Defective circular coloring

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Abstract

The concept of a defective circular coloring is introduced, and results shown for planar, series-parallel, and outerplanar graphs. Numerous problems are also stated.

1 Introduction

A (k, d) *defective coloring* of a graph is an assignment of k colors to the vertices such that each vertex v is adjacent to at most d vertices having the same color as v [1, 2]. Defective coloring is sometimes known as "improper coloring," as it is called in the book by Jensen and Toft [4]. *Circular coloring* is a well-studied refinement of graph coloring: a graph $G = (V, E)$ has a k/q coloring if, when $k \geq 2q$, there exists a function $c : V \rightarrow \{0, \ldots, k-1\}$ such that for each pair of adjacent vertices u and v, we have $q \leq |c(v) - c(u)| \leq k - q$ [9, 10]. Vince's famous result shows that circular coloring is, in fact, a refinement of the usual notion of graph coloring: G has a k/q coloring implies G has an r/s coloring if $\frac{k}{q} \leq \frac{r}{s}$ [9]. Furthermore, Vince showed that every graph has a rational circular chromatic number, i.e., the circular chromatic number can be defined as the *minimum* rational number k/q for which the graph can be k/q colored, rather than the infimum [9].

Define a *defective circular coloring* for a simple graph $G = (V, E)$ to be a function $c : V \rightarrow \{0, \ldots, k-1\}$ such that each vertex v is adjacent to at most d vertices u where $q \leq |c(v) - c(u)| \leq k - q$ does not hold. If such a defective circular coloring exists, we say G is $(k/q, d)$ colorable. A *defect* for a vertex v is an adjacent vertex u such that $q \leq |c(v) - c(u)| \leq k - q$ does not hold. We shall also use the term "defect" in the context of the colors themselves: color i is said to be a defect for color j if $q \leq |i - j| \leq k - q$ does not hold. When $q = 1$ (and $d = 0$), circular coloring conforms to the usual, integer version of graph coloring [9].

Define the *odd-girth* of a graph to be the length of the shortest odd-length cycle in a graph. Thus, triangle-free graphs have odd-girth at least five. Previously, it has been shown that planar graphs exist that are not $(3, 1)$ colorable [2]; that all planar graphs are $(3, 2)$ colorable [2]; that all toroidal graphs are $(3, 2)$ colorable [1]; that triangle-free planar graphs are $(3, 0)$ colorable [3]; that planar graphs with odd-girth at least 17 are $(5/2, 0)$ colorable [5]; that there exist planar graphs with odd girth seven that are not $(5/2, 0)$ colorable (a graph due to Albertson and Moore mentioned in [5]); that triangle-free outerplanar graphs are $(5/2, 0)$ colorable [5]; and that outerplanar graphs are $(2, 2)$ colorable [7]. Pan and Zhu [8] proved that series-parallel graphs with odd-girth at least five are $(8/3, 0)$ colorable and that series-parallel graphs of odd-girth at least seven are $(5/2, 0)$ colorable.

In this paper, results on defective circular coloring are presented for planar, outerplanar, and series-parallel graphs. Numerous problems are also stated.

2 Fundamental Issues

It is immediate that if G is $(k/q, d)$ colorable, then G is $(k/q, d')$ colorable, for any $d' \geq d$. We extend this below and then pose two questions which are analogues of Vince's results.

A basic lemma is given first that will be used in a few places in this paper. For ease of exposition, we sometimes will use an alternate, but equivalent, definition of circular coloring, that is, k/q-coloring: color each vertex with q adjacent integers from $\{1, 2, \ldots, k\}$ (integers 1 and k are adjacent via wrap-around) so that adjacent vertices are colored with disjoint sets of integers. It is well-known that this definition is equivalent to the definition of circular coloring given in Section 1. Extending this definition to defective circular coloring, we claim a defect exists when two endvertices of an edge are not colored with disjoint sets of integers.

Lemma 1 *The two definitions of defective circular coloring are equivalent.*

Proof: In each of the two definitions, we can map the k integers onto a circle, labelled in a clockwise direction in increasing order. In both definitions, a color assigned to a vertex is an interval containing q integers (for a k/q coloring), and a defect exists when two colors overlap. It follows that the definitions are equivalent. □

Using this alternate definition of defective circular coloring, it is easy to see that a $(k/1, d)$ coloring is equivalent to a (k, d) coloring. Another lemma is needed to simplify the proof of the subsequent theorem.

