Extensions of α-polynomial classes

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Abstract

Let $\alpha(G)$ be the stability number of a graph G. A class of graphs $\mathcal{P}$ is called α-polynomial if there exists a polynomial-time algorithm to determine $\alpha(G)$ for $G \in \mathcal{P}$. For every hereditary α-polynomial class $\mathcal{P}$ we construct a hereditary extension of $\mathcal{P}$ which is either an α-polynomial class or α can be approximated in polynomial time in the extended class.

1 Introduction

Let $G = (V(G), E(G))$ be a graph. A set $S \subseteq V(G)$ is called *stable* if no two vertices in S are adjacent. The *stability number* $\alpha(G)$ of G is the maximum cardinality of a stable set of G. A class $\mathcal{P}$ of graphs is called *α-polynomial* if there exists a polynomial-time algorithm to determine $\alpha(G)$ for $G \in \mathcal{P}$.

For a set $X \subseteq V(G)$, the graph $G(X)$ *induced by X in G* has vertex set X and edge set $\{uv : u, v \in X \text{ and } uv \in E(G)\}$. A graph H is an *induced subgraph* of a graph G if H is isomorphic to $G(X)$ for some $X \subseteq V(G)$. For a set $\mathcal{F}$ of graphs, a graph G is called *$\mathcal{F}$-free* if no induced subgraph of G is isomorphic to a graphs in $\mathcal{F}$. The graphs in $\mathcal{F}$ are called *forbidden induced subgraphs* for the class of all $\mathcal{F}$-free graphs. A class $\mathcal{P}$ of graphs is called *hereditary* if it is closed under taking induced subgraphs. Note that a class $\mathcal{P}$ of graphs is hereditary if and only if there is a (possibly infinite) set $\mathcal{F}$ of graphs such that $\mathcal{P}$ is the class of $\mathcal{F}$-free graphs.

Definition 1. *Let G_0 and G be two graphs and let $I_0 \subseteq V(G_0)$ be a maximal stable set of G_0, i.e. $V(G_0) = N_{G_0}[I_0] := I_0 \cup N_{G_0}(I_0)$.*

*(i) A (G_0, I_0)-*extension *$G_{(G_0,I_0)}$* of *G is a graph that arises from disjoint copies of G_0 and G by adding edges (possibly none) between the vertices in $V(G_0) \setminus I_0$ and $V(G)$.*

*(ii) A (G_0, I_0)-*stable set of *G is a stable set $I \subseteq V(G)$ of G such that there is an isomorphism $\phi : V(G_0) \to V(\tilde{G}_0)$ where $\tilde{G}_0$ is an induced subgraph of G and $\phi(I_0) = I \cap V(\tilde{G}_0)$ (see Figure 1).*

We define $\alpha_{(G_0,I_0)}(G)$ as the maximum cardinality of a (G_0, I_0)-stable set in G or as 0 if no such set exists.

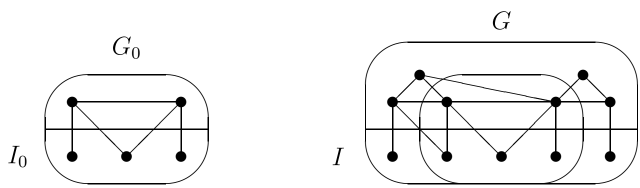

Figure 1

*(iii) Let $\mathcal{F}$ be a set of graphs and let $\mathcal{P}$ be the class of $\mathcal{F}$-free graphs. The (G_0, I_0)-*extension of *$\mathcal{P}$ is the class $\mathcal{P}_{(G_0,I_0)}$ of $\mathcal{F}_{(G_0,I_0)}$-free graphs where $\mathcal{F}_{(G_0,I_0)} = \{F_{(G_0,I_0)} : F \in \mathcal{F}\}$ is the set of all (G_0, I_0)-extensions of graphs in $\mathcal{F}$.*

Proposition 1. *Let G_0, I_0 and $\mathcal{P}$ be as in Definition 1. Let $G \in \mathcal{P}_{(G_0,I_0)}$ and let $\phi : V(G_0) \to V(\tilde{G}_0)$ be an isomorphism where $\tilde{G}_0$ is an induced subgraph of G. Then $G \setminus N_G[\phi(I_0)] \in \mathcal{P}$.*

Proof. Let $G' = G \setminus N_G[\phi(I_0)]$. If $G' \notin \mathcal{P}$, then G' contains an induced subgraph $F \in \mathcal{F}$ where $\mathcal{F}$ is as in Definition 1. By the maximality of I_0, $V(\tilde{G}_0) \cap V(F) = \emptyset$. Now $G(V(\tilde{G}_0) \cup V(F))$ is a (G_0, I_0)-extension of F, a contradiction to $G \in \mathcal{P}_{(G_0,I_0)}$. □

