The spectrum of triangle-free regular graphs containing a cut vertex

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Dedicated to the memory of my mentor, my colleague, and my very dear friend Norman J. Pullman 1931–1999.

Abstract

We determine, for all $n > 0$, the set $C(n) = \{k\colon$ there exists a triangle-free k-regular graph on n vertices containing a cut vertex$\}$.

1 Introduction

In a recent paper [3] the authors determined all values $c(n) = \max\,\{k\colon$ there exists a triangle-free k-regular graph on n vertices containing a cut vertex$\}$. We will make use of the extremal graphs constructed in that paper to determine the complete spectrum $C(n) = \{k\colon$ there exists a triangle-free k-regular graph on n vertices containing a cut vertex$\}$.

We refer the reader to [1] for standard definitions and notations. The degree of a vertex x in the graph G, denoted $deg_G(x)$, is the number of vertices in G to which x is adjacent. A graph G is called k-regular if $deg_G(x) = k$ for all vertices x in G. A graph G will be called *almost k-regular* if one vertex (called the *special vertex*) in G has degree $k-2$ and every other vertex in G has degree k. A *k-factor* in a graph G is a subgraph of G each of whose vertices has degree k, while a *near-k-factor* is a subgraph of G in which all but one vertex has degree k with the remaining vertex having degree 0 (i.e. is isolated). Note that an almost 2-regular graph is equivalent to a near-2-factor.

The following theorem of Petersen is well-known.

Theorem 1.1

Every $2t$-regular graph has a 2-factor.

A *k-factorization* of a graph G is an edge-decomposition of G into k-factors. Thus the following is an immediate consequence of Theorem 1.1.

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Corollary 1.2 (Petersen)

Every $2t$-regular graph has a 2-factorization.

The classification of which k-regular graphs (on an even number of vertices) have one-factorizations is a very difficult open problem. The well-known One-Factorization Conjecture, for example, asserts that the largest k for which there exists a k-regular graph of order $2m$ without a one-factorization is $F(2m) = 2\lfloor \frac{m-1}{2} \rfloor$; we refer the reader to [2] for further discussion. In fact, one of the motivations for [3] was to determine a lower bound on the quantity $f(2m)$, which denotes the largest k for which a triangle-free k-regular graph of order $2m$ without a one-factorization exists. (Note that regular graph with a cut-vertex cannot be one-factorizable, whence $f(2m) \geq c(2m)$.) As a consequence of Theorem 1.5 (see ahead) it was determined that $f(2m) \geq \alpha(2m) = \lfloor \frac{4}{9}(m-1) \rfloor + 1$ if $m = 7, 12, 16$ or 21, and $f(2m) \geq \lfloor \frac{4}{9}(m-1) \rfloor$ for all other $m \geq 8$. Of course $f(2m)$ is bounded above by the largest k for which there exists a k-regular graph on $2m$ vertices with odd girth $\gamma \geq 5$; i.e. $f(2m) \leq 5$ if $m = 7, f(2m) \leq 9$ if $m = 12$, and $f(2m) \leq 2\lfloor \frac{2m}{5} \rfloor$ for all other $m \geq 8$ (see Shi [4]).

Let $t'(n)$ denote the largest k for which there exists a triangle-free almost k-regular graph on n vertices. Let $S = \{8, 11, 14, 15, 18, 21, 24\}$ and define

$$a(n) = \begin{cases} 4 & \text{if } n = 9, \\ \lfloor \frac{2n-4}{5} \rfloor + 1 & \text{if } n \in S, \\ \lfloor \frac{2n-4}{5} \rfloor - 1 & \text{if } n \equiv 1 \text{ or } 5 \ mod\ 10 \text{ and } n \notin S, \\ \lfloor \frac{2n-4}{5} \rfloor & \text{for all other } n \geq 10. \end{cases}$$

The following result was determined in [3]:

Theorem 1.3

$t'(n) = a(n)$ for all $n \geq 8$. Moreover, $t'(n)$ does not exist if $n = 1, 2, 3, 4$, and $t'(5) = t'(6) = t'(7) = 2$.

We will begin by determining, in Section 2, the spectrum $T'(n) = \{k\colon$ there exists a triangle-free almost k-regular graph on n vertices$\}$:

Theorem 1.4

(i) For odd $n \geq 5, T'(n) = \{k : k \text{ even and } 2 \leq k \leq t'(n)\}$.

(ii) For even $n \geq 6, T'(n) = \{k : 2 \leq k \leq t'(n)\}$.

