$K_{1,pq}$-factorization of complete bipartite graphs

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Abstract

Let $K_{m,n}$ be a complete bipartite graph with two partite sets having m and n vertices, respectively. A $K_{1,k}$-factorization of $K_{m,n}$ is a set of edge-disjoint $K_{1,k}$-factors of $K_{m,n}$ which partition the set of edges of $K_{m,n}$. When k is a prime number p, Wang [Discrete Math. 126 (1994)] investigated the $K_{1,p}$-factorization of $K_{m,n}$ and gave a sufficient condition for such a factorization to exist. Du [Discrete Math. 187 (1998) and Appl. Math. J. Chinese Univ. 17B (2001)] extended Wang's result to the case k is a prime power p^u. In this paper, it is shown that the conclusion in Wang's 1994 paper is true for any prime product pq. We will give a sufficient condition for the existence of the $K_{1,pq}$-factorization of $K_{m,n}$, whenever p and q are prime numbers, that is (1) $m \leq pqn$, (2) $n \leq pqm$, (3) $pqm - n \equiv pqn - m \equiv 0 \pmod{p^2q^2 - 1}$ and (4) $(pqm - n)(pqn - m) \equiv 0 \pmod{pq(pq-1)(p^2q^2-1)(m+n)}$.

1 Introduction

Let $K_{m,n}$ be a complete bipartite graph with two partite sets having m and n vertices. A subgraph F of $K_{m,n}$ is called a spanning subgraph of $K_{m,n}$ if F contains all the vertices of $K_{m,n}$. It is clear that a graph with no isolated vertices is uniquely determined by the set of its edges. So in this paper, we consider a graph with no isolated vertices to be a set of 2-element sets of its vertices. Let k be a positive integer. A $K_{1,k}$-factor of $K_{m,n}$ is a spanning subgraph F of $K_{m,n}$ such that every component of F is a $K_{1,k}$. A $K_{1,k}$-factorization of $K_{m,n}$ is a set of edge-disjoint $K_{1,k}$-factors of $K_{m,n}$ which partition the set of edges of $K_{m,n}$. In paper [7] the $K_{1,k}$-factorization of $K_{m,n}$ is defined as a resolvable $(m,n,k,1)$ bipartite S_{k+1} design. The graph $K_{m,n}$ is called $K_{1,k}$-factorizable whenever it has a $K_{1,k}$-factorization. For graph theoretical terms see [1].

The $K_{1,k}$-factorization of $K_{m,n}$ can be applied to combinatorial multiple-valued index-file organization schemes of order two in database systems (see [7]). So the

$K_{1,k}$-factorization of $K_{m,n}$ has been studied by several researchers. Using simple computation, we can establish the following trivial necessary condition for the existence of a $K_{1,k}$-factorization of $K_{m,n}$.

Theorem 1.1 *If $K_{m,n}$ has a $K_{1,k}$-factorization, then*
(1) $m \leq kn$,
(2) $n \leq km$,
(3) $km - n \equiv kn - m \equiv 0 \pmod{(k^2-1)}$ *and*
(4) $(km-n)(kn-m) \equiv 0 \pmod{k(k^2-1)(m+n)}$.

There are some known results on the existence of the $K_{1,k}$-factorization of $K_{m,n}$. When $k = 2$, the spectrum problem for $K_{1,2}$-factorization of $K_{m,n}$ has been completely solved by Ushio [4]. When k is a prime number p, the spectrum problem for $K_{1,p}$-factorization of $K_{m,n}$ has been partially solved (see [6] and [8]). Wang [8] investigated the $K_{1,p}$-factorization of $K_{m,n}$ and gave a sufficient condition for such a factorization to exist.

Theorem 1.2 [8] *For any prime number p, if*
(1) $m \leq pn$,
(2) $n \leq pm$,
(3) $pm - n \equiv pn - m \equiv 0 \pmod{(p^2-1)}$ *and*
(4) $(pm-n)(pn-m) \equiv 0 \pmod{p(p-1)(p^2-1)(m+n)}$,
then $K_{m,n}$ is $K_{1,p}$-factorizable.

