# Spectra and cuts

C. Delorme

LRI, Bât. 490
Université Paris-Sud
91405 Orsay Cedex
France

**Abstract**

We show that the spectrum, as well as the Laplacian spectrum, does not suffice to compute exactly the max-cut and the bisection. We examine what happens if switch-equivalence is required.

We also show that in general no two of the three usual spectra (adjacency, Laplacian, Seidel) suffice to compute the third one.

## 1 Introduction

To each finite simple graph $G$ is associated its adjacency matrix $A$, whose entry $a_{ij}$ is 1 if $\{ij\}$ is an edge and 0 otherwise. The spectrum of $G$ is the spectrum of that matrix. We will call it $\mathrm{Spec}(G)$.

One also associates to $G$ the Laplacian matrix $L$ defined by the property that $L + A$ is the diagonal matrix whose entries are the degrees of the corresponding vertices. The spectrum of $L$ will be denoted $\mathrm{Spec}_L(G)$.

If $G$ is regular of degree $d$, the spectrum $\mathrm{Spec}_L$ is obtained from Spec by the mapping $x \mapsto d - x$. Morover, in that case, it is easy to compute the spectrum of the complement $\bar{G}$ of $G$ from the spectrum of $G$: since $A(\bar{G}) = J - I - A(G)$ and $A$ commutes with the matrix $J$ filled with 1's, each eigenvalue $\lambda$ is replaced by $-1 - \lambda$, except for one occurrence of the degree $d$, replaced by $n - 1 - d$.

For example, from the spectrum of the graph $G$ made from two disjoint edges, namely $1, 1, -1, -1$, we deduce its Laplacian spectrum $0, 0, 2, 2$, the spectrum of the 4-cycle (that is the complement of $G$), namely $2, 0, 0, -2$ and the Laplacian spectrum of the 4-cycle, that is $0, 2, 2, 4$.

The max-cut $\mathbf{m}(G)$ of a graph is the maximum number of edges between $X$ and $V \setminus X$, where $X$ is a part of the vertex set $V$ of $G$. In other words, it is the maximum number of edges in a bipartite subgraph of $G$. If $G$ has even order $n$, its bisection $\mathbf{b}(G)$ is the minimum number of edges between $X$ and $V \setminus X$, with the extra condition that $X$ has $n/2$ vertices.

The largest and smallest elements in Spec and $\mathrm{Spec}_L$ give bounds for the max-cut $\mathbf{m}$ and the bisection $\mathbf{b}$, namely

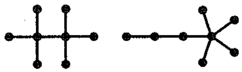

Figure 1: Two cospectral trees

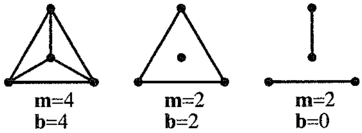


Figure 2: Graphs switch-equivalent to $K_4$

$$\mathbf{m} \leq \frac{n}{4} \max(\mathrm{Spec}_L) \text{ and } \mathbf{m} \leq \frac{m}{2} - \frac{n}{4} \min(\mathrm{Spec}).$$

The Laplacian spectrum provides also the inequality

$$\mathbf{b} \geq \frac{n}{4} \min(\mathrm{Spec}_L \setminus 0).$$

More generally, if the vertex set $V$ is cut into two parts having $n_1$ and $n_2 = n - n_1$ vertices, the number of edges between these two parts is between $\frac{n_1 n_2}{n} \max(\mathrm{Spec}_L)$ and $\frac{n_1 n_2}{n} \min(\mathrm{Spec}_L \setminus 0)$.

Let us be precise: spectra are multisets, so removing 0 from $\mathrm{Spec}_L$ is just decreasing the multiplicity of 0 by 1.

So in the former example of $C_4$ we find $\mathbf{m} \leq 4$ and $\mathbf{b} \geq 2$; these bounds are the actual values. We also find for the complement $\mathbf{m} \leq 2$ and $\mathbf{b} \geq 0$, also the actual values.

We may ask whether knowing the whole spectrum, and not only its ends, helps to determine the max-cut and bisection. In other words, do cospectral graphs have the same max-cut, or bisection?

