Light paths in large polyhedral maps with prescribed minimum degree

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Abstract

Let k be an integer and $\mathbb{M}$ be a closed 2-manifold with Euler characteristic $\chi(\mathbb{M}) \leq 0$. We prove that each polyhedral map G on $\mathbb{M}$ with minimum degree δ and large number of vertices contains a k-path P, a path on k vertices, such that:

(i) for $\delta \geq 4$ every vertex of P has, in G, degree bounded from above by $6k - 12, k \geq 8$ (It is also shown that this bound is tight for k even and that for k odd this bound cannot be lowered below $6k - 14$);

(ii) for $\delta \geq 5$ and $k \geq 68$ every vertex of P has, in G, a degree bounded from above by $6k - 2\log_2 k + 2$. For every $k \geq 68$ and for every $\mathbb{M}$ we construct a large polyhedral map such that each k-path in it has a vertex of degree at least $6k - 72\log_2(k-1) + 112$.

(iii) The case $\delta = 3$ was dealt with in an earlier paper of the authors (Light paths with an odd number of vertices in large polyhedral maps. Annals of Combinatorics 2(1998), 313-324) where it is shown that every vertex of P has, in G, a degree bounded from above by $6k$ if $k = 1$ or k even, and by $6k - 2$ if $k \geq 3$, k odd; these bounds are sharp.

The paper also surveys previous results in this field.

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1. Introduction

This paper continues the investigations of [7, 8, 9]. Some of the definitions of [7] are repeated.

In this paper all manifolds are compact 2-dimensional manifolds. If a graph G is embedded in a manifold $\mathbb{M}$ then the closure of the connected components of $\mathbb{M} - G$ are called *the faces* of G. If each face is a closed 2-cell and each vertex has valence at least three then G is called a *map* in $\mathbb{M}$. If, in addition, no two faces have a multiply connected union then G is called a *polyhedral map* in $\mathbb{M}$. This condition on the union of two faces is equivalent to saying that any two faces that meet, meet on a single vertex or a single edge. When two faces in a map meet in one of these two ways we say that they *meet properly.*

In the following, let $\mathbb{S}_g$ ($\mathbb{N}_q$) be an orientable (a non-orientable) surface of genus g (genus q) respectively. We say that H is a *subgraph* of a polyhedral map G if H is a subgraph of the underlying graph of the map G.

The degree of a face α of a polyhedral map is the number of edges incident to α. Vertices and faces of degree j are called j-valent vertices and j-valent faces, respectively. Let $v_i(G)$ and $p_i(G)$ denote the number of i-valent vertices and i-valent faces, respectively. For a polyhedral map G let $V(G)$, $E(G)$ and $F(G)$ be the vertex set, the edge set and the face set of G, respectively. The cardinality of the set $V(G)$ is called the order of G. The degree of a vertex A in G is denoted by $\deg_G(A)$ or $\deg(A)$ if G is known from the context. A path and a cycle on k vertices is defined to be the *k-path* and the *k-cycle*, respectively. The length $\rho(p)$ and $\rho(C)$ of a path p and a cycle C, respectively, is the number of its edges. A k-path passing through vertices $A_1, \dots, A_k$ is denoted by $[A_1, A_2, \dots, A_k]$ provided that $A_iA_{i+1} \in E(G)$ for any $i = 1, 2, \dots, k-1$.

It is an old classical consequence of the famous Euler's formula that each planar graph contains a vertex of degree at most 5. A beautiful theorem of Kotzig [11, 12] states that every 3-connected planar graph contains an edge with degree-sum of its endvertices being at most 13. This result was further developed in various directions and served as a starting point for discovering many structural properties of embeddings of graphs, see e.g. [1, 4, 5, 7, 8, 13].

Recently the following problem has been investigated.

Problem 1. *For a given connected graph H let $\mathcal{G}(H,\mathbb{M})$ be the family of all polyhedral maps on a closed 2-manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M})$ having a subgraph isomorphic with H. What is the minimum integer $\phi(H,\mathbb{M})$ such that every polyhedral map $G \in \mathcal{G}(H,\mathbb{M})$ contains a subgraph K isomorphic with H for which*

$$\deg_G(A) \le \phi(H,\mathbb{M}) \text{ for every vertex } A \in V(K)?$$

If such a minimum does not exist we write $\phi(H,\mathbb{M}) = \infty$. If such a minimum exists H is called *light.*

The answer to this question for $\mathbb{S}_0$ and $\mathbb{N}_1$ is given in Theorem 1; the answer for each 2-manifold other then $\mathbb{S}_0$ and $\mathbb{N}_1$ is given in Theorem 2.

Theorem 1. (Fabrici and Jendrol', [1]) *Let k be an integer, $k \ge 1$. Then*

$$\phi(P_k, \mathbb{S}_0) = \phi(P_k, \mathbb{N}_1) = 5k, \quad \textit{for any } k \geq 1$$

$$\phi(H, \mathbb{S}_0) = \phi(H, \mathbb{N}_1) = \infty, \ \textit{for any } H \neq P_k.$$

Theorem 2. (Jendrol' and Voss, [7]) *Let k be an integer, $k \geq 1$, and $\mathbb{M}$ be a closed 2-manifold with Euler characteristic $\chi(\mathbb{M}) \notin \{1, 2\}$. Then*

(i) $\phi(P_1, \mathbb{M}) \leq \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor$.

(ii) $2 \left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor \leq \phi(P_k, \mathbb{M}) \leq k \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor$, $k \geq 2$.

(iii) $\phi(H, \mathbb{M}) = \infty$, *for any* $H \neq P_k$.

In Theorem 2 the upper bound is sharp for even k.

For odd $k \geq 3$ the behaviour of $\phi(P_k, \mathbb{M})$ has been investigated in [10]. If $\mathbb{M}$ is the torus $\mathbb{S}_1$ or Klein's bottle $\mathbb{N}_2$ then Theorem 2 implies:

$$\phi(P_k, \mathbb{S}_1) = \phi(P_k, \mathbb{N}_2) = 6k \text{ if } k \text{ is even, and}$$
$$6k - 6 \leq \phi(P_k, \mathbb{S}_1), \phi(P_k, \mathbb{N}_2) \leq 6k, \text{ if } k \geq 3 \text{ is odd.}$$

The exact result is

Theorem 3. (Jendrol' and Voss, [9]) *Let k be an integer, $k \geq 1$. Then*

$$\phi(P_k, \mathbb{S}_1) = \phi(P_k, \mathbb{N}_2) = \begin{cases} 6k, & \textit{if } k = 1 \textit{ or } k \textit{ is even}, \\ 6k - 2 & \textit{if } k \textit{ is odd}, k \geq 3. \end{cases}$$

This result is also valid for polyhedral maps on 2-manifolds $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M}) < 0$, if these maps have enough vertices. Thus the following problem has been investigated.

