Degree conditions for the existence of $[k, k+1]$-factors containing a given Hamiltonian cycle

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Abstract

Let $k \geq 2$ be an integer and G a 2-connected graph of order $|G| \geq 3$ with minimum degree at least k. Suppose that $|G| \geq 8k - 16$ for even $|G|$ and $|G| \geq 6k - 13$ for odd $|G|$. We prove that G has a $[k, k+1]$-factor containing a given Hamiltonian cycle if $\max\{\deg_G(x), \deg_G(y)\} \geq |G|/2$ for each pair of nonadjacent vertices x and y in G. This is best possible in the sense that there exists a graph having no k-factor containing a given Hamiltonian cycle under the same conditions. The lower bound of $|G|$ is also sharp.

1 Introduction

We consider finite undirected graphs without loops and multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, we denote by $\deg_G(x)$ the degree of x in G and by $|G|$ the order of G. Put $\delta(G) = \min\{\deg_G(x) \mid x \in V(G)\}$. For $S \subseteq V(G)$, $G[S]$ is the subgraph of G induced by S. We write $G - S$ for $G[V(G) \setminus S]$. We define the *distance* $d(x, y)$ between two vertices x and y as the minimum of the lengths of the x–y paths of G.

Let a and b be two integers such that $1 \leq a \leq b$. Then a spanning subgraph F of G is called an $[a, b]$-*factor* if $a \leq \deg_F(x) \leq b$ for all $x \in V(G)$. If $a = b = k$, then a $[k, k]$-factor is just a k-factor.

Let us introduce some well-known results relating our theorem.

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Theorem 1 (G. Fan [3]) *Let G be a 2-connected graph with $|G| \geq 3$. If for any two nonadjacent vertices x and y of G such that $d(x,y) = 2$,*

$$\max\{\deg_G(x), \deg_G(y)\} \geq \frac{|G|}{2},$$

then G has a Hamiltonian cycle.

Theorem 2 (T. Nishimura [6]) *Let $k \geq 3$ be an integer and G a connected graph with $|G| \geq 4k-3$, $k|G|$ even, and $\delta(G) \geq k$. If for each pair of nonadjacent vertices x and y of $V(G)$,*

$$\max\{\deg_G(x), \deg_G(y)\} \geq \frac{|G|}{2},$$

then G has a k-factor.

Combining the two theorems above, we can easily recognize that if a 2-connected graph G satisfies the conditions in Theorem 2, then G has a $[k, k+2]$-factor containing a given Hamiltonian cycle C, which is the union of C and a k-factor of G.

The purpose of this paper is to extend a "$[k, k+2]$-factor containing a given Hamiltonian cycle" to a "$[k, k+1]$-factor containing a given Hamiltonian cycle".

Our main result is the following theorem.

Theorem 3 *Let $k \geq 2$ be an integer. Suppose that G is a 2-connected graph of order $|G| \geq 3$ with $|G| \geq 8k-16$ for even $|G|$ and $|G| \geq 6k-13$ for odd $|G|$. If $\delta(G) \geq k$ and*

$$\max\{\deg_G(x), \deg_G(y)\} \geq \frac{|G|}{2}$$

for each pair of nonadjacent vertices x and y of $V(G)$, then G has a $[k, k+1]$-factor containing a given Hamiltonian cycle.

As seen in [2], we know the condition $\delta(G) \geq |G|/2$ does not guarantee the existence of a k-factor containing a given Hamiltonian cycle of G, which is shown in the following example: Suppose that $|G| \geq 5$ and $k \geq 3$. Put

$$m = \begin{cases} \dfrac{|G|}{2} + 2 & \text{for even } |G| \text{ and} \\ \dfrac{|G|+3}{2} & \text{for odd } |G|. \end{cases}$$

Let $C := v_1 v_2 \ldots v_m$ be a cycle of order m and $P := v_{m+1} v_{m+2} \ldots v_{|G|}$ a path of order $|G| - m$. Then the join $G := C + P$ has no k-factor containing Hamiltonian cycle $v_1 v_2 \ldots v_{|G|}$, but satisfies $\delta(G) \geq |G|/2$.

Moreover, the lower bounds of the conditions $|G| \geq 8k-16$ for even $|G|$ and $|G| \geq 6k-13$ for odd $|G|$ are sharp. In the example above, put $2k \leq |G| < 8k-16$ for even $|G|$ and $2k-1 \leq |G| < 6k-13$ for odd $|G|$. Then the join $G := C + P$ has no k-factor containing Hamiltonian cycle $v_1 v_2 \ldots v_{|G|}$, but satisfies $\delta(G) \geq |G|/2$.