Lemma 2 *If G is $(k/q, d)$ colorable, then G is $(ck/cq, d)$ colorable, for any $c \geq 1$.*

Proof: Consider a $(k/q, d)$ coloring of G. That is, the vertices are colored with q adjacent colors each (from the set $1, 2, \ldots, k$) and a defect exists when neighboring vertices have a color in common. To form the $(ck/cq, d)$ coloring, map the q colors assigned to vertex v: $i, i + 1, \ldots, i + q - 1$, to the colors $c(i - 1) + 1, c(i - 1) + 2, \ldots, c(i + q - 1)$. It is easy to see that u and v are defects in the k/q coloring if and only if they are defects in the ck/cq coloring. □

By the same token, the converse of Lemma 2 is also true:
If G is $(ck/cq, d)$ colorable, then G is $(k/q, d)$ colorable, for any $c \geq 1$.

Theorem 3 *A graph $G = (V, E)$ is $(k/q, d)$ colorable if it is $(r/s, d)$ colorable, whenever $k/q \geq r/s$.*

Proof: Let c be the least common denominator of r/s and k/q. Re-write these fractions as r'/c and k'/c. As $k/q \geq r/s$, $k' \geq r'$. Consider an $(r'/c, d)$ coloring of G, which exists by Lemma 2. That is, the vertices are colored with c adjacent colors each (from the set $1, 2, \dots, r'$) and a defect exists when neighboring vertices have a color in common. Such a coloring is, by definition, also a $(k'/c, d)$ coloring. Therefore, G is $(k/q, d)$ colorable. □

In other words, defective circular coloring is a refinement of defective coloring. We next ask whether the defective circular chromatic number always exists, as a rational number. That is, must we define the defective circular chromatic number of a graph as the infimum over all its defective circular colorings (with a fixed number of defects), or may it be defined as the minimum?

Conjecture 1 *Let G be a graph and $d \geq 0$ an integer. There exists a rational number k/q such that G is $(k/q, d)$ colorable and such that G is not $(r/s, d)$ colorable for any $r/s < k/q$.*

The next question is likely more difficult, as similar questions are not yet fully understood for defective coloring.

Question 2 *Let G be $(k/q, d)$ colorable. Can it be characterized when G is $(r/s, d')$ colorable, based on the relationship between k/q and r/s and the relationship between d and d'?*

3 Planar Graphs

A k-face in a planar graph is a face with k edges. We begin this section with a negative result.

Theorem 4 *Let c be a positive integer. There exists a planar graph G that is not $(5/2, c)$ colorable.*

Proof: We first show that not all planar graphs have a $(5/2, 1)$ coloring. Start with a triangle v_1, v_2, v_3. Now 5/2-coloring the vertices forces at least two of the vertices to have one defect. If all three have one defect, adding a new vertex, v_4, inside the interior of this 3-face such that v_4 is adjacent to each of v_1, v_2, v_3, forces a vertex to have a second defect (which is illegal). Otherwise, add a new vertex v_4 that is adjacent to each vertex of the triangle. Coloring v_4 produces a 3-face either with at least one vertex having two defects (which is illegal) or a 3-face with all three vertices having exactly one defect. In the latter case, add a new vertex, v_5, to the interior of that 3-face so that v_5 is adjacent to each vertex on the 3-face, and a second defect will be forced when we attempt to color v_5.

This construction can be extended to show that not all planar graphs have a $(5/2, c)$ coloring, for any constant c, since each vertex of some face, f, in the constructed graph has at least one defect: repeating the main step of the construction on the interior of face f yields a graph that is not $(5/2, 2)$ colorable (i.e., a 3-face of that graph will have all vertices with at least two defects, and the next step forces a third defect upon some vertex). One additional step must be considered when $c > 1$. Consider the case when $c = 2$ as an illustration. If a 3-face f' has all three vertices the same color (which was not possible when $c = 1$, as this would have caused each vertex on the 3-face to have two defects), then we add a new vertex v to the inside of this 3-face which is adjacent to each vertex in the 3-face. If v is colored differently than its three neighbors (as it must be), iterate the construction again: create a 3-face with v, a vertex from f' and a new vertex u, which forces a defect upon v and u. Repeating this step forces a second defect upon v and u and one more iteration of the main step yields the desired configuration. Further iteration can then be used to prove the theorem for any $c \geq 2$. □

We next state a conjecture and a question.

