Homogeneous cartesian products

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Abstract

A graph G is 1-homogeneous if certain isomorphisms between similarly embedded induced subgraphs of G extend to automorphisms of G. We show that the only connected composite 1-homogeneous graphs are the cube, and $K_n \times K_2$ and $K_n \times K_n$ with $n \geq 2$.

1 Introduction

The homogeneity of a graph G may be measured in terms of which isomorphisms between its induced subgraphs extend to automorphisms of G. In the extreme we may insist that all such isomorphisms so extend. Sheehan [10] and Gardiner [3] studied these graphs to find, not surprisingly, that there are only a few. Gardiner called them *ultrahomogeneous* and proved the following characterisation.

Theorem 1 *The ultrahomogeneous graphs are:*

1. *the disjoint union, tK_r, of t copies of the complete graph on r vertices,*
2. *the complete multipartite graph $K_{t;r}$, the complement of tK_r,*
3. *the cartesian product $K_3 \times K_3 = L(K_{2;3})$,*
4. *the pentagon C_5.*

Although C_3 ($= K_3$), C_4 ($= K_{2;2}$), and C_5 are ultrahomogeneous, the cycles C_n for $n \geq 6$ are large enough to allow isomorphic induced subgraphs to be embedded in different ways. For example, take two vertices at distance 2 in C_n as one subgraph and two at distance 3 for the other. Both subgraphs are isomorphic to $2K_1$ but no isomorphism between them extends to an automorphism of C_n because automorphisms preserve distance.

Size permits a similar situation in $K_n \times K_n$ for $n \geq 4$. To see how exactly we recall the definition of the cartesian product and the nature of its automorphism group. The *cartesian product*, $G \times H$, of graphs G and H is the graph with vertex set $VG \times VH$ and edges $\langle (g_1, h_1), (g_2, h_2) \rangle$ only where $g_1 = g_2$ and h_1 is adjacent to h_2 or $h_1 = h_2$ and g_1 is adjacent to g_2. A graph is *composite* if it is the product of at least two non-trivial graphs. A graph is *prime* if non-trivial and not composite. We note that K_n is prime for $n \geq 2$ because it is connected and has no chordless square. Two graphs are *relatively prime* if they have no non-trivial factor in common. $G \times H$ has obvious automorphisms that are essentially permutations of copies of G induced by automorphisms of H or permutations of copies of H induced by automorphisms of G. The cartesian product G^n has additional automorphisms induced by permuting the coordinates of every vertex with the same permutation. The following theorems tell us that these automorphisms generate all others.

Theorem 2 ([9]) *If G and H are relatively prime connected graphs, then every automorphism of $G \times H$ has the form $\gamma = (\alpha, \beta)$ where α and β are automorphisms of G and H respectively and $\gamma(g, h) = (\alpha g, \beta h)$.*

Theorem 3 ([5]) *For a connected prime graph G, the automorphism group of G^m is the set of all permutations $\gamma = (\alpha; \beta_1, \ldots, \beta_m)$ where $\alpha \in S_m$ and $\beta_1, \ldots, \beta_m$ are automorphisms of G, and γ is defined by $\gamma(g_1, \ldots, g_m) = (\beta_1 g_{\alpha 1}, \ldots, \beta_m g_{\alpha m})$.*

An edge in $G \times H$ of the form $\langle (g_i, h_k), (g_j, h_k) \rangle$ will be called *horizontal* and an edge of the form $\langle (g_k, h_i), (g_k, h_j) \rangle$ will be called *vertical*. Two edges are *parallel* if they are both horizontal or both vertical. From Theorems 2 and 3, any automorphism will map parallel edges to parallel edges. Now consider two induced subgraphs of $K_n \times K_n$ that are isomorphic to $2K_2$. We suppose both edges in one subgraph are horizontal (this is possible for $n \geq 4$) and in the other subgraph one edge is horizontal and the other is vertical. No isomorphism between them will extend to an automorphism of $K_n \times K_n$.