In Section 3, we then use the almost k-regular graphs constructed in Section 2 to determine the spectrum $C(n)$, using a construction from [3]. Let

$$\alpha(n) = \begin{cases} \lfloor \frac{2n-4}{9} \rfloor + 1 & \text{if } n \in \{14, 24, 32, 42\}, \\ \lfloor \frac{2n-4}{9} \rfloor_e + 2 & \text{if } n \in \{17, 19, 27, 37\}, \\ \lfloor \frac{2n-4}{9} \rfloor & \text{for all other even } n \geq 16, \\ \lfloor \frac{2n-4}{9} \rfloor_e & \text{for all other odd } n \geq 21, \end{cases}$$

where $\lfloor x \rfloor_e$ denotes the largest even integer not exceeding x. The following is the main result from [3].

Theorem 1.5

$c(n) = \alpha(n)$ for all $n \geq 16$. Moreover, $c(14) = \alpha(14) = 3$ while $c(n)$ does not exist if $n \leq 13$ or if $n = 15$.

Corollary 1.6

If G is a connected k-regular triangle-free graph on n vertices with $k > 3$ when $n = 14$, or $k > \alpha(n)$ when $n \geq 16$, then G is 2-connected.

Now Theorem 1.5 implies that $C(n) = \emptyset$ if $n \leq 13$ or if $n = 15$. In Section 3 we will obtain the following main result.

Theorem 1.7

(i) For odd $n \geq 17, C(n) = \{k : 4 \leq k \leq c(n) \text{ and } k \text{ even }\}$.

(ii) For even $n \geq 14, C(n) = \{k : 3 \leq k \leq c(n)\}$.

In all of the cases in Theorem 1.4 we will use as a starting point the (extremal) almost $t'(n)$-regular graphs from [3], removing one-factors and/or two-factors as needed. Throughout we rely heavily on Petersen's Theorem.

2 The spectrum of triangle-free almost k-regular graphs.

In this section we determine $T'(n)$ for all $n \geq 5$. We first obtain the following preliminary result.

Theorem 2.1

Let k be an even integer. Then every almost k-regular graph G has a near 2-factor whose isolated vertex is the special vertex in G.

Proof

Let G be an almost k-regular graph ($k \geq 2$) with special vertex x. Let H be any k-regular graph with $V(G) \cap V(H) = \emptyset$. Let $\{a, b\}$ be an edge in H. Form a graph J whose vertex set is $V(J) = V(G) \cup V(H)$ and whose edge set is $E(J) = ((E(G) \cup E(H)) \setminus \{\{a, b\}\}) \cup \{\{x, a\}, \{x, b\}\}$. Then J is a k-regular graph and since k is even, J has a 2-factorization (Corollary 1.2). Let F be a 2-factor in J containing the edge $\{x, a\}$; then it is clear that F also contains the edge $\{x, b\}$, since x is a cut-vertex in J. Then the restriction of F to the vertices $V(G) - \{x\}$ is the near 2-factor in G whose isolated vertex is x, as desired.

□

Theorem 1.4(i) now follows immediately:

Lemma 2.2

For all odd $n \geq 5, T'(n) = \{k : k \text{ even and } 2 \leq k \leq t'(n)\}$.

Proof

Let $n \geq 5$ be odd and let G be an almost $t'(n)$-regular triangle-free graph on n vertices (e.g. from [3]). Let k be even, $2 \leq k \leq t'(n)$. By Theorem 2.1, G has a near 2-factor F whose isolated vertex is the special vertex in G. Removing the edges of F from G yields a $(t'(n) - 2)$-regular graph H on n vertices. Since $t'(n)$ is even, $t'(n) - 2$ is even and so H has a 2-factorization (Corollary 1.2). Therefore, we can remove $(t'(n) - k)/2$ edge-disjoint 2-factors from H to obtain a $(k - 2)$-regular graph H', and then replace the edges of F to obtain a graph G' on n vertices (that is, $V(G') = V(H')$ and $E(G') = E(H') \cup F$) which is almost k-regular (its special vertex is the same as that of G) and triangle-free (G' is a subgraph of G). Hence $k \in T'(n)$, and the result follows.

□

Now when n is even, an almost k-regular graph can have k odd (or even). We summarize our strategy in this case in the following observation.

Lemma 2.3

Let n be an even integer, $n \geq 6$, and let G be an almost $t'(n)$-regular triangle-free graph on n vertices (e.g. from [3]).