In papers [2] and [3], Du extended Wang's result to the case when k is a prime power p^u. Du [3] investigated the K_{1,p^u}-factorization of $K_{m,n}$ and gave a sufficient condition for such a factorization to exist.

Theorem 1.3 [3] *Suppose k is a prime power p^u. If*
(1) $m \leq kn$,
(2) $n \leq km$,
(3) $km - n \equiv kn - m \equiv 0 \pmod{(k^2-1)}$ *and*
(4) $(km-n)(kn-m) \equiv 0 \pmod{k(k-1)(k^2-1)(m+n)}$,
then $K_{m,n}$ is $K_{1,k}$-factorizable.

In this paper, it is shown that the conclusion in [8] is true for any prime product pq. We will give a sufficient condition for the existence of the $K_{1,pq}$-factorization of $K_{m,n}$, whenever p and q are prime numbers; see the next theorem.

Theorem 1.4 *For any prime numbers p and q, if*
(1) $m \leq pqn$,
(2) $n \leq pqm$,
(3) $pqm - n \equiv pqn - m \equiv 0$ $(mod\ (p^2q^2-1))$ *and*
(4) $(pqm-n)(pqn-m) \equiv 0$ $(mod\ pq(pq-1)(p^2q^2-1)(m+n))$,
then $K_{m,n}$ is $K_{1,pq}$-factorizable.

2 Proof of Theorem 1.4

In this section we shall give the proof of Theorem 1.4. For our main result, we need the following lemmas. The first two lemmas and the corollary are easy observations and they are used in [8] also. For any two integers x and y, we use $\gcd(x, y)$ to denote the greatest common divisor of x and y.

Lemma 2.1 *Let u, v and w be positive integers. If* $\gcd(u, v) = 1$ *then* $\gcd(uv, u + vw) = \gcd(u, w)$.

Lemma 2.2 *If $K_{m,n}$ has a $K_{1,k}$-factorization, then $K_{sm,sn}$ has a $K_{1,k}$-factorization for any positive integer s.*

A corollary of Lemma 2.2 is as follows.

Corollary 2.3 *$K_{s,sk}$ is $K_{1,k}$-factorizable for any positive integer s.*

Corollary 2.3 implies that we only need to treat the case $m < pqn$ and $n < pqm$. Let

$$a = \frac{pqn - m}{p^2q^2 - 1}, \quad b = \frac{pqm - n}{p^2q^2 - 1}, \quad r = \frac{(pq + 1)mn}{pq(m + n)}, \quad c = \frac{(pqn - m)(pqm - n)}{pq(pq + 1)(m + n)}.$$

Then, from conditions (1)–(4) in Theorem 1.4, a, b, and c are positive integers. It is easy to see that $m = a + pqb$, $n = pqa + b$ and $r = a + b + c$ hold, and then r is also a positive integer. Let $d = \gcd(a, pqb)$, $a = du$ and $pqb = dv$ for some positive integers u and v with $\gcd(u, v) = 1$. Since $c \equiv 0$ $(mod\ (pq - 1)^2)$, let $e = \dfrac{c}{(pq - 1)^2}$. These equalities imply the following equalities:

$$d = \frac{pq(pqu + v)e}{uv}, \quad r = \frac{(u + v)(p^2q^2u + v)e}{uv}, \quad m = \frac{pq(pqu + v)(u + v)e}{uv},$$

$$n = \frac{(pqu + v)(p^2q^2u + v)e}{uv}, \quad a = \frac{pqu(pqu + v)e}{uv}, \quad b = \frac{v(pqu + v)e}{uv}.$$

Now we can establish the following lemma.

Lemma 2.4 *(1) If* $\gcd(v, p^2q^2) = 1$, *then*

$$m = pq(pqu + v)(u + v)s,\ n = (pqu + v)(p^2q^2u + v)s,$$

$$a = pqu(pqu + v)s,\ b = v(pqu + v)s,\ r = (u + v)(p^2q^2u + v)s,$$

for some positive integer s.