Conjecture 3 *Let G be a planar, triangle-free graph. Then G can be $(5/2, 2)$ colored.*

Question 4 *Does every planar triangle-free graph have a $(5/2, 1)$ coloring? If not, what girth or odd-girth lower bound ensures such a coloring exists?*

Theorem 5 *Let G be a graph with $\Delta(G) \leq 3$. Then G is $(5/2, 2)$-colorable.*

Proof: Suppose the theorem were false and let G be a counterexample with the minimum number of vertices. G must contain a cycle $C = v_1, \ldots, v_k, v_1$, or else G is a forest and thus $(4/2, 0)$ colorable. Now $(5/2, 2)$ color the subgraph G' of G induced by $V(G) - V(C)$. Extend this coloring to a coloring of G as follows. Let $N(C) = \{v \mid v \in V(G) - V(C), vv_i \in E(G), 1 \leq i \leq k\}$. Note that each vertex of $N(C)$ may receive up to two defects in the coloring of G'. Color v_1 so that v_1 is not a defect for u_1, where u_1 is the neighbor (provided one exists) of v_1 in $N(C)$. Then color v_2 similarly, noting that now v_1 and v_2 may be defects for one another. Continuing around C, each vertex on C is colored with at most two defects, which is a contradiction. □

Question 5 *Can every planar, triangle-free graph G with $\Delta(G) = 3$ be $(5/2, 1)$ colored?*

A partial solution is given next. Some properties of planar graphs related to Euler's formula are reviewed in the appendix. These properties are used in the proof of the next result.

Theorem 6 *Let G be a planar graph with girth at least six and $\Delta(G) \leq 3$. Then G is $(5/2, 1)$-colorable.*

Proof: Suppose the theorem were false and let G be a counterexample with the minimum number of vertices. Let $\hat{G}$ be the underlying graph of G, obtained from G by replacing each maximal induced path with an edge.

By Lemma 10 (see the appendix), let $\hat{C}$ be a facial circuit of $\hat{G}$ with a positive Euler contribution (*Euler contribution* is defined in the appendix). Note that the Euler contribution of a facial cycle of length at least six is at most zero, since $\delta(\hat{G}) \geq 3$. Therefore $\hat{C}$ is of length at most five. Let C be the cycle of G corresponding to $\hat{C}$ in the underlying graph $\hat{G}$. Therefore, as the girth of G is at least six, length(C) ≥ 6.

It is easy to see than G cannot contain an induced P_4 (or else we could delete the two interior vertices of the P_4, color the remaining vertices of G and extend the coloring to include the two deleted vertices). We shall assume for the time being that the length of $\hat{C}$ is exactly five and the length of C is exactly six. The remaining cases, namely when $\hat{C}$ is of length five and the length of C is seven or more, or when $\hat{C}$ is of length four or three, are dealt with below. Hence there exists an induced P_3 in G whose three vertices are on C. Let v be the degree two vertex on this P_3, i.e., $\deg(v) = 2$ in G and $v \in V(C)$. Let the neighbors of v be u, w; each of these has degree three (or else G has an induced P_4). Let the neighbor of u (w) which is not on C be u_1 (w_1). Note that we assume that u_1, w_1 do not lie on C, or else the argument proceeds in a similar fashion (since there is no neighbor of u (w) whose color from the coloring of $V(G) - V(C)$ could create a defect). And let us assume without loss of generality that u_1 has two additional neighbors u_2, u_3 and w_1 has two additional neighbors w_2, w_3; possibly $u_2 = w_2$.

Denote the vertices of C, in order around C, as a, b, c, u, v, w, a and let the neighbors of a, b, c that are not on C be a_1, b_1, c_1, respectively. Let G' be the subgraph of G induced by $V(G) - V(C)$. Now $(5/2, 1)$ color G'. We extend the coloring of G' to a coloring of G as follows. If u_1 and w_1 do not have disjoint colors, we claim the coloring of G' can be so extended (see below). Otherwise, assume without loss of generality that $c(u_1) = \{1, 2\}$ and $c(w_1) = \{3, 4\}$ and that each of these two vertices has one defect, or else the coloring of G' can easily be extended to a coloring of G (by having, say, u have a defect with u_1, if necessary). Assume, without loss of generality, that u_1 has a defect with u_2 and no defect with u_3. Then we may modify the color of u_1 to another color so that u_1 still has a defect with u_2 and not with u_3. For example, if $c(u_1) = \{1, 2\}$, $c(u_2) = \{1, 2\}$, and $c(u_3) = \{3, 4\}$, we can change the color of u_1 to $\{1, 5\}$.