(i) If $t'(n)$ is odd and G contains a one-factor, then $T'(n) = \{k : 2 \leq k \leq t'(n)\}$.

(ii) If $t'(n)$ is even and G contains two edge-disjoint one-factors, then $T'(n) = \{k : 2 \leq k \leq t'(n)\}$.

Proof

(i) Let F_1 be a one-factor in G. By Theorem 2.1, the graph obtained by removing the edges of F_1 from G has a near 2-factor F_2 whose isolated vertex is the special vertex in G. Removing also from G the edges of F_2 yields a $(t'(n) - 3)$-regular graph H on n vertices. Since $t'(n)$ is odd, $t'(n) - 3$ is even and so by Corollary 1.2 H has a 2-factorization. Now remove $(t'(n) - k)/2$ (if k is odd) or $(t'(n) - k - 1)/2$ (if k is even) edge-disjoint 2-factors from H. In the former case we obtain a $(k - 3)$-regular graph H', to which we add the edges of $F_1 \cup F_2$ to yield an almost k-regular triangle-free graph G' on n vertices, while in the latter case we obtain a $(k - 2)$-regular graph H'', to which we add the edges of F_2 to yield an almost k-regular triangle-free graph G'' on n vertices. In either case we have $k \in T'(n)$, and the result follows.

(ii) Let F_1 and F_1' be edge-disjoint one-factors in G. By Theorem 2.1, the graph obtained by removing the edges of $F_1 \cup F_1'$ from G has a near 2-factor F_2 whose isolated vertex is the special vertex in G. Removing also from G the edges of F_2

yields a $(t'(n)-4)$-regular graph H on n vertices which, since $t'(n)$ is even, has a 2-factorization (Corollary 1.2). We may assume that $k < t'(n)$. Now remove $(t'(n)-k-2)/2$ (if k is even) or $(t'(n)-k-1)/2$ (if k is odd) edge-disjoint 2-factors from H. In the former case we obtain a $(k-2)$-regular graph H', to which we add the edges of F_2 to yield an almost k-regular triangle-free graph G' on n vertices, while in the latter case we obtain a $(k-3)$-regular graph H'', to which we add the edges of $F_1 \cup F_2$ to yield an almost k-regular triangle-free graph G'' on n vertices. Again in either case, we have $k \in T'(n)$, and the result follows.

□

Thus we will now proceed to show that for each even integer $n > 6$ there is an almost $t'(n)$-regular triangle-free graph on n vertices which has a one-factor (if $t'(n)$ is odd) or two edge-disjoint one-factors (if $t'(n)$ is even). In fact we will show that in each case the almost $t'(n)$-regular graphs from [3] have this property. (Note that the case $n = 6$ is trivial, as clearly $T'(6) = \{2\}$.) To do this we must first reconstruct from [3] all of the almost $t'(n)$-regular triangle-free graphs on an even number $n > 6$ of vertices. For the sake of brevity we will henceforth adopt the notation (k, n)-graph (from [3]) to denote an almost k-regular triangle-free graph on n vertices.

There are three categories of these graphs:

(C1) $n \equiv 0$ or $6 \; mod \; 10, n \geq 10$.

Let H be the $2m$-regular graph of order n shown in Figure 1 of the Appendix. Select a vertex $b_i \in B_i$ for $i = 0, 1, 2$. Let H' be the subgraph of H obtained by deleting the vertices b_0 and b_2, and let M be a one-factor in H'. Then the graph G obtained from H by deleting the set of edges $M \cup \{\{b_0, b_1\}, \{b_1, b_2\}\}$ is a $(t'(n), n)$-graph (with special vertex $x = b_1$). Note that in all of these cases $t'(n)$ is odd.

(C2) $n \in \{8, 14, 18, 24\}$.

The four $(t'(n), n)$-graphs are given in Figure 2 of the Appendix. Note that in all of these cases $t'(n)$ is odd.

(C3) $n \equiv 2, 4$ or $8 \; mod \; 10, n \geq 12$ and $n \notin \{14, 18, 24\}$.

These graphs are given in Figure 3 of the Appendix. Note that in all of these cases $t'(n)$ is even.

Lemma 2.4

Each of the graphs in categories (C1) and (C2), as defined above, has a one-factor.

Proof

For a one-factor in each graph in category (C2), see Figure 4 of the Appendix.