But already the trees of Figure 1, although cospectral, with common characteristic polynomial $T^4(T^2 - T - 3)(T^2 + T - 3)$, do not have the same bisection, 1 for the tree on the left, 2 for the one on the right. Of course they have the same max-cut 7. Their Laplacian spectra differ: the characteristic polynomials of their Laplacian matrices are $T(T-4)(T^2-6T+2)(T-1)^4$ (left) and $T(T^4 - 11T^3 + 36T^2 - 38T + 8)(T-1)^3$ (right).

## 2 Switch-equivalence

We will use also another kind of similarity between graphs, that is switch-equivalence. Given a graph $G$, and a part $U$ of its vertex-set, we build the graph $G_U$ by switching with respect to $U$, that is the graph with edge-set the symmetric difference between the set of edges in $G$ and the pairs of vertices with one element in $U$ and the other in $V \setminus U$. Two graphs $G, H$ will be said to be switch-equivalent if there exists $U$ such that $H$ is isomorphic to $G_U$.

It is worth noticing that $(G_{U_1})_{U_2} = G_{U_3}$ where $U_3$ is the symmetric difference between $U_1$ and $U_2$.

For example, the graphs switch-equivalent to $K_4$ are presented in Figure 2.

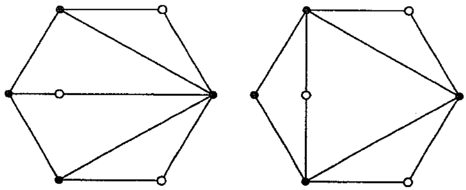

Figure 3: A cospectral pair

It can be seen that switch-equivalent graphs can have different sizes (and therefore different spectra and Laplacian spectra, since the size $e$ of a graph satisfies $2e = \mathrm{tr}(A^2) = \mathrm{tr}(L)$), and different values for the parameters $\mathbf{m}$ and $\mathbf{b}$.

However, switch-equivalent graphs have the same Seidel spectrum. Indeed, the effect of switching with respect to $S$ on the Seidel matrix is the same as the conjugation by the diagonal matrix with 1's at places corresponding to vertices in $S$ and $-1$'s in the remaining places.

Let us use the Seidel matrix of the graph $S = J - I - 2A$ (see [3, p. 26]); if the graph is regular, the matrices $J$ and $A$ commute, and hence the spectrum of the Seidel matrix consists of $n - 1 - 2d$ and the image of $\mathrm{Spec}(G) \setminus d$ by $\lambda \mapsto -1 - 2\lambda$. Since switching does not modify the spectrum of the Seidel matrix, we can conclude that

switch-equivalent regular graphs having the same degree are cospectral.

Thus switching provides a way to obtain graphs with the same spectrum, but not necessarily non-isomorphic ones.

# 3 Independence of the three spectra

We give here examples showing that graphs having two spectra in common may have a different third one. These examples of course use non-regular graphs.

## 3.1 Adjacency and Seidel spectra do not determine the Laplacian one

We have a classical example (see [4]) with graphs on 7 vertices (see Figure 3).

The common adjacency spectrum of the two graphs is described by the characteristic polynomial $T(T-1)(T+1)(T^4 - 9T^2 - 4T + 8)$; the equality of their Seidel spectra follows from their being switch-equivalent (Figure 3 shows what set of vertices should be swapped).

The Laplacian characteristic polynomials are respectively $T(T^2 - 6T + 7)(T^2 - 9T + 17)(T^2 - 5T + 5)$ and $T(T-2)(T-6)(T^2 - 6T + 7)^2$ for the left hand and right hand graphs.