Now suppose that u_1 and w_1 do not have disjoint colors. Furthermore, modify the colors of a_1, b_1, c_1, if necessary, as we did for u_1, in order that $c(c_1) \neq c(b_1)$ and $c(b_1) \neq c(a_1)$. It may be that as a result, for example, $c(c_1)$ and $c(b_1)$ are not disjoint, but we only require that they not be the same color. We can now color a, b and c so that they have no defects with one another, nor with a_1, b_1, or c_1, respectively. We may then color u, v, w so that each vertex on C has at most one defect. This resolves the case when the length of C is six.

Next, assume $\hat{C}$ is of length five and the length of C is seven. This case basically proceeds as above: we assume G contains no induced P_4, color the subgraph induced by $V(G) - V(C)$ and extend the coloring. Focus again on vertex v (of degree two in

G) and its neighbors, u and w (of degree three in G). As above, modify the colors of u_1 and w_1 (the other neighbors of u and w, respectively), if necessary, so that they do not have disjoint colors. Likewise, modify the colors of the vertices a_1, b_1 and c_1 so that $c(c_1) \neq c(b_1)$ and $c(b_1) \neq c(a_1)$. Then the modified coloring can be extended to the seven vertices of C so that each vertex on C has at most one defect and none of the colors assigned to vertices on C are defects for any of the vertices in $V(G) - V(C)$. To see this, there are two cases to consider. One, suppose ab and bc are edges in C. In this case, we may assume by symmetry that v' (a vertex of degree two in G which lies on C) is adjacent to w, i.e., $C = a, b, c, u, v, w, v', a$. Color a, b and c so that they have no defects with one another, nor with a_1, b_1, or c_1. It is now simple to color u, v, v' and w so that each vertex on C has at most one defect, as we did in the case when the length of C was six (by coloring v' first so that it has no defect with a, then the problem of extending the coloring reduces to the same situation as when the length of C was six). On the other hand, if ab and bc are not both edges on C, then C is of the form u, v, w, a, v', b, c, u, where v and v' have degree two in G. By coloring a, b and c so that they have no defects with one another, nor with a_1, b_1, or c_1, we can again easily extend the coloring to include u, v, v' and w. The cases when the length of C is greater than seven are identical.

Finally, we have the cases when $\hat{C}$ is of length four or three. If the length of $\hat{C}$ is 3, the only way to avoid an induced P_4 is for C to be of the form a, b, c, d, e, f, a where b, d and f are degree two vertices in G. In this case, color the subgraph induced by $V(G) - V(C)$. Then color a, c, e so they have no defects with their neighbors from $V(G) - V(C)$. Such a coloring can then be extended to include b, d and f. Now suppose the length of $\hat{C}$ is 4. This proceeds exactly as in the case when the length of $\hat{C}$ is 5, by treating one of the degree two vertices as vertex "u" (the degree two vertex in the argument above) and one of the degree two vertices as either "a", "b", or "c", depending on its location in C. As this vertex has no neighbors outside of C, we can color, in the same manner that a, b, or c were colored, the case when the length of $\hat{C}$ is 5. This completes the proof. □

4 Outerplanar Graphs

Theorem 7 *There exists an outerplanar graph not having a* $(5/2, 1)$ *coloring.*

Proof: It is easy to verify that the graph in Figure 1 cannot be $(5/2, 1)$ colored. □

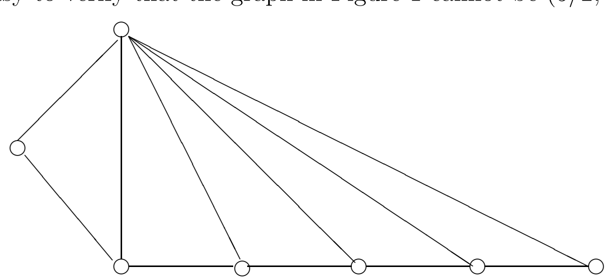

Figure 1. Outerplanar Graph

Theorem 8 *Let G be an outerplanar graph with no adjacent triangles (i.e., no 3-cycles share an edge). Then G is $(5/2, 1)$ colorable.*

Proof: Assume without loss of generality that $\delta(G) > 1$, because degree one vertices can be colored easily. The proof is by induction on the number of vertices. Denote the closed walk along the exterior face of G as C (if G is 2-connected, then C is a simple cycle). Of course, since G is outerplanar, $V(G) = V(C)$. Now G is either a chordless cycle (in which case the result immediately follows), or G contains a cut-vertex v, or C contains a chord, xy, such that the cut-vertex/chord "cuts" G into two graphs, G_1, G_2 (both containing the cut-vertex v/the chord xy), such that one of them, say G_2, is 2-regular.