(2) If $\gcd(v, p^2q^2) = p$, *let $v = pv_1$. Then*

$$m = pq(qu + v_1)(u + pv_1)s,\ n = p(qu + v_1)(pq^2u + v_1)s,$$

$$a = pqu(qu + v_1)s,\ b = pv_1(qu + v_1)s,\ r = (u + pv_1)(pq^2u + v_1)s,$$

for some positive integer s.

(3) If $\gcd(v, p^2q^2) = p^2$, let $v = p^2v_2$. Then

$$m = q(qu + pv_2)(u + p^2v_2)s, \ n = p(qu + pv_2)(q^2u + v_2)s,$$

$$a = qu(qu + pv_2)s, \ b = pv_2(qu + pv_2)s, \ r = (u + p^2v_2)(q^2u + v_2)s,$$

for some positive integer s.

(4) If $\gcd(v, p^2q^2) = pq$, let $v = pqv_3$. Then

$$m = pq(u + v_3)(u + pqv_3)s, \ n = pq(u + v_3)(pqu + v_3)s,$$

$$a = pqu(u + v_3)s, \ b = pqv_3(u + v_3)s, \ r = (u + pqv_3)(pqu + v_3)s,$$

for some positive integer s.

(5) If $\gcd(v, p^2q^2) = p^2q$, let $v = p^2qv_4$. Then

$$m = q(u + pv_4)(u + p^2qv_4)s, \ n = pq(u + pv_4)(qu + v_4)s,$$

$$a = qu(u + pv_4)s, \ b = pqv_4(u + pv_4)s, \ r = (u + p^2qv_4)(qu + v_4)s,$$

for some positive integer s.

(6) If $\gcd(v, p^2q^2) = p^2q^2$, let $v = p^2q^2v_5$. Then

$$m = (u + pqv_5)(u + p^2q^2v_5)s, \ n = pq(u + pqv_5)(u + v_5)s,$$

$$a = u(u + pqv_5)s, \ b = pqv_5(u + pqv_5)s, \ r = (u + p^2q^2v_5)(u + v_5)s,$$

for some positive integer s.

Proof Recall that p and q are prime numbers and $\gcd(u, v) = 1$.

(1) By Lemma 2.1, we see that $\gcd(uv, u + v) = 1$ and $\gcd(uv, p^2q^2u + v) = \gcd(v, p^2q^2) = 1$. Since

$$r = \frac{(u + v)(p^2q^2u + v)e}{uv}$$

is an integer, we see that $\frac{e}{uv}$ must be an integer. Let $s = \frac{e}{uv}$. Then the equalities in (1) hold.

(2) By Lemma 2.1, we see that $\gcd(uv_1, u + pv_1) = \gcd(u, p) = 1$ and $\gcd(uv_1, pq^2u + v_1) = \gcd(v_1, pq^2) = 1$. Since

$$r = \frac{(u + pv_1)(pq^2u + v_1)e}{uv_1}$$

is an integer, we see that $\frac{e}{uv_1}$ must be an integer. Let $s = \frac{e}{uv_1}$. Then the equalities in (2) hold.

(3) By Lemma 2.1, we see that $\gcd(uv_2, u+p^2v_2) = \gcd(u,p^2) = 1$ and $\gcd(uv_2, q^2u+v_2) = \gcd(v_2,q^2) = 1$. Since

$$r = \frac{(u+p^2v_2)(q^2u+v_2)e}{uv_2}$$

is an integer, we see that $\frac{e}{uv_2}$ must be an integer. Let $s = \frac{e}{uv_2}$. Then the equalities in (3) hold.

(4) By Lemma 2.1, we see that $\gcd(uv_3, u+pv_3) = \gcd(u,p) = 1$ and $\gcd(uv_3, pqu+v_3) = \gcd(v_3,pq) = 1$. Since

$$r = \frac{(u+pv_3)(pqu+v_3)e}{uv_3}$$

is an integer, we see that $\frac{e}{uv_3}$ must be an integer. Let $s = \frac{e}{uv_3}$. Then the equalities in (4) hold.

(5) By Lemma 2.1, we see that $\gcd(uv_4, u+p^2qv_4) = \gcd(u,p^2q) = 1$ and $\gcd(uv_4, qu+v_4) = \gcd(v_4,q) = 1$. Since

$$r = \frac{(u+p^2qv_4)(qu+v_4)e}{uv_4}$$

is an integer, we see that $\frac{e}{uv_4}$ must be an integer. Let $s = \frac{e}{uv_4}$. Then the equalities in (5) hold.