The following theorem is deduced from Theorem 3.

Theorem 4 (M. Cai, Y. Li, and M. Kano [2]) *Let $k \geq 2$ be an integer and let G be a graph of order $|G| \geq 3$ with $|G| \geq 8k-16$ for even $|G|$ and $|G| \geq 6k-13$ for odd $|G|$. Suppose that*

$$\deg_G(x) + \deg_G(y) \geq |G|$$

for each pair of nonadjacent vertices x and y of $V(G)$. Then G has a $[k, k+1]$-factor containing a given Hamiltonian cycle.

Since a "$[k, k+1]$-factor containing a Hamiltonian cycle" is a 2-connected $[k, k+1]$-factor, our theorem is an extension of the following theorem on a 2-connected graph.

Theorem 5 (M. Cai [1]) *Let $k \geq 3$ be an odd integer and let G be a graph of odd order $|G| \geq 4k-3$. Suppose that $\delta(G) \geq k$ and*

$$\max\{\deg_G(x), \deg_G(y)\} \geq \frac{|G|}{2}$$

for each pair of nonadjacent vertices x and y of $V(G)$. Then G has a connected $[k, k+1]$-factor.

2 Proof of Theorem 3

Our proof depends on the following theorem, which is a special case of Lovász's (g, f)-factor theorem.

Theorem 6 (L. Lovász [5]) *Let G be a graph and let a and b be integers such that $1 \leq a < b$. Then G has an $[a, b]$-factor if and only if*

$$\gamma(S, T) := b|S| + \sum_{x \in T} (\deg_{G-S}(x) - a) \geq 0$$

for all disjoint subsets S and T of $V(G)$.

Proof of Theorem 3 By Theorem 1, G has a Hamiltonian cycle C. For $k = 2$, Theorem 3 holds since C itself is a desired factor. Hence we may assume that $k \geq 3$. Put $H := G - E(C)$ and $r := k - 2$. Note that $r \geq 1$, $V(H) = V(G)$, and $\delta(H) = \delta(G) - 2 \geq r$.

Obviously, G has a required factor if and only if H has an $[r, r+1]$-factor. In order to prove the theorem by reduction to absurdity, we assume that H has no $[r, r+1]$-factor. Then, by Theorem 6, there exist disjoint subsets S and T of $V(H) = V(G)$ satisfying

$$\gamma(S, T) = (r+1)|S| + \sum_{x \in T} (\deg_{H-S}(x) - r) \leq -1. \tag{1}$$

We choose such subsets S and T so that $|T|$ is as small as possible.

Claim 1 $|T| \geq r + 2$.

Proof If $|T| \leq r+1$, then by (1) and $|S| + \deg_{H-S}(x) \geq \deg_H(x) \geq \delta(H) \geq r$ for all $x \in T$, we obtain

$$\gamma(S,T) = (r+1)|S| + \sum_{x \in T}(\deg_{H-S}(x) - r) \geq \sum_{x \in T}(|S| + \deg_{H-S}(x) - r) \geq 0,$$

which is a contradiction. ∎

Claim 2 $\deg_{H-S}(x) \leq r-1$ *for all* $x \in T$.

Proof If $\deg_{H-S}(x) \geq r$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (1). This contradicts the choice of S and T. ∎

Claim 3 $|S| \geq 1$.

Proof By Claim 2, it follows that

$$|S| + r - 1 \geq |S| + \deg_{H-S}(x) \geq \deg_H(x) \geq \delta(H) \geq r$$

for all $x \in T$, implying $|S| \geq 1$. ∎

Write

$$U := \left\{ x \in V(G) \,\middle|\, \deg_G(x) \geq \frac{|G|}{2} \right\} \quad \text{and} \quad L := V(G) \setminus U.$$

Claim 4 $G[L]$ *is a complete graph.*

Proof For any two vertices x and y in L, we obtain $\max\{\deg_G(x), \deg_G(y)\} < |G|/2$ by the definition of L. Thus, $xy \in E(G)$. ∎

Claim 5 $|S| \leq \left\lceil \frac{|G|}{2} \right\rceil - 3$.