We first argue the case when C contains a chord xy; the case when there is a cut-vertex is handled below. Inductively color G_1. We claim that there exists such a coloring of G_1 that can be extended to a coloring of G (i.e., extended to a coloring including the remaining vertices of G_2). If x and y are defects for each other in the coloring of G_1, then the claim follows easily. So suppose x and y are not defects for each other and each has one defect in the coloring of G_1 (if at least one of x, y has no defects, the claim follows easily). We examine two cases. First suppose xy borders a k-face, $k > 3$, $ab \ldots xy \ldots ab$ in G_1. Then either ab is a chord of C or G_1 is a cycle. The latter case is easily handled. So assume ab is a chord of C. Let G_3 be the subgraph of G_1 induced by $V(G_1) - \{x, y\}$. Inductively color G_3. We can then extend this coloring to G_1 in such a way that either x, y are defects for one another, or x, y have no defects whatsoever. To do this, suppose this k-face in question is $a, a_1, \ldots, a_i, x, y, b_j, b_{j-1}, \ldots, b, a$. Start with a's color and color $a_1, a_2, \ldots, a_i, x$ so that none of these vertices have defects. Then color $b_j, b_{j-1}, \ldots, b_1$ so that none of these vertices have defects. Now color y so that it either has no defects, or only a defect with x. This coloring of G_1 can then be extended to include G_2. Therefore we may assume that xy lies on a 3-face in G_1. But then G_2 cannot be a triangle, and thus the coloring of G_1 is easily extended to G_2.

To complete the proof, consider the case when G contains a cut-vertex. If C contains a chord, we can apply the argument above. So we may assume that G is a set of chordless cycles joined at cut-vertices. Remove one of these chordless cycles and inductively color the remaining graph. This coloring can be trivially extended to the remaining cycle. □

5 Series-Parallel Graphs

Let G be a (two-terminal) series-parallel graph with terminals x and y. Let $\ell_o(G)$ denote the length of the shortest odd-length path from x to y in G and $\ell_e(G)$ denote the length of the shortest even-length path from x to y in G; the parameter G is omitted when the meaning is clear from the context.

Theorem 9 *Let G be a triangle-free series-parallel graph. Then G can be $(5/2, 2)$ colored.*

Proof: Let x, y be the two terminals of G. Define a set $L(G)$ as follows; the parameter G is omitted when the meaning is clear from the context.

If $\ell_o = 1$ and $\ell_e = 2$ then $L = \emptyset$ (G contains a triangle);
if $\ell_o = 1$ and $\ell_e \geq 4$ then let $L = \{0, 1, 2, 3, 4\}$;
if $\ell_o = 3$ and $\ell_e = 2$ then let $L = \{1, 2, 3, 4\}$;
if $\ell_o = 3$ and $\ell_e \geq 4$ then let $L = \{1, 2, 3, 4\}$;
if $\ell_o \geq 5$ and $\ell_e = 2$ then let $L = \{0, 1, 4\}$;
if $\ell_o \geq 5$ and $\ell_e \geq 4$ then let $L = \{0, 1, 4\}$.

We prove that for any $q \in L(G)$, G can be $(5/2, 2)$ colored in such a way that x is assigned color 0, y is assigned color q, and each of x, y has no defects, except for possibly with each other. The proof is by induction on the number of series or parallel constructions, k, used in the construction of G. If $k = 0$, then $G = K_2$ and the theorem is true. For the inductive step, consider two cases, depending on whether the k^{th} step in the construction of G is a parallel or series construction.

Case 1 The k^{th} step is a parallel construction. Let the k^{th} step take two series-parallel graphs G_1 and G_2, with terminals x_1, y_1 and x_2, y_2, respectively, and identify x_1 with x_2 and y_1 with y_2. Observe that if there exists a path of length l between x and y in G, then there exists a path of length l between x and y in at least one of G_1, G_2. Let $q \in L(G)$. Note that $L(G) \subseteq L(G_1) \cap L(G_2)$ and $L(G) \neq \emptyset$. Color G_1 so that $c(x_1) = 0$ and $c(y_1) = q$ and x_1 and y_1 having no defects except possibly with each other. Likewise, color G_2 with $c(x_2) = 0$ and $c(y_2) = q$ and x_2 and y_2 having no defects except possibly with each other. When the two pairs of terminals are identified, the resulting coloring of G will be such that $c(x) = 0, c(y) = q$ and x, y have no defects except possibly with each other.