Problem 2. Let $N \geq 1$ be an integer. For a given connected graph H let $\mathcal{G}_N(H, \mathbb{M})$ be the family of all polyhedral maps of order $\geq N$ on a closed 2-manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M})$ having a subgraph isomorphic with H. What is the minimum integer $\phi_N(H, \mathbb{M})$ such that every polyhedral map $G \in \mathcal{G}_N(H)$ contains a subgraph K isomorphic with H for which

$$\deg_G(A) \leq \phi_N(H, \mathbb{M}) \text{ for every vertex } A \in V(K)?$$

Obviously, $\phi_1(H, \mathbb{M}) = \phi(H, \mathbb{M})$.

Let N_k denote the largest number of vertices in a connected graph with maximum degree $\leq 6k$ containing no path with k vertices. Obviously, $N_k \leq (6k)^{k/2+2}$.

A solution of Problem 2 gives

Theorem 4. (Jendrol' and Voss, [9]) *For any* 2*-manifold* $\mathbb{M}$ *with Euler characteristic* $\chi(\mathbb{M}) < 0$, *any integer* $k \geq 1$ *and any integer* $N > 30000\ (|\chi(\mathbb{M})| + 1)^3(N_k + 3(|\chi(\mathbb{M})| + 1))$,

(i) $\phi_N(P_k, \mathbb{M}) = \begin{cases} 6k, & \text{if } k = 1 \text{ or } k \text{ is even} \\ 6k - 2, & \text{if } k \geq 3 \text{ is odd.} \end{cases}$

(ii) $\phi_N(H, \mathbb{M}) = \infty$ *for any* $H \neq P_k$.

In this paper we shall investigate the subclasses which contain all graphs of $\mathcal{G}_N(H, \mathbb{M})$ with a given minimum degree $\delta, \delta \geq 3$.

Problem 3. Let $N \geq 1$ be an integer. For a given connected graph H let $\mathcal{G}_N(\delta, H, \mathbb{M})$ be the family of all polyhedral maps of minimum degree $\geq \delta$ and order $\geq N$ on a closed 2-manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M})$ having a subgraph isomorphic with H. What is the minimum integer $\phi_N(\delta, H, \mathbb{M})$ such that every polyhedral map $G \in \mathcal{G}_N(\delta, H, \mathbb{M})$ contains a subgraph K isomorphic with H for which

$$\deg_G(A) \leq \phi_N(\delta, H, \mathbb{M}) \text{ for every vertex } A \in V(K)?$$

Let $\phi_N(\delta, H, \mathbb{M}) := \infty$ if such a bound does not exists, and $\phi(\delta, H, \mathbb{M}) := \phi_1(\delta, H, \mathbb{M})$. Obviously, $\phi(H, \mathbb{M}) = \phi_1(3, H, \mathbb{M})$ and $\phi_N(H, \mathbb{M}) = \phi_N(3, H, \mathbb{M})$. Large graphs of $\mathcal{G}_N(\delta, H, \mathbb{M})$ with $\delta \geq 7$ do not exist, i.e., $\mathcal{G}_N(7, H, \mathbb{M}) = \emptyset$ for large N.

The case $\delta = 3$ has been dealt with in Theorems 1–4. For $\delta = 4$ it is known

Theorem 5. (Fabrici, Hexel, Jendrol' and Walther, [2]) *Let* k *be an integer,* $k \geq 1$. *Then*

(a) $\phi(4, P_1, \mathbb{S}_0) = 5, \phi(4, P_2, \mathbb{S}_0) = 7, \phi(4, P_3, \mathbb{S}_0) = 9, \phi(4, P_4, \mathbb{S}_0) = 15,$
$\phi(4, P_5, \mathbb{S}_0) = 19, \phi(4, P_6, \mathbb{S}_0) = 23, \phi(4, P_7, \mathbb{S}_0) = 27;$

(b) $\phi(4, P_k, \mathbb{S}_0) = 5k - 7$ *for* $k \geq 8$;

(c) $\phi(4, H, \mathbb{S}_0) = \infty$ *for every connected planar graph* $H \neq P_k (k \geq 1)$.

In a forthcoming paper we shall show that large triangulations of minimum degree ≥ 5 on compact 2-manifolds $\mathbb{M}$ contain light triangles, light 4-cycles with one inner chord, and 5-cycles with two inner chords. Here we shall prove a generalization of Theorem 5 to large polyhedral graphs on compact 2-manifolds $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M}) \leq 0$.

Theorem 6. *Let* $\mathbb{M}$ *be a compact* 2*-manifold of Euler characteristic* $\chi(\mathbb{M}) \leq 0$, *and let* $N > 30000(|\chi(\mathbb{M})| + 1)^3 \cdot (N_k + 3(|\chi(\mathbb{M})| + 1))$ *be an integer. Then*

$$\phi_N(4, P_k, \mathbb{M}) = 6k - 12 \text{ for all even } k \geq 8$$

$$6k - 14 \leq \phi_N(4, P_k, \mathbb{M}) \leq 6k - 12 \text{ for all odd } k \geq 9.$$

Theorem 7. *Let* k *be an integer. Then*

$$5k - 235 \leq \phi(5, P_k, \mathbb{S}_0) \leq 5k - 7 \text{ for all } k \geq 68.$$

Theorem 8. *For any 2-manifold* $\mathbb{M}$ *with Euler characteristic* $\chi(\mathbb{M}) \leq 0$*, any integer* $k \geq 66$*, and any integer* $N > 30000(|\chi(\mathbb{M})|+1)^3 \cdot (N_k + 3(|\chi(\mathbb{M})|+1))$*,*

$$6k - 72\log_2(k-1) + 112 \leq \phi_N(5, P_k, \mathbb{M}) \leq 6k - \log_2 k + 2.$$

We can even prove:

Corollary 8.1.

$$\phi_N(5, P_k, \mathbb{M}) \leq 6k - 2\log_2 k + 2, \quad k \geq 68.$$

An obvious assertion is Theorem 9 (it can be proved in a similar way as Lemma 9).

Theorem 9. *For each integer* $k \geq 1$ *there exists an integer* $N = N(k)$ *so that*

$$\phi_N(6, P_k, \mathbb{M}) = 6.$$

2. Minimum degrees of graphs on $\mathbb{M}$

In this paper $\chi(\mathbb{M}) \leq 0$. Let G be a graph embedded in a compact 2-dimensional manifold $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M})$. If G is a map, i.e. each face is a 2-cell then G fulfils Euler's formula

$$n - e + f = \chi(\mathbb{M}),$$

where

$$\chi(\mathbb{M}) = \begin{cases} 2(1-g) & \text{if } \mathbb{M} = \mathbb{S}_g, \\ 2-q & \text{if } \mathbb{M} = \mathbb{N}_q. \end{cases}$$

If G contains a face F which is not a 2-cell than add an edge to its interior so that F is not subdivided. Add edges in this way until a 2-cell embedding is obtained. Let e^* denote the number of these edges then Euler's formula is fulfilled with

$$n - (e + e^*) + f = \chi(\mathbb{M}),$$

where n, e and f denote the number of vertices, edges and faces of G, respectively. We summarize this in