For $n = 10, 16$, a one-factor in the (3,10) and (5,16)-graphs is given in Figure 5 of the Appendix. In each case the solid lines indicate the edges in the one-factor, while the dotted lines indicate the edges of $M \cup \{\{b_0, b_1\}, \{b_1, b_2\}\}$ (see the foregoing construction for the graphs in this category). For $n \equiv 0 \; mod \; 10, n \geq 20$, a one-factor in the $(t'(n), n)$-graph can be obtained by adjoining $\frac{1}{10}(n-10)$ vertex-disjoint copies of $\mathcal{A}$ to the one-factor in the (3,10)-graph, while for $n \equiv 6 \; mod \; 10, n \geq 26$, a one-factor in the $(t'(n), n)$-graph can be obtained by adjoining $\frac{1}{10}(n-16)$ vertex-disjoint copies of $\mathcal{A}$ to the one-factor in the (5,16)-graph. In the graph $\mathcal{A}$ (also in Figure 5), the solid lines indicate the edges in the one-factor, while the dotted lines indicate the edges of M.

□

Lemma 2.5

Each of the graphs in category (C3) has two edge-disjoint one-factors.

Proof

In each case we will construct a 2-factor which is composed of even length cycles.

For $n = 12$, a 2-factor in the (4,12)-graph is given in Figure 6 of the Appendix. Now for $n \equiv 2 \; mod \; 10, n \geq 22$, a 2-factor in the $(t'(n), n)$-graph can be obtained by adjoining $\frac{1}{10}(n-12)$ vertex-disjoint copies of $\mathcal{B}$ (also in Figure 6) to the 2-factor in the (4,12)-graph.

For $n = 28$, a 2-factor in the (10,28)-graph can be obtained by adjoining a copy of $\mathcal{B}$ to the 2-factor in the (non-extremal) (6,18)-graph given in Figure 6. Similarly, for $n = 34$ a 2-factor in the (12,34)-graph can be obtained by adjoining two vertex-disjoint copies of $\mathcal{B}$ to the 2-factor in the (non-extremal) (4,14)-graph given in Figure 6. (Note that these non-extremal (6,18)- and (4,14)-graphs are obtained from Figure 3 with $m = 3$ and 2, respectively.) Then for $n \equiv 4$ or $8 \; mod \; 10, n \geq 38$, a 2-factor in the $(t'(n), n)$-graph can be obtained by adjoining an appropriate number of vertex-disjoint copies of $\mathcal{B}$ to the 2-factor in the (12,34)-graph or the (10,28)-graph, respectively.

□

Theorem 1.4(ii) now follows from Lemmas 2.3, 2.4 and 2.5:

Lemma 2.6

For all even $n \geq 6, T'(n) = \{k : 2 \leq k \leq t'(n)\}$.

3 The spectrum $C(n)$

In this section we will determine $C(n)$ for all $n \geq 14, n \neq 15 (C(15) = \emptyset$, see Theorem 1.5). Following [3] we define

$$m = \begin{cases} \lceil \frac{5\alpha(n)}{2} \rceil_e & \text{if } n \in \{24, 32, 42\}, \\ \lceil \frac{5\alpha(n)}{2} \rceil_e + 2 & \text{for all other even } n \geq 16, \\ \lceil \frac{5\alpha(n)}{2} \rceil_0 & \text{if } n \in \{19, 27, 37\}, \text{ or 9 if } n = 17, \\ \lceil \frac{5\alpha(n)}{2} \rceil_0 + 2 & \text{for all other odd } n \geq 21, \end{cases}$$

where $\alpha(n)$ is as defined in Section 1 and $\lceil x \rceil_e$ (resp. $\lceil x \rceil_0$) denotes the smallest even (resp. odd) integer not less than x.

Lemma 3.1

For each odd integer $n \geq 17, C(n) = \{k : 4 \leq k \leq c(n) \text{ and } k \text{ even }\}$.

Proof

Let k be even, $4 \leq k \leq c(n)$. It is straightforward to verify that $n - m \geq 2\alpha(n) = 2c(n)$ (Theorem 1.5) $\geq 2k$. Since $n - m$ is even we can construct a k-regular bipartite graph G_1 of order $n - m$. On the other hand, it is again straightforward to verify that $t'(m) = \alpha(n) = c(n)$ for every odd $n \geq 17$. Hence since k is even and $4 \leq k \leq t'(m)$ we can construct an almost k-regular triangle-free graph G_2 of order m, with special vertex x, such that $V(G_1) \cap V(G_2) = \emptyset$, see Lemma 2.2. Now select two adjacent vertices a and b in G_1. We then obtain a k-regular graph on $V(G_1) \cup V(G_2)$ by deleting the edge $\{a, b\}$ and adding the new edges $\{x, a\}$ and $\{x, b\}$. This k-regular graph on n vertices is triangle-free and, since $k \geq 4$, has x as a cut-vertex. Hence $k \in C(n)$, and the result follows.