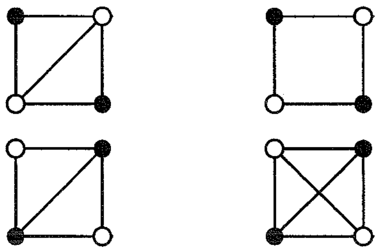

Figure 4: A cospectral pair

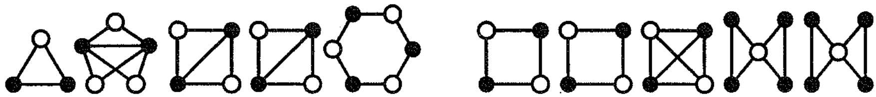

Figure 5: A cospectral pair

### 3.2 Laplacian and Seidel spectra do not determine the adjacency one

Here is a simple example. The Seidel spectra are equal, owing to the switch-equivalence (indicated in Figure 4); the common Laplacian spectrum is $4^3, 2^3, 0^2$, and the adjacency characteristic polynomials are respectively $T^2(T+2)^2(T^2-T-4)^2$ and $(T+1)^3(T-3)(T-2)(T+2)T^2$ for the left hand and right hand graphs.

### 3.3 Laplacian and adjacency spectra do not determine the Seidel one

Here is a slightly larger example (Figure 5). The connected components easily provide the common adjacency spectrum, corresponding to the polynomial $T^4(T-3)(T-1)^2(T+2)^2(T^2-T-4)^2(T+1)^7$ and the common Laplacian spectrum, corresponding to the polynomial $T^5(T-1)^2(T-2)^4(T-3)^4(T-4)^5(T-5)^2$.

Maple [6] gives the Seidel spectra $T^4(T+4)(T+2)^2(T-2)^7(T^2+2T-16)(T^5-14T^4-92T^3+520T^2+1344T-5376)(T-4)$ and $T^4(T+4)(T+2)^2(T-2)^7(T^2+2T-16)(T^4-10T^3-132T^2-8T+1312)(T-4)^2$ for the left hand and right hand graphs respectively.

## 4 Adjacency and Laplacian Spectra give neither m nor b

Looking closer at the graphs above, we see that the left hand one has $\mathbf{m} = 22$, the right hand one has $\mathbf{m} = 20$ (the colors of the vertices in the graphs show an appropriate vertex partition (Figure 5)). The bisections are 0 (take the triangle and the two components on 4 vertices to build one side of the vertex partition) and $\geq 0$,

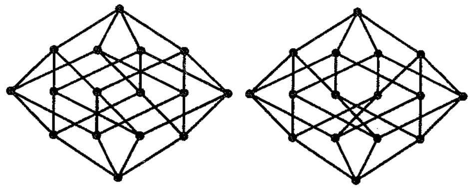

Figure 6: The 4-cube and Hoffmann graph

since 11 is not attainable as a sum of sizes of connected components (actually it is 2, with a side consisting of the two components on 5 vertices and a vertex of a 4-cycle).

# 5 Do the equality of the three spectra suffice?

One may now wonder whether graphs having the same spectrum and the same Laplacian spectrum also have the same max-cut and the same bisection.

For regular graphs of the same degree, the equality of adjacency spectra is equivalent to the equality of Laplacian spectra and also to the the equality of Seidel spectra. Hence we will now look at regular graphs; maybe however there exist non-regular graphs with the same three spectra?

## 5.1 A case of equality

A well-known pair of regular cospectral graphs consists of the cartesian sum $K_4 \times K_4$ and Shrikhande graph. Their common spectrum is $\{6, 2^6, (-2)^9\}$ (see [1, p. 104-105]), but they also have the same max-cut 32 and the same bisection 16.

On the other hand, these two graphs are also switch-equivalent. It suffices to switch with respect to a stable on 4 vertices in one of these graph to obtain a graph isomorphic to the other one.

## 5.2 A case of inequality

We consider here the cube of dimension 4 and the Hoffman graph. These two graphs are described in [1, p. 263], and drawn in Figure 6. They are bipartite on 16 vertices and have the same adjacency spectrum, namely $\{4, 2^4, 0^6, (-2)^4, -4\}$. Hence they have the same max-cut, namely 32, but their bisections differ, 8 for the cube and 10 for the Hoffman graph.

Maple [6] was used to obtain the statistics of the number of partitions giving cuts with a prescribed number of edges; see Table 1.