Lemma 1. *Let* G *be the embedding of a graph in a compact* 2*-dimensional manifold* $\mathbb{M}$ *of Euler characteristic* $\chi(\mathbb{M})$*. Let* e^* *denote the maximum number of edges which can be added to* G *without changing the number of its faces (loops and multiple edges can be added). Then the Euler sum is*

$$n - e + f = \chi(\mathbb{M}) + e^*,$$

where n, e *and* f *denote the number of vertices, edges and faces of* G*, respectively.*

Lemma 2. *Let G be the embedding of a simple graph with minimum degree $\delta(G) \geq 2$ in a compact 2-dimensional manifold $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M})$. Let e^* denote the maximum number of edges which can be added to G without changing the number of its faces. Then $p_0 = p_1 = p_2 = 0$, and the number of edges of G is*

$$e \leq 3(n + |\chi(\mathbb{M})| - e^*).$$

Proof. By Lemma 1 we have

$$n - e + f = \chi(\mathbb{M}) + e^* \tag{1}$$

On the boundary of each face F a vertex, say B lies. Since $\delta(G) \geq 2$ and the graph G is simple at least two edges incident with B belong to F. For the endvertices of these edges different from B the same is true. Hence F is bounded by at least three edges of G. $p_0 = p_1 = p_2 = 0$, and

$$3f \leq 2e. \tag{2}$$

The formulas (1) and (2) imply

$$3(\chi(\mathbb{M}) + e^*) = 3n - 3e + 3f \leq 3n - 3e + 2e,$$

and

$$e \leq 3(n + |\chi(\mathbb{M})| - e^*). \quad \square$$

Lemma 3. *Let G be the embedding of a simple graph in a compact 2-dimensional manifold $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M})$. Let e^* denote the maximum number of edges which can be added to G without changing the number of faces. If $e^* > |\chi(\mathbb{M})|$ then G has minimum degree $\delta(G) \leq 5$.*

Proof. Assume that $\delta(G) \geq 6$ for some embedding G on $\mathbb{M}$. Then

$$2e = \sum_{X \in V(G)} \deg_G(X) \geq 6n.$$

By Lemma 2 we have

$$e \leq 3(n + |\chi(\mathbb{M})| - e^*).$$

The assumption $e^* > |\chi(\mathbb{M})|$ implies

$$6n \leq 2e \leq 6(n + |\chi(\mathbb{M})| - e^*),$$

and

$$0 \leq 6(|\chi(\mathbb{M})| - e^*) < 0.$$

This contradiction proves the lemma. $\square$

Lemma 4. *Let G be the embedding of a graph in a compact 2-dimensional manifold of Euler characteristic $\chi(\mathbb{M})$. Let e^* denote the maximum number of edges, which can be added to G without changing the number of faces of G. Then*

$$\sum_{j\geq 0}(6-j)v_j + 2\sum_{j\geq 0}(3-j)p_j = 6(\chi(\mathbb{M}) + e^*).$$

Proof. By Lemma 1 we have

$$n - e + f = \chi(\mathbb{M}) + e^*.$$

With $2e = \sum_{j\geq 0} jv_j = \sum_{j\geq 0} jp_j$, $n = \sum_{j\geq 0} v_j$, and $f = \sum_{j\geq 0} p_j$ the assertion of the lemma is true. □

3. Proof of Theorem 8 – The upper bound

The proof follows the ideas of [1] and [8]. Suppose that there is a counterexample to our Theorem 9 having $n > 3 \cdot 10^4(|\chi(\mathbb{M})| + 1)^3 \cdot (N_k + 3(|\chi(\mathbb{M}| + 1))$ vertices, $k \geq 66$. Let G be a counterexample with the maximum number of edges among all counterexample having n vertices. A vertex A of the graph G is *major (minor)* if $\deg_G(A) > 6k - \lfloor \log_2 k \rfloor + 2$ ($\leq 6k - \lfloor \log_2 k \rfloor + 2$, respectively). Note that each path on k vertices in G contains a major vertex.

Lemma 5. *Every v-valent face α, $v \geq 4$, of G is incident only with minor vertices.*

Proof. Suppose there is a major vertex B incident with an v-valent face α, $v \geq 4$. Let C be a diagonal vertex on α with respect to B. Because G is a polyhedral map we can insert an edge BC into the v-valent face α. The resulting embedding is again a counterexample but with one edge more, a contradiction. □

Each path with k vertices contains a major vertex.

Let H be the subgraph of G induced on the major vertices of G.

Lemma 6. *The minimum degree of H is $\delta(H) \geq 6$.*

Proof. Assume that H contains a vertex A of degree $\deg_H(A) \leq 5$. On the other hand A is a major vertex in G, so the degree of A in G is $\deg_G(A) \geq 6k - 1$. Because of Lemma 5 the subgraph of G induced on the set of vertices consisting of A and its neighbours contains a wheel of length $\deg_G(A)$. The major vertices of the cycle of the wheel partition the minor vertices of this cycle into $\deg_H(A) \leq 5$ paths, and one of these paths has a length

$$\geq \left\lceil \frac{\deg_G(A) - \deg_H(A)}{\deg_H(A)} \right\rceil \geq \left\lceil \frac{\deg_G(A) - 5}{\deg_H(A)} \right\rceil \geq \left\lceil \frac{6k - 1 - 5}{5} \right\rceil = k + \left\lceil \frac{k-6}{5} \right\rceil \geq k.$$

This contradiction proves Lemma 6. □

Lemma 7. $\sum_{j>6}(j-6)v_j(H) + 2\sum_{j>3}(j-3)p_j(H) \leq 6|\chi(\mathbb{M})|$.

Proof. The subgraph H induced by the major vertices of G is possibly not a 2-cell embedding in $\mathbb{M}$. Thus $e^* \geq 0$ edges have to be successively added so that the number of faces remains unchanged, and a 2-cell embedding is obtained. Lemmas 3 and 6 imply

$$0 \leq e^* \leq |\chi(\mathbb{M})|, \tag{1}$$

and with Lemma 4

$$\sum_{j\geq 0}(6-j)v_j(H) + 2\sum_{j\geq 0}(3-j)p_j(H) = 6(\chi(\mathbb{M}) + e^*).$$

By Lemmas 6 and 2 we have $p_0(H) = p_1(H) = p_2(H) = 0$ and $v_j(H) = 0$, $j = 0, 1, 2, \ldots, 5$. This implies $\sum_{j>6}(6-j)v_j(H)+2\sum_{j>3}(3-j)p_j(H) = 6(\chi(\mathbb{M})+e^*)$. $\chi(\mathbb{M}) + e^*$ ranges between 0 and $-|\chi(\mathbb{M})|$, and $\sum_{j>6}(j-6)v_j(H) + 2\sum_{j>3}(j-3)$ $p_j(H) \leq 6|(\chi(\mathbb{M})|$. □

Let H' denote the subgraph of G generated by the minor vertices.