Following Myers [7, 8] we relax the homogeneity condition by limiting the isomorphic induced subgraphs to only those that, in some sense, are similarly embedded in the graph. He made the following inductive definition. Let X and Y be induced subgraphs of a graph G. Any isomorphism f from X to Y is called a *0-isomorphism.* For positive integer k, f is also called a *k-isomorphism* if for each vertex x in G there is a vertex y in G such that the mapping $f \cup (x, y)$ is a $(k-1)$-isomorphism from the induced subgraph $\langle VX \cup \{x\} \rangle$ to $\langle VY \cup \{y\} \rangle$ and f^{-1} satisfies the analagous condition from Y to X. Graph G is *k-homogeneous* if, for every pair of its induced subgraphs X and Y, every k-isomorphism from X to Y can be extended to an automorphism of G. The notion of k-homogeneity has been studied in other areas such as logic [2] and geometry [1]. The 0-homogeneous graphs are precisely the ultrahomogeneous graphs of Theorem 1. Clearly, if G is k-homogeneous, then it is l-homogeneous for $l \geq k$ and every component is k-homogeneous. Also the complement of G is k-homogeneous if and only if G is k-homogeneous. Exploiting the near uniqueness of G for a given line graph $L(G)$, Myers [7] showed that $L(K_n)$ is 1-homogeneous for all n and $L(K_{t;r})$

is 4-homogeneous for all t and r. In what follows we will confine our attention to 1-homogeneous graphs.

The study of 1-homogeneous graphs can be reduced to certain simpler cases. For a graph G let $F(G)$ denote the set of all vertices in G of full valency, that is, adjacent to all other vertices in G. From the definition of 1-homogeneous graphs we have the following immediate results.

Lemma 4 ([4], [8]) *For any graph G:*

1. *G is 1-homogeneous if and only if its complement is 1-homogeneous.*
2. *If G is 1-homogeneous and connected, then $G - F(G)$ is 1-homogeneous and transitive.*
3. *If G is 1-homogeneous and connected, then the diameter of G is at most* 3.

Thus the investigation of 1-homogeneous graphs reduces to the study of connected, transitive graphs of diameter at most 3. By routine argument the only connected 1-homogeneous graphs regular of degree two are the cycles C_n for $n \leq 7$. Myers [8] classifed the trivalent variety.

Theorem 5 *The connected cubic 1-homogeneous graphs are:*

1. *K_4,*
2. *$K_3 \times K_2$, the complement of C_6,*
3. *$K_{2;3}$, the complement of $2K_3$,*
4. *the Petersen graph, the complement of $L(K_5)$,*
5. *the cube, K_2^3,*
6. *the Heawood graph.*

We note that only the cube and the Heawood graph have diameter 3 and all except $K_3 \times K_2$ are distance transitive. It was shown in [4] that all connected 1-homogeneous graphs are distance transitive or almost so. This result, Lemma 4, and the comprehensive catalogue of transitive graphs in [6] can be used to find all connected transitive 1-homogeneous graphs on fewer than 20 vertices. Inspection of these graphs suggests various infinite families of 1-homogeneous graphs. In the following section we look at composite graphs.

2 Composite graphs

Lemma 6 *If a connected composite graph is 1-homogeneous, then it is one of the following:*

1. *the cube,* K_2^3*,*
2. $K_n \times K_2$ *with* $n \geq 2$*,*
3. $K_n \times K_n$ *with* $n > 2$*.*

Proof. Let P be a connected composite 1-homogeneous graph. From Lemma 4 the diameter of P is at most 3 so it has at least one factor of diameter 1. Thus $P = K_r \times H$ where H has diameter 1 or 2.

Case 1: $P = K_r \times K_s$ with $r \geq s \geq 2$.

If $s > 2$ then any edge in a copy of K_r is 1-isomorphic to any edge in a copy of K_s. Hence there is an automorphism of P that maps one edge to the other. This contradicts Theorem 2 unless $r = s$.