□

Lemma 3.2

For each even integer $n \geq 14, C(n) = \{k : 3 \leq k \leq c(n)\}$.

Proof

For $n = 14$ we have $c(n) = 3$ (Theorem 1.5) and so clearly $C(14) = \{3\}$. Suppose now that $n \geq 16$ and that $3 \leq k \leq c(n)$. As with the proof of Lemma 3.1 we have $n - m \geq 2\alpha(n) = 2c(n) \geq 2k$ and so, since $n - m$ is even, we can construct a k-regular bipartite graph G_1 of order $n - m$. Again as with the proof of Lemma 3.1 we have $t'(m) = \alpha(n) = c(n)$ for all even $n \geq 16$, except that $t'(18) = 7 = \alpha(30) + 1 = c(30) + 1$. In any case since $3 \leq k \leq c(n) \leq t'(m)$ we can construct an almost k-regular triangle-free graph G_2 of order m, with special vertex x, such that $V(G_1) \cap V(G_2) = \emptyset$, see Lemma 2.6. Now proceed exactly as in the proof of Lemma 3.1 to obtain a k-regular graph on the n vertices $V(G_1) \cup V(G_2)$ which is triangle-free and, since $k \geq 3$, has x as a cut-vertex. Hence $k \in C(n)$, and the result follows.

□

4 Conclusion

As mentioned in the introduction, part of the motivation for the determination of $c(n)$ in [3] was to determine a lower bound on $f(2m)$. Let $TF(2m)$ denote the spectrum

$\{k$: there exists a k-regular triangle-free graph on $2m$ vertices that does not have a one-factorization $\}$. Then $C(2m) \subseteq TF(2m)$; furthermore, $2 \in TF(2m)$ for all $m \geq 5$ (just take the union of a pair of vertex-disjoint odd cycles, each of length larger than 3). Hence from Theorem 1.7(ii) we have $\{k : 2 \leq k \leq c(2m)\} \subseteq TF(2m)$ for all $m \geq 7$. This interval covers a little more than the bottom half of the possible spectrum for $TF(2m)$.

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Appendix

The following notations apply only to Figures 1 to 3. A solid circle (i.e. a dot) denotes a single vertex, while a hollow circle with the number t inside denotes an independent set of t vertices. A solid line between two circles indicates the prescence of all possible edges between the corresponding sets of vertices; a dotted line indicates the prescence of all possible edges except those of a one-factor between the corresponding sets of vertices, while two dotted lines indicate the prescence of all edges except those of two disjoint one-factors.