Lemma 8. *The subgraph H induced by the major vertices of G has $n(H)$ vertices, where*

$$n(H) > 15000(|\chi(M)| + 1)^3 - |\chi(M)|.$$

Proof. By the maximality of G each face F of H contains no or precisely one component K of H'. This component K has $\leq N_k$ vertices because it contains no path P_k on k vertices. By Lemma 7 each face F of H is bounded by $\leq 3(|\chi(\mathbb{M})|+1)$ vertices. Hence in each face F and its boundary lie $\leq N_k + 3(|\chi(\mathbb{M})| + 1)$ vertices of G. A lower bound for the number $f(H)$ of faces of H is obtained by dividing the number n of vertices of G by an upper bound for the number of vertices of G lying in the interior or on the boundary of a face. Therefore,

$$f(H) \geq \frac{n}{N_k + 3(|\chi(\mathbb{M})| + 1)}. \tag{2}$$

By Lemma 2 each face of H is bounded by at least three edges, and

$$3f(H) \leq 2e(H) \leq \sum_{j\geq 6} jv_j(H) = \sum_{j\geq 6}(j-6)v_j(H) + \sum_{j\geq 6} 6v_j(H).$$

The sum $\sum_{j\geq 6} 6v_j(H) = 6n(H)$, and by Lemma 7 the sum $\sum_{j\geq 6}(j-6)v_j(H) \leq 6|\chi(\mathbb{M})|$. Consequently,

$$3f(H) \leq 6(|\chi(\mathbb{M}) + n(H)). \tag{3}$$

Finally we get with (2)

$$n(H) \geq \frac{1}{2}f(H) - |\chi(\mathbb{M})| \geq \frac{n}{2}(N_k + 3(|\chi(\mathbb{M})| + 1))^{-1} - |\chi(\mathbb{M})|.$$

With $n > 30000(|\chi(\mathbb{M})| + 1)^3(N_k + 3(|\chi(\mathbb{M})| + 1))$ we obtain

$$n(H) \geq 15000(|\chi(\mathbb{M})| + 1)^3 - |\chi(\mathbb{M})|. \quad \square$$

A face is said to be a triangle if it is a 2-cell bounded by a 3-cycle.

Lemma 9. *The subgraph H contains a vertex X with the property: X and all vertices Z having a distance at most three from X have degree* 6 *and are incident only with triangles.*

Proof. By Lemma 7 we have

$$\sum_{j>6}(j-6)v_j(H) + 2\sum_{j>3}(j-3)p_j(H) \leq 6|\chi(\mathbb{M})|.$$

The largest vertex degree is $\leq 6(|\chi(\mathbb{M})|+1)$ and the number of d-valent vertices, $d>6$, is $\leq 6|\chi(\mathbb{M})|$. The largest face size is $\leq 3(|\chi(\mathbb{M})|+1)$, the number of d-valent faces, $d>3$, is $\leq 3|\chi(\mathbb{M})|$ and by (1) the number of faces which are not 2-cells is $\leq |\chi(\mathbb{M})|$.
Let M_0 denote the set of vertices having a degree >6, or lying on a face of size >3, or lying on a face which is no 2-cell. All vertices outside M_0 have degree 6 (see Lemma 6) and are only incident with triangles. The number of vertices of M_0 is bounded by

$$\begin{aligned}|M_0| &\leq 6|\chi(\mathbb{M})| + (3|\chi(\mathbb{M})| + |\chi(\mathbb{M})|)\cdot 3(|\chi(\mathbb{M})|+1)\\ |M_0| &\leq 20|\chi(\mathbb{M})|(|\chi(\mathbb{M})|+1).\end{aligned}$$

Let M_i denote the number of vertices having from M_0 distance i. Since the maximum degree of the vertices of M_0 is at most $6(|\chi(\mathbb{M})|+1)$ and all other vertices have degree 6 we have $|M_1| \leq |M_0|6(|\chi(\mathbb{M})|+1)$, $|M_2| \leq |M_1|\cdot 5$, $|M_3| \leq |M_2|\cdot 5$. This implies

$$\sum_{j=0}^{3}|M_i| \leq 4000|\chi(M)|(|\chi(M)|+1)^2.$$

By Lemma 8 the number of vertices is

$$n(H) > 15000(|\chi(M)|+1)^3 - |\chi(M)|.$$

Hence $\bigcup_{j=0}^{3} M_i$ does not contain all vertices of H, and H contains a vertex X having a distance at least four from M_0. So X has the required properties. □

Next we study more precisely the properties of the components of the subgraph H' of G induced by the minor vertices of G.

Lemma 10. *Each triangle D of H contains a vertex $V \in V(H)$ which is adjacent only with $< k - \lfloor \log_2 k \rfloor + 2$ minor vertices lying in D.*

Proof. Assume the contrary, i.e., there exists a triangle $[P,Q,R]$ of H such that each of its vertices is joint with $\geq k - \log k + 2$ minor vertices inside of $[P,Q,R]$.

Let K denote the subgraph of G induced by the minor vertices of G lying in the interior of $[P,Q,R]$. Since G is a maximal counterexample K is a component of the subgraph H' of G induced by the minor vertices of G. By Lemma 5 the vertex P and all its neighbours induce a wheel W_P. Correspondingly Q and R

are the naves of a wheel W_Q and W_R, respectively. Let p, q and r denote the path of $W_P \cap K$, $W_Q \cap K$, and $W_R \cap K$, respectively. Then p, q and q, r and r, p have a common endvertex Q', R', and P', respectively (a sketch of the situation is depicted in Fig. 1). Let p^*, q^* and r^* denote the longest $P'Q'$-path, $Q'R'$-path and $R'P'$-path, respectively. p and q have at least one second common vertex because otherwise $p \cup q$ would form a $P'R'$-path with $\geq 2(k - \log k + 2) \geq k$ vertices. Each common vertex V of p and q and the vertices P and Q induce a separating path PVQ of the subgraph of G induced by $[P, Q, R] \cup K$. Therefore by walking on p or q from Q' to the other end P' or R', respectively, the common vertices appear on p and q in the same order.

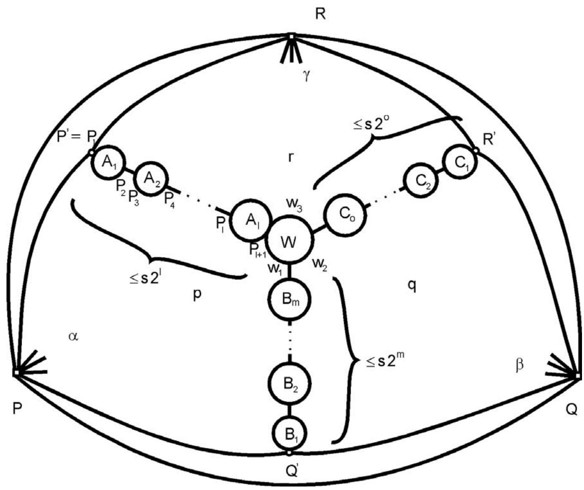

FIG. 1

Let $P_1 = P'$, $P_2, \ldots, P_{l+1}$, $l \geq 1$, be the common vertices of p and q. Between P_i and P_{i+1}, $1 \leq i \leq l$, lies a block A_i of K.