(6) By Lemma 2.1, we see that $\gcd(uv_5, u+p^2q^2v_5) = \gcd(u,p^2q^2) = 1$ and $\gcd(uv_5, u+v_5) = 1$. Since

$$r = \frac{(u+p^2q^2v_5)(u+v_5)e}{uv_5}$$

is an integer, we see that $\frac{e}{uv_5}$ must be an integer. Let $s = \frac{e}{uv_5}$ Then the equalities in (6) hold.

This proves the lemma.

We are now in a position to prove Theorem 1.4. For our main result, we only need the following direct constructions.

First of all, using the constructions which were devised by Wang (Lemma 3.5 and Lemma 3.6 in [8]), we have the following lemmas.

Lemma 2.5 *For any positive integers u and v, let*

$$\begin{aligned} m &= pq(pqu+v)(u+v), \\ n &= (pqu+v)(p^2q^2u+v). \end{aligned}$$

Then $K_{m,n}$ has a $K_{1,pq}$-factorization.

Lemma 2.6 *For any positive integers u and v, let*

$$\begin{aligned} m &= pq(u+v)(u+pqv), \\ n &= pq(u+v)(pqu+v). \end{aligned}$$

Then $K_{m,n}$ has a $K_{1,pq}$-factorization.

We then only need the following lemmas.

Lemma 2.7 *For any positive integers u and v, let*

$$\begin{aligned} m &= pq(qu+v)(u+pv), \\ n &= p(qu+v)(pq^2u+v). \end{aligned}$$

Then $K_{m,n}$ has a $K_{1,pq}$-factorization.

Proof Let $a = pqu(qu+v),\ b = pv(qu+v),\ r = (u+pv)(pq^2u+v),\ r_1 = u+pv$ and $r_2 = pq^2u+v$. Let X and Y be the two partite sets of $K_{m,n}$ and set

$$\begin{aligned} X &= \{x_{i,j} \mid 1 \le i \le r_1;\ 1 \le j \le pq(qu+v)\}, \\ Y &= \{y_{i,j} \mid 1 \le i \le r_2;\ 1 \le j \le p(qu+v)\}. \end{aligned}$$

We will construct a $K_{1,pq}$-factorization of $K_{m,n}$. We remark in advance that the additions in the first subscripts of $x_{i,j}$'s and $y_{i,j}$'s are taken modulo r_1 and r_2 in $\{1,2,\ldots,r_1\}$ and $\{1,2,\ldots,r_2\}$, respectively, and the additions in the second subscripts of $x_{i,j}$'s and $y_{i,j}$'s are taken modulo $pq(qu+v)$ and $p(qu+v)$ in $\{1,2,\ldots,pq(qu+v)\}$ and $\{1,2,\ldots,p(qu+v)\}$, respectively.

For each i and h, $1 \le i \le u$, $1 \le h \le pq$. Let $s(i,h) = pq^2(i-1)+q(h-1)+1$, and $t(i,h) = pq(i-1)+h-1$, and set

$$E_i = \{x_{i,kp(qu+v)+j}y_{s(i,h)+k,t(i,h)+j} \mid 1 \le j \le p(qu+v);\ 1 \le h \le pq;\ 0 \le k \le q-1\}.$$

For each i and h, $1 \le i \le v$, $1 \le h \le p$. Let $\varphi(i,h) = u+p(i-1)+h$ and $\psi(i,h) = pqu+p(i-1)+h-1$, and set

$$E_{u+i} = \{x_{\varphi(i,h),j}y_{pq^2u+i,\psi(i,h)+j} \mid 1 \le j \le pq(qu+v);\ 1 \le h \le p\}$$