Case 2 The k^{th} step is a series construction. Let the k^{th} step take two series-parallel graphs G_1 and G_2, with terminals x_1, y_1 and x_2, y_2 respectively, and identify y_1 with x_2. Keep in mind that, in terms of G, we let $x = x_1$ and $y = y_2$. We consider four subcases depending on the values of $\ell_o(G)$ and $\ell_e(G)$.

Subcase 1 $\ell_o = 1$. Then the k^{th} step cannot be a series construction.

Subcase 2 Suppose x_1 is adjacent to y_1 and x_2 is adjacent to y_2. That is, $\ell_e = 2$. If $\ell_o = 3$, then since G is triangle-free, the k^{th} step cannot be series construction. So we may assume that $\ell_o \geq 5$. Since G is triangle-free and $\ell_o(G_1) = \ell_o(G_2) = 1$, we have $\ell_e(G_1) \geq 4$ and $\ell_e(G_2) \geq 4$. Then $L(G) = \{0, 1, 4\}$. Color G_1 and G_2 inductively so that $c(y_1) = c(x_2)$, noting that the colorings of G_1 and G_2 are based on $L(G_1)$ and $L(G_2)$, and thus we may choose $c(y_1)$ and $c(x_2)$ to be any of $\{0, 1, 2, 3, 4\}$. When we color G_1, we need $c(x_2) \neq 0$, in order that no defect exists with $x = x_1$ in the future (and also, $c(y_1) \neq q$, in order that no defect exists with $y = y_2$ in the future). This requirement that $c(x_2) \neq 0$ seems to violate the needs of the inductive hypothesis that we color the "x" terminal of a graph with color 0. However, in this case, we *shift* the colors by subtracting $c(x_2)$ from each color, modulo 5, (do this shifting to each $v \in V(G_2)$) so that we color x_2 with the desired non-zero color. The color assigned to x_2 will also be assigned to y_1. Since $L(G_2) = \{0, 1, 2, 3, 4\}$, the coloring of y_2 and G_2 can be completed. In other words, (inductively) color G_2 with $c(x_2) = 0$ and $c(y_2) = q - s$ (modulo 5), where s, $1 \leq s \leq 4$, is the ultimate color we wish to assign x_2 in G (thereby satisfying the induction requirements). We then modify the coloring of each vertex in G_2, by shifting, to one in which $c(x_2) = s \neq 0$ and $c(y_2) = q$, and use the modified coloring to complete the coloring of G.

In this manner, we inductively color G_1 and G_2 with $c(y_1) = c(x_2)$, with $c(y_1)$ not being a defect for 0, and with $c(x_2)$ not being a defect for $c(y_2) = q$. Combining the colorings of G_1 and G_2 into a coloring of G, we have that x and y have the desired colors and no defects.

Subcase 3 Suppose $\ell_o = 3$ and $\ell_e \geq 4$. First assume that $\ell_o(G_1) = 1$. This implies $\ell_e(G_1) \geq 4$, $\ell_e(G_2) = 2$, and $\ell_o(G_2) \geq 3$. Based on $L(G_1)$ and $L(G_2)$, we can color G_1 and G_2 such that $c(x_1) = 0, c(y_2) = q$, $c(y_1) = c(x_2) \neq 0$ (so y_1 is not a defect for x_1) and $c(x_2)$ is not a defect for $c(y_2) = q$. Note that in this case, since $c(y_1) \neq 0$, we need to use the shifting technique discussed in Subcase 2. In particular, $L(G)$ is equal to $\{1, 2, 3, 4\}$. And $L(G_2) = \{1, 2, 3, 4\}$ or $L(G_2) = \{0, 1, 4\}$ and $L(G_1) = \{0, 1, 2, 3, 4\}$. We may restrict our consideration by setting $L(G_2) = \{1, 4\}$ in this case (since $\{1, 4\}$ is a subset of the two possible values of $L(G_2)$ mentioned in the previous sentence). Color G_2 as follows. Color y_2 with 0 and let $c(x_2) \in \{1, 4\}$ (by treating y_2 as the "x" terminal of G_2, i.e., the terminal that we always color with color 0). Now shift the coloring of G_2 by changing y_2's color to q (the desired color of y). Since (in this case) $L(G_2) \supseteq \{1, 4\}$, at least one color from the shifted $L(G_2)$ will be non-zero. Assign this color to x_2 and subsequently to y_1 in G_1. These two colorings can then be combined to a coloring of G. A similar argument can be applied if $\ell_o(G_2) = 1$.