Correspondingly, let $Q_1 = Q'$, $Q_2, \ldots, Q_{m+1}$, $m \geq 1$, and $R_1 = R'$, $R_2, \ldots, R_{o+1}$, $o \geq 1$, be the common vertices of q and r or r and p, respectively. Between Q_i and Q_{i+1}, $1 \leq i \leq m$, and R_j and R_{j+1}, $1 \leq j \leq o$, lies a block B_i or C_j of K, respectively. The intersection $V(p) \cap V(q) \cap V(r)$ is either empty or contains precisely one vertex $P_{l+1} = Q_{m+1} = R_{o+1}$. In the first case P_{l+1}, Q_{m+1} and R_{o+1} are pairwise distinct vertices of a block, say W (for this case see Fig. 1).

We need the concept of an *H-bridge*. Let H be an arbitrary subgraph of a graph G, $H \neq G$. There are two types of H-bridges $\mathcal{L}$. Firstly, $\mathcal{L} \cong K_2$, and the only edge of $\mathcal{L}$ is not in H, but it joins two vertices of H. Secondly, $\mathcal{L}$ is obtained from a component K of $G \setminus H$ by adding all K, H-edges and all endvertices of such edges.

The vertices of $H \cap \mathcal{L}$ are called the vertices of attachment of $\mathcal{L}$.

Next we show

(1)
If A_i is a block with at least two edges then $1 \leq \rho(p[P_i, P_{i+1}]) \leq \rho(p^*[P_i, P_{i+1}]) - 1$.

Proof of (1). For convenience let $w := p[P_i, P_{i+1}]$.

Since $A_i \ncong K_2$ and A_i is 2-connected there exists at least one w-bridge; each w-bridge has at least two vertices of attachment (Note: a w-bridge can be a K_2).

Let $\mathcal{L}$ be a w-bridge so that the partial path of w between two vertices of attachment has smallest length. Let A and A' be these two vertices of attachment.

If the partial path $w[A, A']$ has a length ≥ 2 then each inner vertex of $w[A, A']$ has degree 3 in G – a contradiction(since $\delta(G) \geq 5$). Hence A and A' are neighbours on w, $\mathcal{L} \ncong K_2$, and each A, A'-path of $\mathcal{L}$ has a length ≥ 2. Replacing in w the edge (A, A') by an A, A'-path of $\mathcal{L}$ meeting no attaching vertex different from A and A' we obtain a P_i, P_{i+1}-path v of A_i of length $> \rho(w)$. Consequently, $\rho(p[P_i, P_{i+1}]) = \rho(w) \leq \rho(v) - 1 \leq \rho(p^*[P_i, P_{i+1}]) - 1$. □

Correspondingly, it can be proved

(2) *If B_i or C_i is a block with at least two edges then*

$$\rho(q[Q_i, Q_{i+1}]) \leq \rho(q^*[Q_i, Q_{i+1}]) - 1, \textit{ and}$$
$$\rho(r[R_j, R_{j+1}]) \leq \rho(r^*[R_j, R_{j+1}]) - 1.$$

Moreover,

$$\rho(p[P_{l+1}, Q_{m+1}]) \leq \rho(p^*[P_{l+1}, Q_{m+1}]) - 1,$$
$$\rho(q[Q_{m+1}, R_{o+1}]) \leq \rho(q^*[Q_{m+1}, R_{o+1}]) - 1, \textit{ and}$$
$$\rho(r[R_{o+1}, P_{l+1}]) \leq \rho(r^*[R_{o+1}, P_{l+1}]) - 1.$$

Let $s = \lfloor \log_2 k \rfloor - 2$. Then each of the vertices P, Q, and R is adjacent to at least $k - s$ vertices of K, i.e., $\rho(p) \geq k - s - 1$, $\rho(q) \geq k - s - 1$, and $\rho(r) \geq k - s - 1$. We consider all blocks touching p, i.e., we consider the chain of blocks $P_1 A_1 P_2 A_2 \dots A_l P_{l+1} W B_{m+1} Q_m B_m \dots\ Q_2 B_2 Q_1$ where W is empty if $P_{l+1} = B_{m+1}$. Let

$$V_1 D_1 V_1' V_2 D_2 V_2' V_3 \dots V_{\alpha-1} D_{\alpha-1} V_{\alpha-1}' V_\alpha D_\alpha V_\alpha'$$

be a new notation of this chain, where $D_1, \dots,\ D_\alpha$ are the blocks with at least two edges, and $V_i' = V_{i+1}$ if D_i and D_{i+1} have a common cut vertex or $V_i' \neq V_{i+1}$ and D_i and D_{i+1} are joined by a path of length 1 or 2 consisting of one or two K_2-blocks (The latter case is only possible if $P_{l+1} = Q_{m+1}$). Since D_i is 2-connected, it is bounded by an outer cycle C_i. Let $d_i := p[V_i, V_i']$ and $d_i' := C_i \backslash (p[V_i, V_i'] \backslash \{V_i, V_i'\})$. Thus $d_i \cup d_i' = C$ and $V(d_i) \cap V(d_i') = \{V_i, V_i'\}$.

By assumption

(3) $$\rho(p), \rho(q), \rho(r) \geq k - s - 1.$$

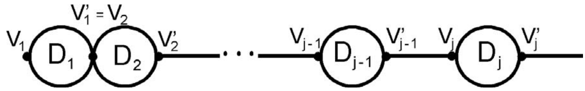

FIG. 2

Let p^* denote a longest V_1, V'_α-path. Then $p^*[V_i, V'_i]$ is a longest V_i, V'_i-path in the block D_i. The path $(d'_j \backslash \{V'_j\}) \cup d_j \cup p^*[V'_j, V'_\alpha]$ of K, $1 \leq j \leq \alpha$, has a length $\leq k-2$ (see Fig. 2). By (3) and (1) with $V_{\alpha+1} = V'_\alpha$ we have:

$$k - s - 1 \leq \rho(p) = \sum_{i=1}^{\alpha} \rho(p[V_i, V_{i+1}])$$
$$\leq \sum_{i=1}^{\alpha} (\rho(p^*[V_i, V_{i+1}]) - 1) = \rho(p^*) - \alpha \leq (k-1) - \alpha.$$

Hence

(4) $$\alpha \leq s.$$

By (1) the length of the partial path $\rho(p^*[V'_j, V'_\alpha]) \geq \rho(p[V'_j, V'_\alpha]) + \alpha - j$. With (3) these conditions imply

$$\begin{aligned} k - 2 &\geq \rho(d'_j \backslash \{V'_j\}) \cup d_j \cup p^*[V'_j, V'_\alpha] \\ &= \rho(d'_j) - 1 + \rho(d_j) + \rho(p^*[V'_j, V'_\alpha]) \\ &\geq \rho(d'_j) - 1 + \rho(p[V_j, V'_j]) + \rho(p[V'_j, V'_\alpha]) + \alpha - j \\ &= \rho(d'_j) - 1 - \rho(p[V_1, V_j]) + \rho(p) + \alpha - j \\ &\geq \rho(d'_j) - 1 - \rho(p[V_1, V_j]) + (k - s - 1) + \alpha - j \\ &= \rho(d'_j) - \rho(p[V_1, V_j]) + k - s + \alpha - j - 2. \end{aligned}$$

Hence
$$\rho(d'_j) \leq \rho(p[V_1, V_j]) + s - \alpha + j.$$

This implies

(5) $$\rho(d'_j) \leq \rho(p[V_1, V_j]) + (s+1) - j \quad \textit{for all } 1 \leq j \leq \left\lfloor \frac{\alpha+1}{2} \right\rfloor$$

With (5) we shall prove

(6) $$\rho(d_j), \rho(d'_j), \rho(d_{\alpha+1-j}), \rho(d'_{\alpha+1-j}) \leq (s+1)2^{j-1} \quad \text{for all } 1 \leq j \leq \lfloor \frac{\alpha+1}{2} \rfloor.$$

Proof of (6). By induction on j.