Let $F = \cup_{1\le i\le u+v}E_i$; then it is easy to see that the graph F is a $K_{1,pq}$-factor of $K_{m,n}$. Define a bijection σ from $X \cup Y$ onto $X \cup Y$ in such a way that $\sigma(x_{i,j}) = x_{i+1,j}$ and $\sigma(y_{i,j}) = y_{i+1,j}$. For each $i \in \{1,2,\ldots,r_1\}$ and each $j \in \{1,2,\ldots,r_2\}$, let

$$F_{i,j} = \{\sigma^i(x)\sigma^j(y) \mid x \in X,\ y \in Y,\ xy \in F\}.$$

It is easy to show that the graphs $F_{i,j}$ $(1 \le i \le r_1,\ 1 \le j \le r_2)$ are $K_{1,pq}$-factors of $K_{m,n}$ and their union is $K_{m,n}$. Thus $\{F_{i,j} \mid 1 \le i \le r_1,\ 1 \le j \le r_2\}$ is a $K_{1,pq}$-factorization of $K_{m,n}$.

This proves the lemma.

Lemma 2.8 *For any positive integers u and v, let*

$$\begin{aligned} m &= q(qu+pv)(u+p^2v), \\ n &= p(qu+pv)(q^2u+v). \end{aligned}$$

Then $K_{m,n}$ has a $K_{1,pq}$-factorization.

Proof Let $a = qu(qu+pv)$, $b = pv(qu+pv)$, $r = (u+p^2v)(q^2u+v)$, $r_1 = u+p^2v$, and $r_2 = q^2u+v$. Let X and Y be the two partite sets of $K_{m,n}$ and set

$$\begin{aligned} X &= \{x_{i,j} \mid 1 \le i \le r_1;\ 1 \le j \le q(qu+pv)\}, \\ Y &= \{y_{i,j} \mid 1 \le i \le r_2;\ 1 \le j \le p(qu+pv)\}. \end{aligned}$$

For each i and h, $1 \le i \le u$, $1 \le h \le q$. Let $s(i,h) = q^2(i-1)+q(h-1)+1$ and $t(i,h) = q(i-1)+h-1$, and set

$$\begin{aligned} E_i = \{ & x_{i,k(qu+pv)+j}\, y_{s(i,h)+k,g(qu+pv)+t(i,h)+j} \mid 1 \le j \le qu+pv; \\ & 1 \le h \le q;\ 0 \le k \le q-1;\ 0 \le g \le p-1\}. \end{aligned}$$

For each i and h, $1 \le i \le v$, $1 \le h \le p$. Let $\varphi(i,h) = u+p^2(i-1)+p(h-1)+1$ and $\psi(i,h) = qu+p(i-1)+h-1$, and set

$$\begin{aligned} E_{u+i} = \{ & x_{\varphi(i,h)+k,j}\, y_{q^2u+i,k(qu+pv)+\psi(i,h)+j} \mid 1 \le j \le q(qu+pv); \\ & 1 \le h \le p;\ 0 \le k \le p-1\} \end{aligned}$$

Let $F = \cup_{1 \le i \le u+v} E_i$; then it is easy to see that the graph F is a $K_{1,pq}$-factor of $K_{m,n}$. Define a bijection σ from $X \cup Y$ onto $X \cup Y$ in such a way that $\sigma(x_{i,j}) = x_{i+1,j}$ and $\sigma(y_{i,j}) = y_{i+1,j}$. For each $i \in \{1,\ 2,\ \cdots,\ r_1\}$ and each $j \in \{1,\ 2,\ \cdots,\ r_2\}$, let

$$F_{i,j} = \{\sigma^i(x)\sigma^j(y) \mid x \in X,\ y \in Y,\ xy \in F\}.$$

It is easy to show that the graphs $F_{i,j}$ $(1 \le i \le r_1,\ 1 \le j \le r_2)$ are $K_{1,pq}$-factors of $K_{m,n}$ and their union is $K_{m,n}$. Thus $\{F_{i,j} \mid 1 \le i \le r_1,\ 1 \le j \le r_2\}$ is a $K_{1,pq}$-factorization of $K_{m,n}$.

This proves the lemma.

Applying Lemma 2.4 with Lemmas 2.5 to 2.8, we see that for the parameters m and n satisfying conditions (1)–(4) in Theorem 1.4, $K_{m,n}$ has a $K_{1,pq}$-factorization. This completes the proof of Theorem 1.4.

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