Case 2: $P = K_r \times H$ where $r \geq 2$ and H has diameter 2.

From Lemma 4 P is transitive and hence H is regular of degree at least 2. We label the vertices of K_r and H with the integers $0, 1, 2, \ldots$ and write ij for the vertex (i, j) in P. Let vertices 0 and 2 in H be at distance 2 and mutually adjacent to vertex 1. The induced subgraphs $\langle 00, 02\rangle$ and $\langle 00, 11\rangle$ are 1-isomorphic in P and hence there is an automorphism of P that maps one to the other. This contradicts Theorem 2 if K_r and H are relatively prime. So $P = K_r \times K_r \times K_s$ where $s \geq 2$. If $r \neq s$ we write $P = K_s \times H$ where $H = K_r \times K_r$ and repeat our argument to contradict Theorem 2.

If $r = s$, then we have $P = K_r^3$. Suppose $r \geq 3$. We label the vertices in K_r with the integers $0, 1, 2, \ldots$, and write ijk for the vertex (i, j, k) in K_r^3. Consider $G_1 = \langle 000, 222, 102, 210\rangle$ and $G_2 = \langle 000, 222, 102, 021\rangle$. Let f be the isomorphism from G_1 to G_2 that fixes 000, 222, 102, and maps 210 to 021. It is easy to check that f is a 1-isomorphism. Therefore f extends to an automorphism α of P. Since 010 is at distance 1 from both 000 and 210, $\alpha(010) = 001$ or 020. But 010 and 001 are respectively at distance 3 and 2 from 102 while 010 and 020 are respectively at distance 3 and 2 from 222 giving us a contradiction. □

By Theorem 5, K_2^3 is 1-homogeneous. We show below that $K_n \times K_2$ and $K_n \times K_n$ are 1-homogeneous for all n. First a general lemma that allows us to move induced subgraphs around.

Lemma 7 *Let* G_1 *and* G_2 *be induced subgraphs in a graph* G *and let* α *and* β *be automorphisms of* G*. An isomorphism* $\sigma : G_1 \to G_2$ *is a 1-isomorphism in* G *if and only if* $\alpha\sigma\beta^{-1} : \beta G_1 \to \alpha G_2$ *is a 1-isomorphism in* G*. Moreover,* σ *extends to an automorphism of* G *if and only if* $\alpha\sigma\beta^{-1}$ *does.*

Proof. Let g_1 be a vertex in G and suppose σ is a 1-isomorphism. There is a vertex g_2 in G such that $\sigma \cup (\beta^{-1}g_1, \alpha^{-1}g_2) : \langle G_1 \cup \{\beta^{-1}g_1\}\rangle \to \langle G_2 \cup \{\alpha^{-1}g_2\}\rangle$

is an isomorphism. Hence $\alpha\sigma\beta^{-1} \cup (g_1, g_2) : \langle \beta G_1 \cup \{g_1\} \rangle \to \langle \alpha G_2 \cup \{g_2\} \rangle$ is an isomorphism. Similarly, if g_2 is a vertex in G, then there exists a vertex g_1 in G such that $\beta\sigma^{-1}\alpha^{-1} \cup (g_2, g_1) : \langle \alpha G_2 \cup \{g_2\} \rangle \to \langle \beta G_2 \cup \{g_1\} \rangle$ is an isomorphism. Thus $\alpha\sigma\beta^{-1}$ is a 1-isomorphism in G. Also, if σ extends to an automorphism σ^* of G, then $\alpha\sigma\beta^{-1}$ extends to the automorphism $\alpha\sigma^*\beta^{-1}$ of G. The converses are immediate. □

The next two lemmas show that parallel edges in an induced subgraph of $K_n \times K_2$ and $K_n \times K_n$ stay parallel under any 1-isomorphism. In both products let H and V denote respectively the set of all their horizontal and vertical edges.