Case 1. Let $D_j \neq W$. For $j = 1$ the validity of (5) is implied by (4):

$$\rho(d'_1) \leq \rho(p[V_1, V_1]) + (s+1) - 1 \leq (s+1).$$

For $j \geq 2$ by (4) it holds

$$\rho(d'_j) \leq (\sum_{i=1}^{j-1} \rho(d_i) + j) + (s+1) - j = \sum_{i=1}^{j-1} \rho(d_i) \ + (s+1).$$

In the latter case the induction hypothesis implies:

$$\rho(d'_j) \leq (s+1) \sum_{i=1}^{j-1} 2^{i-1} \ + (s+1) = (s+1)2^{j-1}.$$

Case 2. Let $D_j = W$. Then we have the situation depicted in Fig. 3.

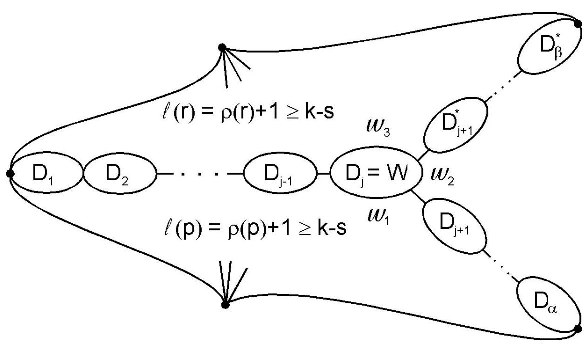

FIG. 3

By the arguments of Case 1 we arrive at $\rho(w_2) + \rho(w_3) = \rho(d'_j) \leq (s+1)2^{j-1}$ and $\rho(w_1) < \rho(w_1) + \rho(w_2) = \rho(d^{*\prime}_j) \leq (s+1)2^{j-1}$. Hence also in Case 2 the proof of (6) for d'_j is complete.

$\rho(d_j) \leq (s+1)2^{j-1}$ can be proved by repeating the proof with the path r. $\rho(d'_{\alpha+1-j}), \rho(d_{\alpha+1-j}) \leq (s+1)2^{j-1}$ can be proved by reversing the block chain $V'_\alpha D_\alpha V_\alpha V'_{\alpha-1} D_\alpha V_\alpha \dots$.

With (6) and $\alpha \leq s$, see(4), the length of p is

$$k - s - 1 \leq \rho(p) \leq 2 \sum_{j=1}^{\lfloor \frac{s+1}{2} \rfloor} (s+1)2^{j-1}$$
$$= 2(s+1)(2^{\lfloor \frac{s+1}{2} \rfloor - 1} - 1).$$

With $s+1 \le 2^{\frac{s+1}{2}}$ for $s = \lfloor \log_2 k \rfloor - 2 \ge 3$ we obtain

$$k \le 2^{s+1} = 2^{\lfloor \log_2 k \rfloor - 1} \le \frac{k}{2}.$$

This contradiction proves Lemma 10.

Next the proof of Theorem 8 will be completed. Lemma 9 implies the existence of a triangle D of H whose vertices have degree 6. By Lemma 10 the triangle D contains a vertex, say P, which is adjacent only with $< k - \lfloor \log_2 k \rfloor + 2$ minor vertices lying in D. In all other triangles adjacent with P the vertex P is joint with $\le k-2$ minor vertices. Hence the major vertex P has a degree

$$\begin{aligned}\deg_G(P) &< \deg_H(P) + (\deg_H(P) - 1)(k-1) + (k - \lfloor \log_2 k \rfloor + 2) \\ &= \deg_H(P) \cdot k - \lfloor \log_2 k \rfloor + 2 \\ &\le 6 \cdot k - \lfloor \log_2 k \rfloor + 2.\end{aligned}$$

This contradicts our assumption that each major vertex has a degree greater than $6 \cdot k - \lfloor \log_2 k \rfloor + 2$.
This contradiction completes the proof of the upper bound of Theorem 8.

It can be proved that there is a major vertex Q incident with two triangles D and D' of H such that Q is incident with $< k - \lfloor \log_2 k \rfloor + 2$ minor vertices in two triangles which proves the validity of the Corollary 8.1 related to Theorem 8.

4. Proof of Theorem 6 - The upper bound

The proof follows the ideas of [1] and [8]. Suppose that there is a counterexample to our Theorem 6 having $n > 3 \cdot 10^4 (|\chi(\mathbb{M})| + 1)^3 \cdot (N_k + 3(|\chi(\mathbb{M})| + 1))$ vertices, $k \ge 8$. Let G be a counterexample with the maximum number of edges among all counterexamples having n vertices. A vertex A of the graph G is *major* (*minor*) if $\deg_G(A) > 6k - 12$ ($\le 6k - 12$, respectively).

The proof follows the ideas of section 3.

First an analogue to Lemma 10 will be proved.

Lemma 11. *In any triangle D of H each vertex V is adjacent only with $\le k-2$ minor vertices lying in the interior of D. If one vertex is incident with $k-2$ minor vertices then one of the other vertices of D is incident with precisely one minor vertex in the interior of D.*

Proof. Assume the contrary, i.e., there exists a triangle $D = [P, Q, R]$ of H such that P is joined with $k-1$ minor vertices of the interior of $[P, Q, R]$.

The notation of the proof of Lemma 10 is used again. The path p of all minor neighbours P in the interior of D belongs to a chain of blocks

$$P' = P_1 A_1 P_2 A_2 \dots A_l P_{l+1} W B_{m+1} Q_{m+1} B_m \dots B_2 Q_2 B_1 Q_1 = Q'.$$

Assertion (1) of the proof of Lemma 10 is again valid. Hence by (1) all blocks A_i and B_j are one–edge blocks K_2 (the part W consists of two one–edge blocks, or is only one vertex). Since both vertices P_1 and Q_1 cannot be joint with all three vertices of $[P, Q, R]$ at least one of these vertices, say Q_1, is joint only with P and Q. Hence Q_1 has degree $\deg_G(Q_1) = 3$, a contradiction. This contradiction proves the validity of the first assertion of Lemma 11. Next let p have precisely $k-2$ vertices. Then B_1 is no one–edge block K_2 but all other blocks of the chain are one–edge blocks K_2 (see Fig. 4 with $B_1 \cong K_4^-$, where K_4^- denotes the complete graph on four vertices with one missing edge). Further $P_1 = P'$ is joint with all three vertices P, Q, R. Consequently, the vertex R has precisely one minor neighbour in the interior of $[P, Q, R]$.