These two graphs are not switch-equivalent. Indeed, since the degree is one fourth of the order, the only way to switch without losing the 4-regularity is to switch with respect to a stable component. This amounts to swapping all 4 pairs of opposite vertices in one stable component of the cube.

| | cube | Hoffman (8,8) | cube | Hoffman (7,9) |
|---|---|---|---|---|
| 8 | 4 | 0 | 0 | 0 |
| 10 | 0 | 24 | 64 | 56 |
| 12 | 384 | 320 | 624 | 676 |
| 14 | 800 | 904 | 1920 | 1776 |
| 16 | 2178 | 2058 | 3680 | 3900 |
| 18 | 1632 | 1736 | 3136 | 2936 |
| 20 | 1056 | 992 | 1392 | 1500 |
| 22 | 288 | 312 | 512 | 480 |
| 24 | 60 | 56 | 96 | 100 |
| 26 | 32 | 32 | 0 | 0 |
| 28 | 0 | 0 | 16 | 16 |
| 32 | 1 | 1 | 0 | 0 |

Table 1: Statistics for the 4-cube and Hoffman graph

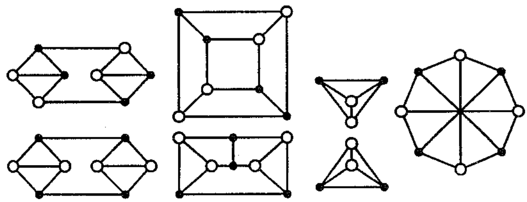

Figure 7: Graphs with 8 vertices, degree 3 and parts suitable for switching

## 6 Do cospectral regular switch-equivalent graphs have the same cuts?

If $G$ and $H$ are switch-equivalent, then their complements are also switch-equivalent. Hence we may assume $d \leq n/2 - 1$, without loss of generality.

Let $G$ be a graph of order $n$, regular of degree $d$. If $U$ is a subset of $V$ containing $u$ vertices, such that switching with respect to $U$ again gives a $d$-regular graph, each vertex of $U$ must be connected to $\frac{n-u}{2}$ vertices in $V \setminus U$, and each vertex of $V - U$ must be connected to $\frac{u}{2}$ vertices in $U$. So it is necessary that $u$ and $n - u$ are even, and $u \leq 2d$ and $n - u \leq 2d$ (hence $d \geq \frac{n}{4}$). We may assume $u \leq n/2$ since we may exchange the roles of $U$ and $V \setminus U$.

If $u = 2$, and thus $d = n/2 - 1$, the suitable parts $U$ are pair of vertices at distance $\geq 3$. Switching then gives isomorphic graphs.

If $u = 4$ and $n = 8$, the case of degree 2 only contains $2C_4$ and $C_8$, and switching gives isomorphic graphs; the cases of degree 3 are displayed in Figure 7, and they also are unchanged up to isomorphism by switching.

If $u = 4$ and $n = 10$, the cases of degree 3 are displayed in Figure 8. They also are unchanged up to isomorphism by switching. There are also fifty odd cases of

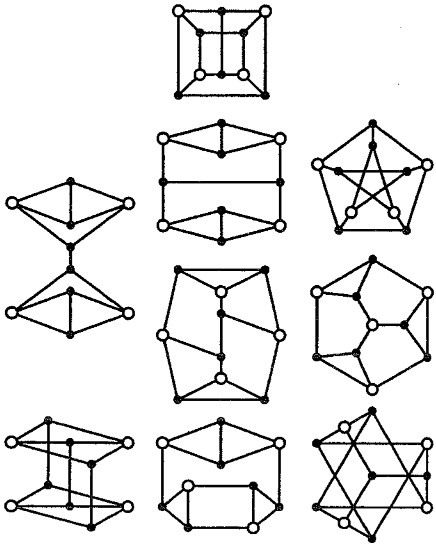

Figure 8: Graphs with 10 vertices, degree 3 and parts suitable for switching

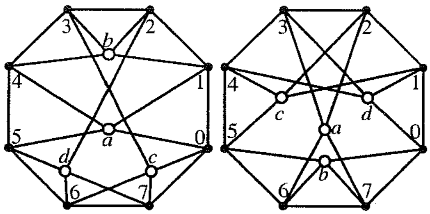


Figure 9: Two switch-equivalent tripartite graphs with 12 vertices of degree 4

degree 4. It seems that they also do not change up to isomorphism when switched with no modification of the degree.