Proof We first consider the case when $|G|$ is even. Suppose that $|S| \geq |G|/2 - 2$. Put $\alpha := |S| - |G|/2 + 2 \geq 0$ and $\beta := |G| - |S| - |T| \geq 0$. Then it follows from

Claim 3 and $|G| \geq 8k-16 = 8r$ that

$$\begin{aligned}\gamma(S,T) &= (r+1)|S| + \sum_{x\in T}(\deg_{H-S}(x) - r)\\ &= (r+1)|S| + \sum_{x\in T}\deg_{H-S}(x) - r(|G|-|S|-\beta)\\ &= (2r+1)|S| - r(|G|-\beta) + \sum_{x\in T}\deg_{H-S}(x)\\ &= (2r+1)\left(\frac{|G|}{2} - 2 + \alpha\right) - r(|G|-\beta) + \sum_{x\in T}\deg_{H-S}(x)\\ &= \frac{|G|}{2} + (2r+1)(\alpha-2) + r\beta + \sum_{x\in T}\deg_{H-S}(x)\\ &\geq (2r+1)\alpha + r\beta + \sum_{x\in T}\deg_{H-S}(x) - 2. \qquad (2)\end{aligned}$$

If $\alpha \geq 1$ or $\beta \geq 2$, then we have $\gamma(S,T) \geq 0$ by the inequalities above. This contradicts (1). Hence we may assume that $\alpha = 0$ and $\beta \leq 1$. We now show that $\sum_{x\in T}\deg_{H-S}(x) \geq 1$ under the assumption $\alpha = 0$ and $\beta \leq 1$.

Suppose that $\sum_{x\in T}\deg_{H-S}(x) = 0$, $\alpha = 0$, and $\beta \leq 1$. Since $\alpha = 0$, it follows that $|S| = |G|/2 - 2$. Write $X := \bar{S} \cap U$ and $Y := \bar{S} \cap L$, where $\bar{S} = V(G) \setminus S$.

Since $\sum_{x\in T}\deg_{H-S}(x) = 0$ and $\beta \leq 1$, we obtain $E(G[\bar{S}]) \subset C$ and $\deg_G(x) \leq \deg_{H-S}(x) + |S| + 2 = |G|/2$ for all $x \in \bar{S}$, implying $\deg_G(x) = |G|/2$ for all $x \in X$. Therefore all the edges of C incident to the vertices in X are contained in $E(G[\bar{S}])$. On the other hand, $G[Y]$ is a complete graph by Claim 4 since $Y \subseteq L$.

Thus we obtain

$$\begin{aligned}|X| + |Y| - 1 = |\bar{S}| - 1 &\geq |E(G[\bar{S}]) \cap E(C)|\\ &\geq |X| + 1 + |E(G[Y])| = |X| + 1 + \frac{|Y|(|Y|-1)}{2},\end{aligned}$$

which implies $|Y| - 1 \geq 1 + |Y|(|Y|-1)/2$, a contradiction.

Consequently, we have $\sum_{x\in T}\deg_{H-S}(x) \geq 1$, $\alpha = 0$, and $\beta \leq 1$. If $\beta = 1$, then by (2), $\gamma(S,T) \geq r + \sum_{x\in T}\deg_{H-S}(x) - 2 \geq 0$, which contradicts (1). If $\beta = 0$, then $V(H) = S \cup T$ and

$$\sum_{x\in T}\deg_{H-S}(x) = \sum_{x\in T}\deg_{H-S}(x) \equiv 2|E(H[T])| \equiv 0 \pmod 2.$$

Hence $\gamma(S,T) \geq \sum_{x\in T}\deg_{H-S}(x) - 2 \geq 0$, which also contradicts (1). Therefore Claim 5 holds for even $|G|$.

We next consider the case when $|G|$ is odd. Suppose that $|S| \geq (|G|-3)/2$. Put $\alpha := |S| - (|G|-3)/2 \geq 0$ and $\beta := |G| - |S| - |T| \geq 0$. Then it follows from Claim 3 and $|G| \geq 6k - 13 = 6r - 1$ that

$$\gamma(S,T) = (r+1)|S| + \sum_{x\in T}(\deg_{H-S}(x) - r) \geq (2r+1)\alpha + r\beta + \sum_{x\in T}\deg_{H-S}(x) - 2.$$

By the same argument above, we may assume that $\alpha = 0$, $\beta \leq 1$, and $\sum_{x \in T} \deg_{H-S}(x) = 0$. Let $X := \{x \in \bar{S} \mid \deg_G(x) \geq (|G|+1)/2\}$ and $Y := \bar{S} \setminus X$. Similarly we obtain

$$\begin{aligned} |X| + |Y| - 1 = |\bar{S}| - 1 \geq |E(G[\bar{S}]) \cap E(C)| \\ \geq |X| + 1 + |E(G[Y])| = |X| + 1 + \frac{|Y|(|Y|-1)}{2}, \end{aligned}$$

which implies $|Y| - 1 \geq 1 + |Y|(|Y|-1)/2$, a contradiction. Therefore Claim 5 holds for odd $|G|$. ∎

Claim 6 $T \cap U \neq \emptyset$.