Lemma 8 *Let G_1 and G_2 be induced subgraphs of $G = K_n \times K_2$ with $n \geq 3$. If $\sigma : G_1 \to G_2$ is a 1-isomorphism in G, then $\sigma(EG_1 \cap H) = EG_2 \cap H$ and $\sigma(EG_1 \cap V) = EG_2 \cap V$.*

Proof. Suppose e is a horizontal edge in G_1 and $\sigma(e)$ is a vertical edge in G_2. There is a vertex in G adjacent to both vertices of e but no corresponding vertex adjacent to both vertices of $\sigma(e)$. Hence σ can not be a 1-isomorphism. Similarly, if e is a vertical edge in G_1 and $\sigma(e)$ is a horizontal edge in G_2, then σ^{-1} is not a 1-isomorphism. □

Lemma 9 *Let G_1 and G_2 be induced subgraphs of $G = K_n \times K_n$ with $n \geq 2$. If $\sigma : G_1 \to G_2$ is a 1-isomorphism in G, then $\{\sigma(EG_1 \cap H), \sigma(EG_1 \cap V)\} = \{EG_2 \cap H, EG_2 \cap V\}$.*

Proof. Suppose e and d are edges in G_1 with $e, d, \sigma(e)$ horizontal and $\sigma(d)$ vertical. Because G_1 and G_2 are isomorphic induced subgraphs of $K_n \times K_n$, simple inspection of the few possible cases shows that $\langle e, d \rangle$ and $\langle \sigma(e), \sigma(d) \rangle$ are both isomorphic to $2K_2$. Hence there is a vertex in G adjacent to both vertices in e but only one vertex in d, while each vertex in G that is adjacent to both vertices in $\sigma(e)$ is adjacent to both or none of the vertices in $\sigma(d)$. Thus σ is not a 1-isomorphism of G. □

For simplicity in what follows we denote the vertices (i, j) of $K_n \times K_2$ and $K_n \times K_n$ by ij.

Theorem 10 *$K_n \times K_2$ is 1-homogeneous for $n \geq 1$.*

Proof. From Theorem 1 $K_n \times K_2$ is 0-homogeneous for $1 \leq n \leq 2$ so we may assume $n \geq 3$. Let G_1 and G_2 be isomorphic induced subgraphs of $G = K_n \times K_2$ and suppose $\sigma : G_1 \to G_2$ is a 1-isomorphism in G. If G_1 has no vertical edge then, by Lemma 8, neither does G_2 and both are therefore isomorphic to a complete graph or the union of two complete graphs. From Theorem 2 and Lemma 7 we may assume that $G_1 = G_2 = \langle A \cup B \rangle$ where $A = \{11, 21, \ldots, r1\}$ and $B = \emptyset$ or $\{(r+1)2, (r+2)2, \ldots, s2\}$ where $1 \leq r \leq n$ and $r < s \leq n$. Thus σ fixes all vertices in G_1 and extends to the identity automorphism of G.

If G_1 does have a vertical edge then, from Theorem 2 and Lemma 7, we may assume $G_1 = \langle \{11, 21, \ldots, r1\} \cup \{12, 22, \ldots, r2\} \cup A \cup B \rangle$ where $A = \emptyset$ and $B = \emptyset$ or $A = \{(r+1)1, (r+2)1, \ldots, s1\}$ and $B = \emptyset$ or $A = \{(r+1)1, (r+2)1, \ldots, s1\}$ and $B = \{(s+1)2, (s+2)2, \ldots, t2\}$ where $1 \leq r < s < t \leq n$. From Theorem 2, Lemma 7, and Lemma 8 we may assume $G_2 = G_1$. Again σ extends to the identity automorphism of G. □

Theorem 11 *$K_n \times K_n$ is 1-homogeneous for $n \geq 1$.*

Proof. From Theorem 1, $K_n \times K_n$ is 0-homogeneous for $1 \leq n \leq 3$. We use induction on n. Let G_1 and G_2 be isomorphic induced subgraphs of $G = K_n \times K_n$ and suppose $\sigma : G_1 \to G_2$ is a 1-isomorphism in G. Let $X = \{11, 21, \ldots, n1\} \cup \{11, 12, \ldots, 1n\}$, $G' = G - X$, $G'_1 = G_1 - X$, and σ' denote σ restricted to G'_1.