Subcase 4 Suppose $\ell_o \geq 5$ and $\ell_e \geq 4$. Then $L(G) = \{0, 1, 4\}$. We analyze several cases.

If $\ell_o(G_1) = 1$, then $\ell_e(G_1) \geq 4$, since G is triangle-free. First suppose that $\ell_o(G_2) \geq 5$. In this case, $L(G_1) = \{0, 1, 2, 3, 4\}$. If $\ell_e(G_2) \geq 2$, then $L(G_2) = \{0, 1, 4\}$. Color $x_1 = 0$. If $q = 1$, then color y_1 with 2. Color x_2 with 0 and y_2 with 4. When we identify y_1 and x_2, shift $c(x_2)$ to 2, which causes $c(y_2)$ to shift to 1, as desired. If $q = 4$, then color y_1 with 3. Color x_2 with 0 and y_2 with 1. When we identify y_1 and x_2, shift $c(x_2)$ to 3, which causes $c(y_2)$ to shift to 4, as desired. If $q = 0$, color x_2 with a 4 and y_2 with a 0 (by treating y_2 as the "x" terminal of G_2). We can then color x_1 with a 0 and y_1 with a 4, and combine the colorings of G_1 and G_2 to produce the desired coloring of G.

Now suppose that $\ell_o(G_2) = 3$ (and $\ell_o(G_1) = 1$, $\ell_e(G_1) \geq 4$). Then $\ell_e(G_2) \geq 2$ and $L(G_2) = \{1, 2, 3, 4\}$. As above, $L(G_1) = \{0, 1, 2, 3, 4\}$. Color x_1 with 0. If $q = 1$, then color y_1 with 2. Color x_2 with 0 and y_2 with 4. When we identify y_1 and x_2, shift $c(x_2)$ to 2, which causes $c(y_2)$ to shift to 1, as desired. If $q = 4$, then color y_1

with 3. Color x_2 with 0 and y_2 with 1. When we identify y_1 and x_2, shift $c(x_2)$ to 3, which causes $c(y_2)$ to shift to 4, as desired. If $q = 0$, color x_2 with 4 and y_2 with a 0 (by treating y_2 as the "x" terminal of G_2). We can then color x_1 with 0 and y_1 with 4, and combine the colorings of G_1 and G_2 to produce the desired coloring of G.

On the other hand, if $\ell_o(G_1) = 1$, $\ell_e(G_1) \geq 4$, $\ell_o(G_2) \geq 5$ and $\ell_e(G_2) \geq 4$, then $L(G_1) = \{0, 1, 2, 3, 4\}$ and $L(G_2) = \{0, 1, 4\}$. The argument is identical to the case above.

If $\ell_o(G_2) = 1$, then $\ell_e(G_2) \geq 4$ and $\ell_o(G_1) \geq 5$. In this case, $L(G_2) = \{0, 1, 2, 3, 4\}$. If $\ell_e(G_1) = 2$, then $L(G_1) = \{0, 1, 4\}$. And if $\ell_e(G_1) \geq 4$, then $L(G_1) = \{0, 1, 4\}$. Both cases proceed easily, since we have freedom of choice for y_2, allowing us to color $y = y_2$ with q.

We can conclude the proof by considering the case when suppose that $\ell_o(G_1) \geq 3$. We need only consider the most restrictive case (i.e., other cases involve supersets of the L sets of this case), which occurs when $L(G_1) = L(G_2) = \{0, 1, 4\}$. Color x_1 with 0. If $q = 0$, then color x_2 with 4 and y_2 with 0 (by treating y_2 as the "x" terminal of G_2). We can then color x_1 with 0 and y_1 with 4, and combine the colorings of G_1 and G_2 to produce the desired coloring of G. If $q = 1$, then color y_1 with 1. Color x_2 with 0 and y_2 with 0 (there is no defect between x_2 and y_2 in this case, as we are assuming that $\ell_o(G_2) > 1$). When we identify y_1 and x_2, shift $c(x_2)$ to 1, which causes $c(y_2)$ to shift to 1, as desired. If $q = 4$, then color y_1 with 0. Color x_2 with 0 and y_2 with 4. The colorings of G_1 and G_2 can then be combined, as desired. □

We conclude with two more questions.