Proof If $T \subseteq L$, then $|E_G[T]| = |T|(|T|-1)/2$ by Claim 4. Since C is a Hamiltonian cycle, $|E_G[T] \cap C_1| \leq |T| - 1$ holds. Hence, we obtain

$$\sum_{x \in T} \deg_{H-S}(x) \geq 2|E_G[T] \setminus E_G(C)| \geq |T|(|T|-1) - 2(|T|-1) = (|T|-1)(|T|-2).$$

Then it follows from Claims 1 and 3 that

$$\begin{aligned} \gamma(S,T) &= (r+1)|S| + \sum_{x \in T} (\deg_{H-S}(x) - r) \\ &\geq (r+1)|S| + (|T|-1)(|T|-2) - r|T| \\ &\geq (r+1)|S| + (|T|-1)r - r|T| = (r+1)|S| - r \geq 1, \end{aligned}$$

which contradicts (1). ∎

Claim 7 $T \cap L \neq \emptyset$.

Proof Suppose that $T \subseteq U$. For every $x \in T$, it follows from Claim 2 that

$$\left\lceil \frac{|G|}{2} \right\rceil \leq \deg_G(x) \leq \deg_{H-S}(x) + |S| + 2 \leq |S| + r + 1,$$

which implies $\deg_{H-S}(x) \geq \lceil |G|/2 \rceil - |S| - 2$ and $\lceil |G|/2 \rceil - |S| - 2 - r \leq -1$. Hence

$$\begin{aligned} \gamma(S,T) &= (r+1)|S| + \sum_{x \in T} (\deg_{H-S}(x) - r) \\ &\geq (r+1)|S| + |T| \left(\left\lceil \frac{|G|}{2} \right\rceil - |S| - 2 - r \right) \\ &\geq (r+1)|S| + (|G| - |S|) \left(\left\lceil \frac{|G|}{2} \right\rceil - |S| - 2 - r \right). \end{aligned}$$

Put $f(|S|) := (r+1)|S| + (|G|-|S|)(\lceil |G|/2 \rceil - |S| - 2 - r)$. Then by Claim 5 and $|G| \geq 6r-1$,

$$\begin{aligned} f'(|S|) &= r+1-\left(\left\lceil \frac{|G|}{2} \right\rceil - |S| - 2 - r\right) - (|G|-|S|) \\ &= 2r+3-\left\lceil \frac{|G|}{2} \right\rceil - |G| + 2|S| \\ &\leq 2r+3-\left\lceil \frac{|G|}{2} \right\rceil - |G| + 2\left(\left\lceil \frac{|G|}{2} \right\rceil - 3\right) = 2r-3+\left\lceil \frac{|G|}{2} \right\rceil - |G| < 0. \end{aligned}$$

Hence we have

$$\begin{aligned} f(|S|) &\geq f\left(\left\lceil \frac{|G|}{2} \right\rceil - 3\right) = (r+1)\left(\left\lceil \frac{|G|}{2} \right\rceil - 3\right) + \left(|G| - \left\lceil \frac{|G|}{2} \right\rceil + 3\right)(1-r) \\ &= r\left(2\left\lceil \frac{|G|}{2} \right\rceil - |G|\right) + |G| - 6r \geq 0. \end{aligned}$$

The last inequality follows from the condition that $|G| \geq 8r$ for even $|G|$ and $|G| \geq 6r-1$ for odd $|G|$. This contradicts (1). ∎

Put

$$T_1 := T \cap U \quad \text{and} \quad T_2 := T \cap L.$$

By Claims 6 and 7, we have $|T_1| \geq 1$, and $|T_2| \geq 1$. It is clear that $\deg_{H-S}(x) \geq \deg_G(x) - |S| - 2$ for all $x \in T$. In particular, for every $x \in T_1$,

$$\deg_{H-S}(x) \geq \begin{cases} \frac{|G|}{2} - |S| - 2 & \text{if } |G| \text{ is even and} \\ \frac{|G|}{2} - |S| - \frac{3}{2} & \text{if } |G| \text{ is odd.} \end{cases} \tag{3}$$

It follows from Claim 2 that

$$\begin{cases} \frac{|G|}{2} - r - |S| - 1 \leq 0 & \text{if } |G| \text{ is even and} \\ \frac{|G|}{2} - r - |S| - \frac{1}{2} \leq 0 & \text{if } |G| \text{ is odd.} \end{cases} \tag{4}$$

By Claim 5 and the inequalities above, we have $r \geq 2$.