Case 1: G_1 contains an isolated vertex.

From Lemma 7 and Theorem 3 we may assume the isolated vertex is 11 and $\sigma 11 = 11$. We claim σ' is a 1-isomorphism in G'. If g_1 is a vertex in G', then there is a vertex g_2 in G such that $\sigma \cup (g_1, g_2)$ is an isomorphism from $\langle VG_1 \cup \{g_1\}\rangle$ to $\langle VG_2 \cup \{g_2\}\rangle$. Because g_1 is not adjacent to 11, neither is g_2 and therefore g_2 is in G'. By induction σ' extends to an automorphism τ' of G'. From Theorem 3, τ' extends to an automorphism τ of G that fixes 11. Thus σ extends to τ.

Case 2: G_1 contains an edge.

From Lemma 7 and Theorem 3 we may assume that the edge is $\langle 11, i1\rangle$, $\sigma 11 = 11$, and $\sigma i1 = j1$ where $1 < i, j \leq n$. From Lemma 9 we may also assume that $VG_1 \cap \{11, 21, \ldots, n1\} = \{11, 21, \ldots, r1\} = VG_2 \cap \{11, 21, \ldots, n1\}$ and $VG_1 \cap \{11, 12, \ldots, 1n\} = \{11, 12, \ldots, 1s\} = VG_2 \cap \{11, 12, \ldots, 1n\}$ for some r and s with $2 \leq r \leq n$ and $1 \leq s \leq n$.

Case 2.1: G'_1 contains no edge.

Let $t = \min(r, s)$. Again from Lemma 7 and Theorem 3 we may assume that G_1 contains at most t vertices $11, 22, \ldots$; at most $r - t$ vertices $(t+1)(s+1), (t+2)(s+2), \ldots$; and at most $s - t$ vertices $(r+1)(t+1), (t+2)(s+2), \ldots$; and every vertex in G_1 is fixed by σ. Thus σ extends to the identity automorphism of G.

Case 2.2: G'_1 contains an edge.

As in Case 1, σ' is a 1-isomorphism in G' and by induction extends to an automorphism τ' of G'. From Lemma 9, under σ the horizontal edges of G_1 remain horizontal and the vertical edges remain vertical. In particular the edge we have in G'_1 remains either horizontal or vertical under σ, hence under σ', and therefore under τ'. For $2 \leq k \leq n$ let $A_k = \{k2, \ldots, kn\}$ and $B_k = \{2k, \ldots, nk\}$. From Theorem 3, $\tau' A_k = A_l$ for some l and $\tau' B_k = B_m$ for some m. To ease notation we assume that σ fixes each vertex in $VG_1 \cap X$ as permitted by Lemma 7. If for some k where $2 \leq k \leq r$, A_k contains a vertex of G_1 or G_2, then τ' fixes A_k. If A_k does not contain a vertex of G_1 or G_2, then we may change $\tau' = (\tau'_1, \tau'_2)$ by deleting the disjoint cycle in τ'_1 that contains k. The resulting automorphism of G' still extends σ' but fixes A_k. So we may assume that τ' fixes A_k for $2 \leq k \leq r$ and B_k for $2 \leq k \leq s$. Thus τ' extends to an automorphism τ of G that fixes each vertex in $VG_1 \cap X$ and so σ extends to τ. □

From Lemma 6 and Theorems 5, 10, and 11 we have our characterisation of connected composite 1-homogeneous graphs.

Theorem 12 *The connected composite 1-homogeneous graphs are:*

1. *the cube,* K_2^3,
2. $K_n \times K_2$ *with* $n \geq 2$,
3. $K_n \times K_n$ *with* $n > 2$.

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