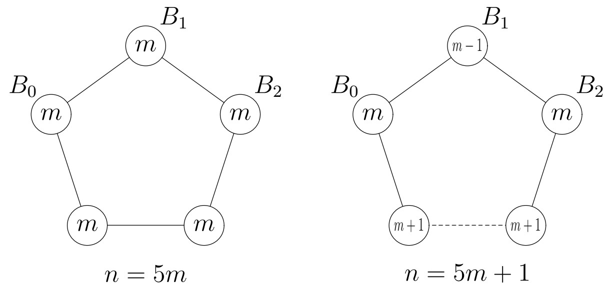

Figure 1

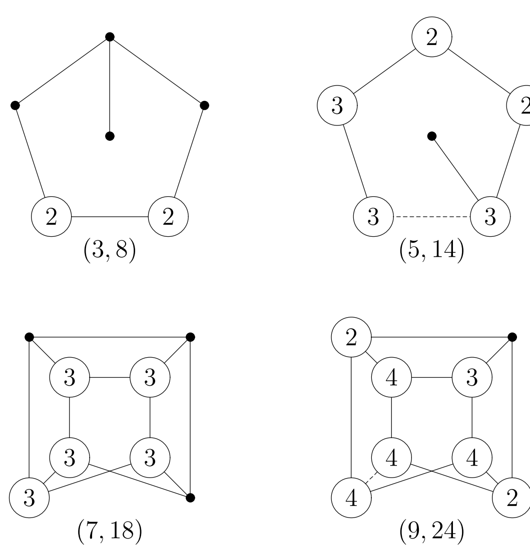

Figure 2

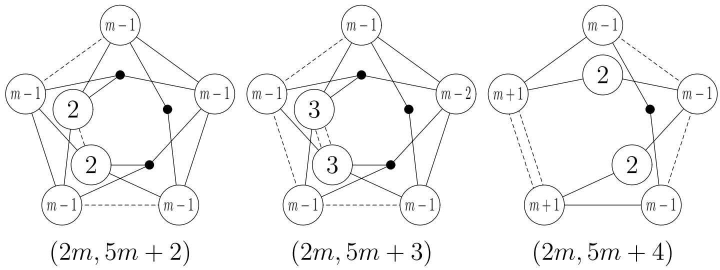

Figure 3

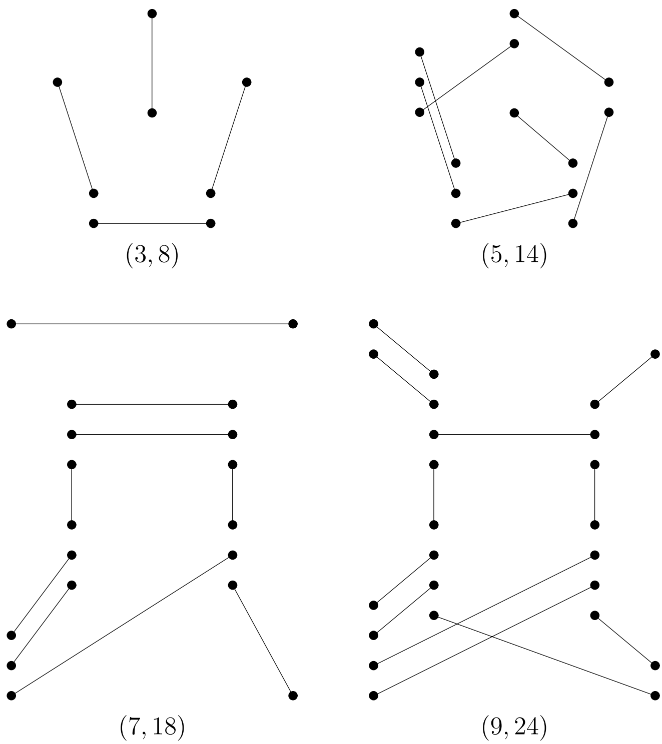

Figure 4

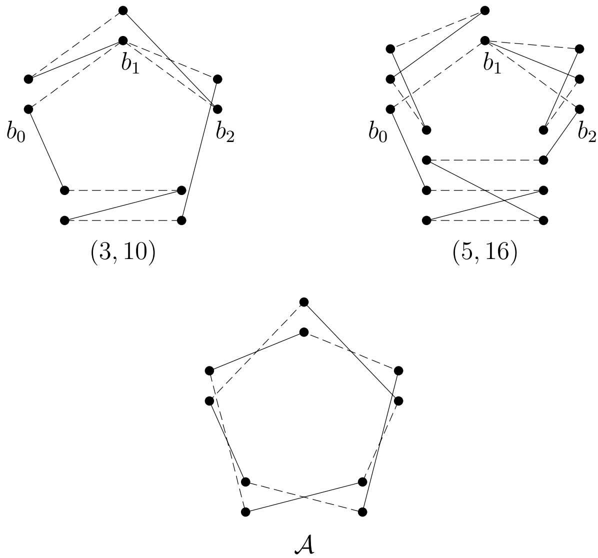

Figure 5

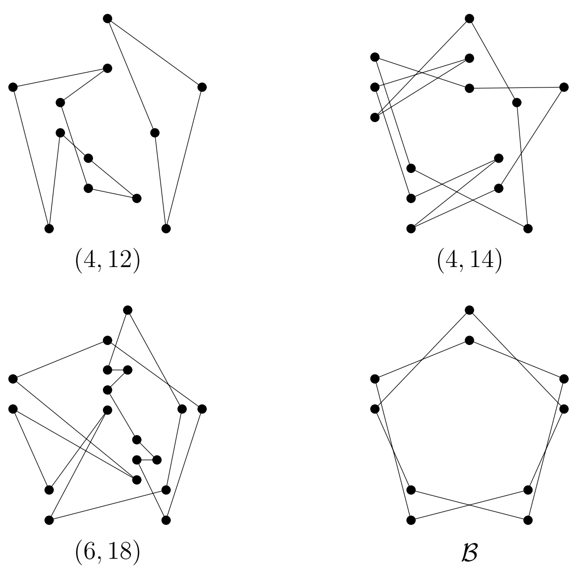

Figure 6

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