Claim 8 $|T_2| \leq r+2$.

Proof Since T_2 is a complete subgraph by Claim 4, $\deg_{H-S}(x) \geq |T_2| - 3$ for all $x \in T_2$. Thus we have $|T_2| - 3 \leq \deg_{H-S}(x) \leq r-1$ by Claim 2, which implies $|T_2| \leq r+2$. ∎

To complete the proof, we consider two cases according to whether $|G|$ is even or odd. For even $|G|$, using (3), (4), Claims 5 and 8, and $|G| \geq 8r$, we obtain

$$\begin{aligned}
\gamma(S,T) &= (r+1)|S| + \sum_{x\in T}(\deg_{H-S}(x) - r) \\
&\geq (r+1)|S| + |T_1|\left(\frac{|G|}{2} - |S| - 2\right) - r(|T_1| + |T_2|) \\
&= (r+1)|S| + |T_1|\left(\frac{|G|}{2} - |S| - 2 - r\right) - r|T_2| \\
&\geq (r+1)|S| + (|G| - |S| - |T_2|)\left(\frac{|G|}{2} - |S| - 2 - r\right) - r|T_2| \\
&= \left(\frac{|G|}{2} - |S| - 3\right)^2 + \left(\frac{|G|}{2} - |S| - 3\right)\left(\frac{|G|}{2} + 3 - 2r - |T_2|\right) + |G| - 6r - |T_2| \\
&\geq 2r - |T_2| \geq 0.
\end{aligned}$$

This contradicts (1).

We next assume $|G|$ is odd. Let $\beta := |G| - |S| - |T|$. Since $G[T_2]$ is a complete graph, we obtain

$$\begin{aligned}
\sum_{x\in T_2} \deg_{H-S}(x) &\geq 2|E(G[T_2]) \setminus E(C)| \\
&\geq |T_2|(|T_2| - 1) - 2(|T_2| - 1) = (|T_2| - 1)(|T_2| - 2). \qquad (5)
\end{aligned}$$

Using (3), (4), Claims 5 and 8, the inequality above, and $|G| \geq 6r - 1$, we obtain

$$\begin{aligned}
\gamma(S,T) &= (r+1)|S| + \sum_{x\in T}(\deg_{H-S}(x) - r) \\
&\geq (r+1)|S| + |T_1|\left(\frac{|G|}{2} - |S| - \frac{3}{2}\right) + (|T_2| - 1)(|T_2| - 2) - r(|T_1| + |T_2|) \\
&= (r+1)|S| + |T_1|\left(\frac{|G|}{2} - |S| - \frac{3}{2} - r\right) - r|T_2| + (|T_2| - 1)(|T_2| - 2) \\
&= (r+1)|S| + (|G| - |S| - |T_2| - \beta)\left(\frac{|G|}{2} - |S| - \frac{3}{2} - r\right) - r|T_2| \\
&\quad + (|T_2| - 1)(|T_2| - 2) \\
&\geq \left(\frac{|G|}{2} - |S| - \frac{5}{2}\right)^2 + r - 1 + (|T_2| - 1)(|T_2| - 2) - |T_2| + \beta.
\end{aligned}$$

By the inequality above, we obtain $\gamma(S,T) \geq 0$ unless $|S| = (|G| - 5)/2$, $|T_2| = 2$, $r = 2$, $\beta = 0$, and (5) holds throughout with equality. Hence we need to consider only the case $|S| = (|G| - 5)/2$, $|T_2| = 2$, $r = 2$, $\beta = 0$, and (5) holds throughout with equality. Since $|T_2| = 2$ and (5) holds throughout with equality, we have $|E(G[T_2])| = |E(G[T_2]) \cap E(C)| = 1$. From $|S| = (|G| - 5)/2$ and $r = 2$, it follows from Claim 2 and (4) that

$$\deg_{H-S}(x) = 1 \quad \text{and} \quad \deg_G(x) = \frac{|G| + 1}{2} \quad \text{for all } x \in T_1.$$

This implies that all the edges of C incident to the vertices in T_1 are contained in $E(G[T]) \setminus E(G[T_2])$. Thus the number of such edges is at least $|T_1| + 1$. Therefore $|E(G[T]) \cap C| \geq |T_1| + 1 + 1 = |T|$, contradicting the fact C is a Hamiltonian cycle of G. Finally, Theorem 1 is proved. ∎

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