Question 6 *Can all triangle-free series-parallel graphs be* $(5/2, 1)$ *colored?*

Question 7 *Can all series-parallel graphs be* $(5/2, 2)$ *colored?*

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Acknowledgments

The author wishes to thank Michelle DeDeo and C.-Q. Zhang for helpful discussions and the anonymous referee for many valuable suggestions, among which were the ideas in Section 2.

Appendix

Let G be a planar graph, without loops or bridges, embedded in the plane with vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$.

Denote the degree of a vertex v by $d(v)$ and the degree of a face f (i.e., the length of the boundary of f) by $d(f)$. For a vertex v of graph G, the set of all edges of G incident with v is denoted by $E(v)$.

Definition 1 *At a vertex $v \in V(G)$, let $\{e_1, \ldots, e_{d(v)}\} = E(v)$ where $e_i, e_{i+1} \pmod{d(v)}$ are on the boundary of a face. An* angle *α (at v) of G is a pair of edges $\{e_i, e_{i+1}\}$.*

Denote the set of all angles of G by $\Lambda(G)$. For an angle $\alpha \in \Lambda(G)$ at a vertex v and at a corner of a face f, denote the vertex v by v_α and the face f by f_α. Note that there are $d(v)$ angles at a vertex v and there are $d(f)$ angles at the corners of a face f and each edge appears in four angles and each angle consists of two edges. It is obvious that

$$\begin{aligned}
|V(G)| &= \sum_{\alpha \in \Lambda(G)} \frac{1}{d(v_\alpha)}, \\
|E(G)| &= \sum_{\alpha \in \Lambda(G)} \frac{1}{2}, \\
|F(G)| &= \sum_{\alpha \in \Lambda(G)} \frac{1}{d(f_\alpha)}.
\end{aligned}$$

By Euler's formula,

$$|F(G)| + |V(G)| = |E(G)| + 2,$$

we have the following *Lebesgue's formula*

$$\sum_{\alpha \in \Lambda(G)} \left(\frac{1}{d(v_\alpha)} + \frac{1}{d(f_\alpha)} - \frac{1}{2}\right) = 2. \tag{1}$$

For each angle α, the general term of equation (1)

$$\Phi(\alpha) = \frac{1}{d(v_\alpha)} + \frac{1}{d(f_\alpha)} - \frac{1}{2} \tag{2}$$

is called *the Euler contribution* of the angle α.

Let f be a face of G. Summing the Euler contributions of all angles at all corners of a face f, one obtains the *Euler contribution* of the face f,

$$\Phi(f) = 1 - \frac{d(f)}{2} + \sum \frac{1}{d(v)}, \tag{3}$$

where the sum is over all the vertices on the boundary of f.

For a vertex v, summing the Euler contributions of all angles at v, one obtains the *Euler contribution* of the vertex v,

$$\Phi(v) = 1 - \frac{d(v)}{2} + \sum \frac{1}{d(f)}, \tag{4}$$

where the sum is over all the faces having v on their boundaries.

For an edge $e = v_1v_2$, let f_1, f_2 be two faces incident with e. Note that e appears in four angles and each angle consists of two edges. When one sums one-half of the Euler contributions of all angles containing e, one obtains the *Euler contribution* of the edge e,

$$\Phi(e) = \frac{1}{d(v_1)} + \frac{1}{d(v_2)} + \frac{1}{d(f_1)} + \frac{1}{d(f_2)} - 1. \tag{5}$$

According to Lebesgue's formula (1), we have the total Euler contributions of angles, vertices, faces and edges as

$$\sum_{\alpha \in \Lambda(G)} \Phi(\alpha) = \sum_{v \in V(G)} \Phi(v) = \sum_{f \in F(G)} \Phi(f) = \sum_{e \in E(G)} \Phi(e) = 2. \tag{6}$$

Since the total Euler contributions of a planar graph is two, we have the following lemma.

Lemma 10 (Lebesgue [6]) *Let G be a planar graph without loops and bridges. There must be an angle, a vertex, a face and an edge such that each of their Euler contributions is positive.*

(Received 1/3/2001)