Theorem 1. *Let G_0, I_0 and $\mathcal{P}$ be as in Definition 1. If $\mathcal{P}$ is an α-polynomial class of graphs then there is a polynomial time algorithm to determine $\alpha_{(G_0,I_0)}$ for graphs in $\mathcal{P}_{(G_0,I_0)}$.*

Proof. First, it is possible to find all induced subgraphs of $G \in \mathcal{P}_{(G_0,I_0)}$ that are isomorphic to G_0 in polynomial time. If G is G_0-free, then $\alpha_{(G_0,I_0)}(G) = 0$. If G is not G_0-free, then for each of the polynomially many isomorphisms ϕ such that $\phi : V(G_0) \to V(\tilde{G}_0)$ for some induced subgraph $\tilde{G}_0$ of G, the graph $G_\phi = G \setminus N_G[\phi(I_0)]$ belongs to $\mathcal{P}$ (Proposition 1). Hence we can determine $\alpha(G_\phi)$ in polynomial time. Obviously, $\alpha_{(G_0,I_0)}(G) = |I_0| + \max\{\alpha(G_\phi) : \phi \text{ as above}\}$. □

Before we proceed to some specific applications for which we consider special choices of G_0 and I_0 we want to point out that $\alpha_{(G_0,I_0)}$ always approximates α within some additive term for graphs of bounded maximum degree that are not G_0-free.

Therefore, let G be some graph that is not G_0-free and let $\phi : V(G_0) \to V(\tilde{G}_0)$ be an isomorphism for some induced subgraph $\tilde{G}_0$ of G. Let I be a maximum independent set of G, i.e. $|I| = \alpha(G)$ and let $I_0' = \phi(I_0) \setminus I$. The set $I' = (I \setminus N_G(I_0')) \cup I_0'$ is a (G_0, I_0)-independent set of G and hence

$$\alpha_{(G_0,I_0)} \geq |I'| \geq |I| - |N_G(I_0')| + |I_0'| \geq \alpha(G) - (\Delta - 1)|I_0'| \geq \alpha(G) - (\Delta - 1)|I_0|$$

where Δ denotes the maximum degree of G.

2 Some applications

Let $K_{1,d}$ denote the star of order $d + 1$.

Proposition 2. *For $d \geq 2$ let $G_0 = K_{1,d}$ and let I_0 consist of the d vertices of degree 1 in $K_{1,d}$. Let G be a graph that is not $K_{1,d}$-free. Then $\alpha_{(K_{1,d},I_0)}(G) \geq \alpha(G) - d(d-2)$.*

Proof. We prove the existence of a $(K_{1,d}, I_0)$-stable set of G of cardinality at least $\alpha(G) - d(d-2)$. Let I be a maximum stable set of G. If $|N_G(u) \cap I| \geq d$ for some vertex $u \in V(G) \setminus I$, then I is a $(K_{1,d}, I_0)$-stable set of G and we are done. Hence, we can assume that $|N_G(u) \cap I| \leq d - 1$ for all vertices $u \in V(G) \setminus I$.

Let the vertices $v_0, v_1, \ldots, v_d \in V(G)$ induce a graph in G that is isomorphic to $K_{1,d}$ such that v_0 is the vertex of degree d. Let $\{v_1, v_2, \ldots, v_d\} \setminus I = \{v_1, v_2, \ldots, v_l\}$ for some $1 \leq l \leq d$. The set $I' = \left(I \setminus \bigcup_{i=1}^{l} N_G(v_i)\right) \cup \{v_1, v_2, \ldots, v_l\}$ is a $(K_{1,d}, I_0)$-stable set of G and $|I'| \geq |I| - l(d-1) + l = |I| - l(d-2) \geq |I| - d(d-2)$. □

Theorem 2 (Berman [1], Halldórsson [4]). *For $d \geq 2$ there is a polynomial time algorithm to approximate α for $K_{1,d}$-free graphs within a factor of $\frac{d}{2}$, i.e. to determine a stable set I of a given $K_{1,d}$-free graph G such that $|I| \geq \frac{2}{d}\alpha(G)$.*

Note that the case $d = 2$ of the above theorem is trivial.