## 6.1 No: a small ad hoc example

For $u = 4$, $n = 12$, $d = 4$, we find a pair of switch-equivalent (and thus cospectral) graphs with characteristic polynomial $(T-1)(T-4)(T^2-T-4)(T^2-1-T)^2(T+2)^4$; thus the lowest eigenvalue is $-2$. The upper bound for the max-cut is thus 18, and can be obtained only with two parts both having 6 vertices.

The max-cut of $G$ (left) is 18; the max-cut of $G_1$ (right) is only 16.

Here again, we used Maple to obtain statistics:

Of course the complements of $G$ and $G_1$ have $\mathbf{m} = 30$ and bisections 18 and 20 respectively.

| | $G$ | $G_1$ |
|---|---|---|
| 6 | 1 | 1 |
| 8 | 13 | 15 |
| 10 | 70 | 60 |
| 12 | 120 | 140 |
| 14 | 167 | 147 |
| 16 | 89 | 99 |
| 18 | 2 | 0 |

Table 2: Cuts for $G$ and $G_1$

| | $T(8)$ | $T'(8)$ | $T''(8)$ | $T'''(8)$ |
|---|---|---|---|---|
| 64 | 840 | 840 | 840 | 840 |
| 68 | 30240 | 27744 | 26880 | 26760 |
| 70 | 70560 | 77856 | 80160 | 80880 |
| 72 | 208320 | 193536 | 192480 | 191040 |
| 74 | 357840 | 408240 | 409920 | 410010 |
| 76 | 910560 | 805440 | 796320 | 800460 |
| 78 | 1139040 | 1272960 | 1280400 | 1274280 |
| 80 | 2328900 | 2096484 | 2101380 | 2101680 |
| 82 | 2772000 | 3230304 | 3228960 | 3237330 |
| 84 | 4844160 | 4253952 | 4243200 | 4234560 |
| 86 | 4841760 | 5433312 | 5437440 | 5437200 |
| 88 | 7111440 | 6392592 | 6401040 | 6407880 |
| 90 | 5404560 | 6278640 | 6271680 | 6266910 |
| 92 | 6108480 | 5379360 | 5379360 | 5379180 |
| 94 | 2963520 | 3330528 | 3332400 | 3334200 |
| 96 | 1024380 | 922908 | 922140 | 921240 |
| 98 | 0 | 11904 | 12000 | 12150 |

Table 3: Statistics for $T(8)$ and Chang graphs

## 6.2 No: a larger example

The graph $T(8)$, in other words the line-graph of the complete graph $K_8$, has order 28, is regular of degree 12, its adjacency spectrum is $12, 4^7, (-2)^{20}$; hence its Laplacian spectrum is $0, 8^7, 14^{20}$ and its Seidel spectrum is $3^{21}, (-9)^7$. It shares these spectra with the three Chang graphs (see [1, p. 105]) that are switch-equivalent to it and are also strongly regular with the same parameters. But the statistics of cuts (with 2 parts of size 14) differ, as shown in Table 3.

## References

[1] A.E. Brouwer, A.M. Cohen and A. Neumaier, *Distance-regular graphs*, Springer-Verlag, Berlin Heidelberg 1989.

[2] J.A. Bondy and U.S.R. Murty, *Graph theory with applications*, North Holland, New York 3rd edition 1980.

[3] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of graphs*, Academic Press, New York 1979.

[4] M. Capobianco and J.C. Moluzzo, *Examples and counterexamples in graph theory*, North Holland, Amsterdam 1978.

[5] C. Delorme. Eigenvalues and weights of induced subgraphs, *Publ. Mat. Inst. (Beograd)* 65(79) (1999), 20–30.

[6] Maple V, Symbolic Computation System, Waterloo Maple Inc.

[7] J.J. Seidel, *A survey of two-graphs* pp. 481–511, in *Teorie combinatorie*, Acad. Nazionale dei Lincei Rome, 1976.

(Received 1/7/2001)