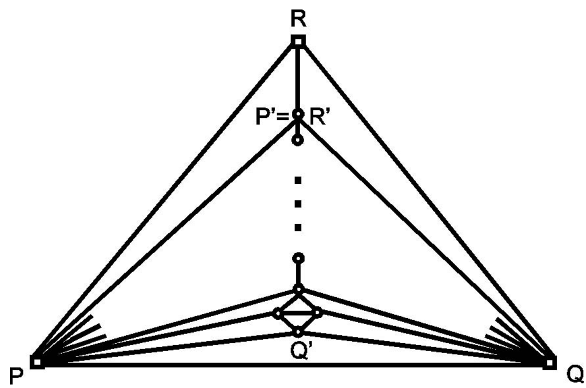

FIG. 4

This completes the proof of Lemma 11.

With Lemma 11 we will complete the proof of Theorem 6. Lemma 9 implies that the subgraph H contains a vertex X with the property: X and all vertices P having a distance at most three from X have degree 6 and are incident only with triangles. If X is adjacent only with $\leq k-3$ minor vertices in each triangle incident with X then

$$\deg_G(X) \leq \deg_H(X) + \deg_H(X)(k-3) = 6k - 12.$$

Next let X be adjacent to precisely $k-2$ minor vertices of some triangle D. By Lemma 11 the triangle D is incident with a vertex Y having only one minor neighbour in D. If Y has $\leq k-3$ neighbours in one triangle different from D then

$$\begin{aligned}\deg_G(Y) &\leq \deg_H(Y) + (\deg_H(Y) - 2)(k-2) + (k-3) + 1 \\ &= 5k - 4 \leq 6k - 12 \text{ for } k \geq 8.\end{aligned}$$

Next let Y be adjacent to precisely $k-2$ minor vertices in each of the five remaining triangles incident with Y. Then

$$\begin{aligned}\deg_G(Y) &\le \deg_H(Y) + (\deg_H(Y) - 1)(k-2) + (k-3) + 1\\ &= 5k - 3 \le 6k - 12 \text{ for } k \ge 9.\end{aligned}$$

In the case $k = 8$ the proof will be continued.

The vertex Y and its neighbours create a wheel $W(Y)$ with the nave Y. Let C denote the cycle $W(Y)\backslash\{Y\}$ of $W(Y)$. If one vertex P of C is incident with two triangles D, D' having only one minor neighbour of P in its interior then

$$\deg_G(P) \le \deg_H(P) + (\deg_H(P) - 2)(k-2) + 2 = 4k \le 6k - 12.$$

Next let each vertex of the cycle C be incident with at most one triangle of $W(Y)$ having precisely one neighbour in its interior. Then C contains three consecutive vertices Z, Z', Z'' being incident with a triangle of $W(Y)$ having precisely one minor neighbour in its interior.

The same arguments applied to the wheel $W(Z')$ lead to a vertex Q of H of valency $\deg_G(Q) \le 6k - 12$. Thus in each case we arrive at a major vertex of a degree $\le 6k - 12$. This contradicts our assumption that each major vertex has a degree $> 6k - 12$. This contradiction completes the proof of the upper bound of Theorem 6. $\square$

5. Proof of Theorem 8 – the lower bound

Let I^- denote the plane graph obtained by embedding the icosahedron minus one vertex so that the outer face has size 5.

The plane graphs R_{2s} and R_{2s+1}, $s \ge 1$, are constructed as follows: In the inner face of the $2s$-cycle $C_{2s} = P_1P_2 \dots P_sQ_s \dots Q_1P_1$ or the $(2s+1)$-cycle $C_{2s+1} = P_1P_2 \dots P_sP_{s+1}Q_s \dots Q_1P_1$ chords are introduced forming the path $Q_1P_2Q_2P_3Q_3 \dots P_{s-1}Q_{s-1}P_s$ or $Q_1P_2Q_2P_3Q_3 \dots P_{s-1}Q_{s-1}P_sQ_s$, respectively (if $s = 1$ then let $R_{2s} \cong K_2$). Finally an edge of the outer face of I^- is identified with the edge P_sQ_s of C_{2s} or $P_{s+1}Q_s$ of C_{2s+1}, respectively (see Fig. 5).

A longest P_1Q_1-path w of R_{2s} and R_{2s+1} has length $l(R_{2s}) = \rho(w) = 2s-1+9 = 2s+8$ and length $l(R_{2s+1}) = \rho(w) = (2s+1)+8 = 2s+9$, respectively. A P_1Q_1-path of R_{2s} and R_{2s+1} bounding the outer face has length $a(R_{2s}) = 2s - 2 + 4 = 2s + 2$ and $a(R_{2s+1}) = 2s - 1 + 4 = (2s+1) + 2 = 2s + 3$, respectively.

The plane graph H_{2s} or H_{2s+1} is obtained from two disjoint copies L' and L'' of R_{2s} or R_{2s+1} by identifying the edge $P_1'Q_1'$ of L' with the edge $P_1''Q_1''$ of L'' so that $P_1' = Q_1''$ and $Q_1' = P_1''$ are identified, respectively. The new vertices are denoted by V and W, respectively (if necessary also by $V(H\dots)$ and $W(H\dots)$). The length of a longest VW-path of $H\dots$ is denoted by $l(H\dots)$.

Next a chain of blocks $O_t = V_0B_0V_1B_1V_2 \dots V_{2t+1}B_{2t+1}V_{2t+2}$ is defined having the following properties:

(1) $B_0 \cong I^-$ and V_1, V_2 are two nonadjacent vertices on the outer face of I^-. The outer face of B_0 has size 5, and the bounding cycle of the outer face is subdivided by V_1 and V_2 into two arcs of lengths 2 and 3.

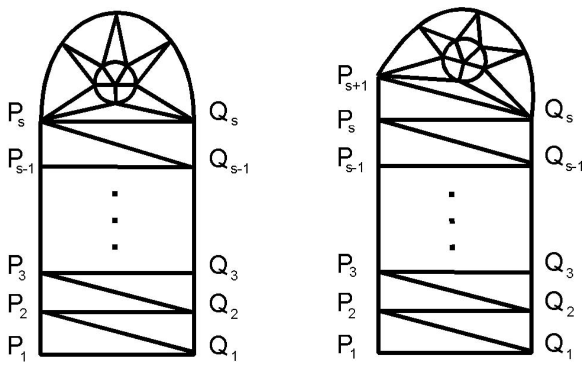

FIG. 5

(2) $B_{2j-1},\ 1 \leq j \leq t+1$, is an one–edge block.
(3) $B_{2j},\ 1 \leq j \leq t$, is isomorphic to some H_{2s} or H_{2s+1}, where s is chosen so that $l(B_{2j}) = 2l(B_{2j-2}) + 1,\ 2 \leq j \leq t,\ l(B_2) = 11$ and $V_{2j} = V$ and $V_{2j+1} = W$. The outer face of B_{2j} has size $2(l(B_{2j}) - 6)$ and the bounding cycle of the outer face is subdivided by V and W into two arcs of length $l(B_{2j}) - 6$.

