	$82t+6i+201,$
$82t+6i-6,$	$84t+6i-3,$	$84t+6i+738,$	$90t+7i+155,$
$91t+7i+207,$	$92t+7i+203,$	$93t+7i-4,$	$94t+7i+738,$
$101t+8i+149,$	$101t+8i+209,$	$103t+8i+205,$	$103t+8i+737,$
$111t+9i+151,$	$112t+9i+211,$	$113t+9i+946,$	$122t+10i+153,$
$122t+10i+952,$	$132t+11i+894.$		

B_0 is a path with an end vertex x and length $n-n_t$. Any other B_i is simply a cycle of length i.

Then $f(n) \geq n+36t$, for $n \geq n_t$. This completes the proof.

From the above theorem, we have

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{2}{5}},$$

which is better than the previous bounds $\sqrt{2}$ (see [2]), $\sqrt{2+\frac{2562}{6911}}$ (see [7]).

Combining this with Boros, Caro, Füredi and Yuster's upper bound, we get

$$1.98 \geq \limsup_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2.4}.$$

We make the following conjecture:

Conjecture.

$$\lim_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} = \sqrt{2.4}.$$

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