Corollary 1. *For $d \geq 2$ let $G_0 = K_{1,d}$ and let I_0 consist of the d vertices of degree 1 in $K_{1,d}$. If $\mathcal{P}$ is an α-polynomial hereditary class of graphs then there is a polynomial time algorithm to approximate α for graphs in $\mathcal{P}_{(K_{1,d},I_0)}$ within a factor of $\frac{d}{2}$.*

Proof. Let $G \in \mathcal{P}_{(K_{1,d},I_0)}$. First, we check in polynomial time whether $\alpha(G) < d^2$ and determine $\alpha(G)$ in this case. Hence we can assume that $\alpha(G) \geq d^2$, or equivalently $\alpha(G) - d(d-2) \geq \frac{2}{d}\alpha(G)$. If G is $K_{1,d}$-free, then Theorem 2 yields the desired result. If G is not $K_{1,d}$-free, then $\alpha_{(K_{1,d},I_0)}(G) \geq \alpha(G) - d(d-2) \geq \frac{2}{d}\alpha(G)$, by Proposition 2. Now, Theorem 1 yields the desired result. $\square$

Note that $\frac{2}{d} = 1$ for $d = 2$, i.e. α can be determined exactly for graphs in $\mathcal{P}_{(K_{1,2},I_0)}$. Below P_n denotes the path of order n.

Example 1. The class $\mathcal{P}$ of all $\{G_1, G_2, \ldots, G_6\}$-free graphs (see Figure 2) is α-polynomial, since $\mathcal{P}$ is the $(K_{1,2}, I_0)$-extension of the α-polynomial class of all P_3-free graphs.

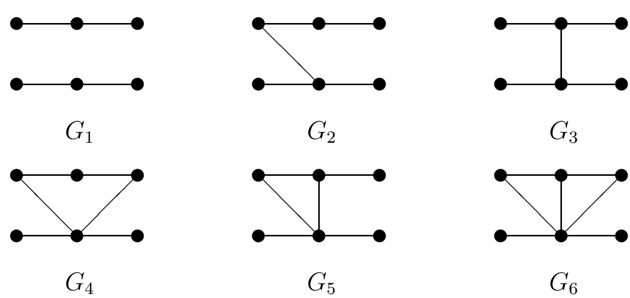

Figure 2

Example 2. The class $\mathcal{P}$ of all $\{F_1, F_2, \ldots, F_{10}\}$-free graphs (see Figure 3) is α-polynomial, since $\mathcal{P}$ is the $(K_{1,2}, I_0)$-extension of the α-polynomial class of all cographs (P_4-free graphs; see [2]).

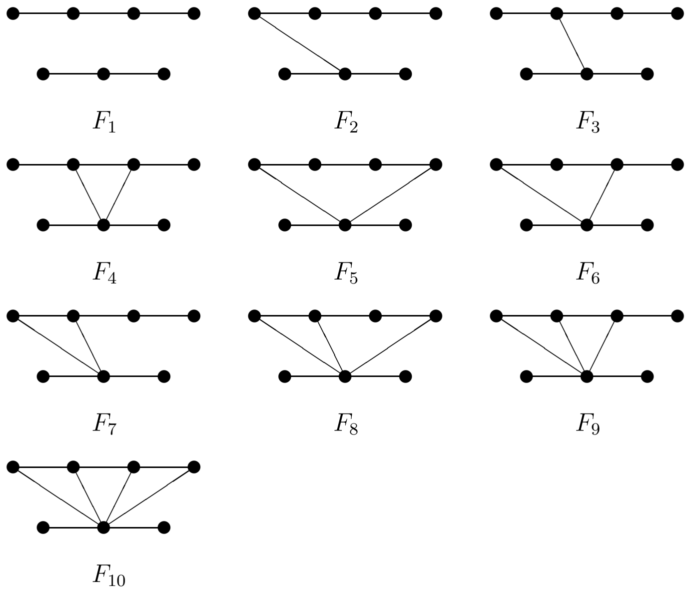

Figure 3

For the case $d = 3$, i.e. *claw-free* graphs, we can actually prove something stronger.

Proposition 3. *Let $G_0 = K_{1,3}$ and let I_0 consist of the three vertices of degree* 1 *in $K_{1,3}$. Let G be a graph that is not $K_{1,3}$-free. Then $\alpha_{(P_3,I_0)}(G) \geq \alpha(G) - 1$.*

Proof. We prove the existence of a $(K_{1,3}, I_0)$-stable set of G of cardinality at least $\alpha(G) - 1$.

Let I be a maximum stable set of G. If some vertex in $V(G) \setminus I$ has three neighbors in I, then we are done. Hence $|N_G(u) \cap I| \leq 2$ for every $u \in V(G) \setminus I$. Since G is not claw-free, let $a, b, c, d \in V(G)$ induce a claw such that d is the vertex of degree three. Obviously, $a, b, c \in I$ does not hold.

If $a, b \in I$ and $c, d \notin I$ then $I' = (I \setminus N_G(c)) \cup \{c\}$ has the desired properties. Hence each claw in G has at most one vertex in I.

Now we assume that $a \in I$ and $b, c, d \notin I$. If $|(N_G(b) \cup N_G(c)) \cap I| \leq 3$, then $I' = (I \setminus (N_G(b) \cup N_G(c))) \cup \{b, c\}$ has the desired properties. Hence $|N_G(b) \cap I| = 2$, $|N_G(c) \cap I| = 2$ and $N_G(b) \cap N_G(c) \cap I = \emptyset$. Since either $N_G(d) \cap N_G(b) \cap I = \emptyset$ or $N_G(d) \cap N_G(c) \cap I = \emptyset$, the set $I' = (I \setminus N_G(d)) \cup \{d\}$ has the desired properties.

Hence we may assume that $a, b, c \notin I$. If $|(N_G(a) \cup N_G(b) \cup N_G(c)) \cap I| \leq 4$ then $I' = (I \setminus (N_G(a) \cup N_G(b) \cup N_G(c))) \cup \{a, b, c\}$ has the desired properties. Hence

$|(N_G(a) \cup N_G(b) \cup N_G(c)) \cap I| \geq 5$ which implies that $\{a, b, c\} \not\subseteq N_G(e)$ for each $e \in I$. Thus $d \notin I$.

If $a, b \notin N_G(e)$ for some $e \in N_G(d) \cap I$ then it is easy to see that $|N_G(a) \cap I| = 2$, $|N_G(b) \cap I| = 2$ and $N_G(a) \cap N_G(b) \cap I = \emptyset$. Since each claw has at most one vertex in I, we have $N_G(d) \cap N_G(a) \cap I \neq \emptyset$ and $N_G(d) \cap N_G(b) \cap I \neq \emptyset$ which implies the contradiction $|N_G(d) \cap I| \geq 3$. Hence $|N_G(e) \cap \{a, b, c\}| = 2$ for each $e \in N_G(d) \cap I$. Since $N_G(d) \cap I \neq \emptyset$, we can assume that $N_G(e) \cap \{a, b, c\} = \{a, b\}$ for some $e \in N_G(d) \cap I$. This implies that $|N_G(c) \cap I| = 2$ and $N_G(c) \cap (N_G(a) \cup N_G(b)) \cap I = \emptyset$.

If $N_G(c) \cap N_G(d) \cap I = \emptyset$, then $I' = (I \setminus N_G(d)) \cup \{d\}$ has the desired properties. Hence there is some $f \in N_G(c) \cap N_G(d) \cap I$. Since $N_G(f) \cap \{a, b\} \neq \emptyset$, we obtain the contradiction $|(N_G(a) \cup N_G(b) \cup N_G(c)) \cap I| \leq 4$. □

Theorem 3 (Minty [3], Sbihi [5]). *The class of all $K_{1,3}$-free graphs is α-polynomial.*

Corollary 2. *Let $G_0 = K_{1,3}$ and let I_0 consist of the three vertices of degree* 1 *in P_3. Let $\mathcal{P}$ be an α-polynomial hereditary class of graphs.*

Then there is a polynomial time algorithm to approximate α for graphs in $\mathcal{P}_{(K_{1,3},I_0)}$ within 1, *i.e. for every graph $G \in \mathcal{P}_{(K_{1,3},I_0)}$ we can determine in polynomial time some α' such that $\alpha(G) - 1 \leq \alpha' \leq \alpha(G)$.*

Proof. Let $G \in \mathcal{P}_{(K_{1,3},I_0)}$. If G is $K_{1,3}$-free, then Theorem 3 yields the desired result. If G is not $K_{1,3}$-free, then $\alpha_{(K_{1,3},I_0)}(G) \geq \alpha(G) - 1$, by Proposition 3. Now the result follows from Theorem 1. □

As an example, we consider the $(K_{1,3}, I_0)$ extension of $K_{1,3}$-free graphs. Let $\mathcal{C}$ be the set of graphs shown in Figure 4 (dotted lines represent potential edges). In fact, $\mathcal{C}$ consists of 8 pairwise non-isomorphic graphs.

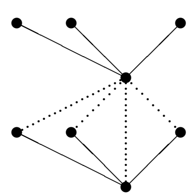

Figure 4. The configuration $\mathcal{C}$

Clearly, the class of all $\mathcal{C}$-free graphs is exactly $\mathcal{P}_{(K_{1,3},I_0)}$, where $\mathcal{P}$ denotes the class of all $K_{1,3}$-free graphs and I_0 is as before. By Corollary 2, there is a polynomial time algorithm to approximate α for $\mathcal{C}$-free graphs within 1 and we pose the following

Open Problem 1. *Is there a polynomial time algorithm to determine α for $\mathcal{C}$-free graphs?*

Finally, we want to point out that our extension operations can obviously be iterated.

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(Received 1/9/2001)