In Fig. 6 the chain O_3 is depicted.

A longest V_0V_{2t+2}-path of $O_t = V_0B_0V_1B_1 \ldots V_{2t}B_{2t}V_{2t+1}B_{2t+1}V_{2t+2}$ has length

$$l(O_t) = \sum_{i=0}^{2t+1} l(B_i) = l(B_0) + \sum_{j=1}^{t+1} l(B_{2j-1}) + \sum_{j=1}^{t} l(B_{2j}).$$

By (1) the length $l(B_{2j-1}) = 1$ and $l(B_0) = 10$:

$$l(O_t) = 10 + t + 1 + \sum_{j=1}^{t} l(B_{2j}).$$

By induction on j the assertions $l(B_{2j+2}) = 2l(B_{2j}) + 1,\ 1 \leq j \leq t-1$, and $l(B_2) = 11$ imply $l(B_{2j+2}) = 11 \cdot 2^j + 2^{j-1} + 2^{j-2} + \cdots + 1$, i.e.,

(4) $$l(B_{2j+2}) = 12 \cdot 2^j - 1, \text{ and } l(O_t) = 12 \cdot 2^t - 1.$$

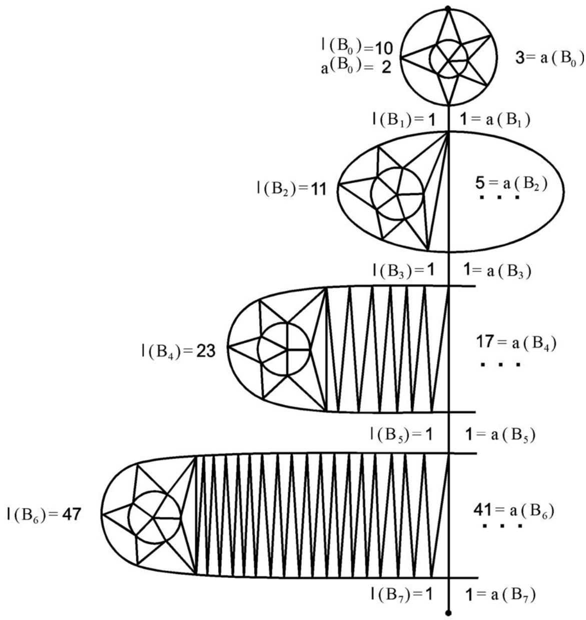

FIG. 6: the chain O_3

An outer V_0V_{2t+2}-path of O_t has length

$$\begin{aligned}
a(O_t) &= a(B_0) + a(O_t[V_2, V_{2t+2}]) \\
&= a(B_0) + \sum_{j=1}^{t+1} a(B_{2j-1}) + \sum_{j=1}^{t} a(B_{2j}) \\
&= a(B_0) + \sum_{j=1}^{t+1} l(B_{2j-1}) + \sum_{j=1}^{t} (l(B_{2j}) - 6) \\
&= a(B_0) + l(O_t[V_2, V_{2t+2}]) - 6t = a(B_0) - l(B_0) + l(O_t) - 6t;
\end{aligned}$$

the length $a(B_0)$ of the outer path of B_0 belonging to w is 2 or 3. Hence $a(B_0) \in \{2, 3\}$. With (4) this implies

(5) $$a(O_t) = a(B_0) + 12 \cdot 2^t - 6t - 11, \text{ where } a(B_0) \in \{2, 3\}.$$

A generalized 3-star S_t is constructed in the following way: three disjoint copies O_t', O_t'', and O_t''' of the chain O_t are embedded in the plane and the vertices

$Z := V'_{2t+2} = V''_{2t+2} = V'''_{2t+2}$ are identified. The obtained plane 3-star is embedded so that to the outer $V'_0V''_0$-path the block B'_0 contributes two edges and the block B''_0 contributes three edges, and the corresponding requirement is also true for the outer $V''_0V'''_0$-path and the outer $V'''_0V'_0$-path (see Fig. 7).

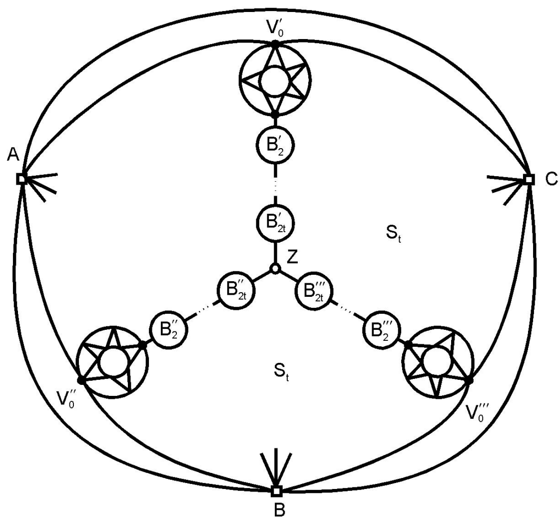

FIG. 7

Next let T be a triangulation of the compact 2-manifold $\mathbb{M}$ of Euler characteristics $\chi(\mathbb{M}) \leq 0$ and minimum degree $\delta(T) \geq 6$ with a large number of vertices (such triangulation exists, see [9]). In each triangle $[ABC]$ of T we insert a generalized 3-star S_t so that A, B, C and V_0, V''_0, V'''_0 appear in the same order around Z. We join each vertex of the outer $V'_0V''_0$-path of T (not containing V'''_0) with A by an edge, each vertex of the outer $V''_0V'''_0$-path with B and each vertex of the outer $V'''_0V'_0$-path with C by an edge (see Fig. 7). The obtained graph is denoted by G_t.

By (5) the outer $V'_0V''_0$-path p has length

$$\begin{aligned}\rho(p) &= a(B'_0) + 12 \cdot 2^t - 6t - 11 + a(B''_0) + 12 \cdot 2^t - 6t - 11 \\ &= 5 + 24 \cdot 2^t - 12t - 22.\end{aligned}$$

Hence the number of AS_t-edges is

$$\rho(p) + 1 = 24 \cdot 2^t - 12t - 16. \qquad (6)$$

The same is true for the number of BS_t-edges and CS_t-edges.

In G each vertex X of the triangulation T has a degree

$$\begin{aligned}\deg_{G_t}(X) &= \deg_T(X) + \deg_T(X)(a(p)+1)\\ &= \deg_T(X) + \deg_T(X)(24\cdot 2^t - 12t - 16)\\ &= \deg_T(X)(24\cdot 2^t - 12t - 15)\\ &\geq \delta(T)(24\cdot 2^t - 12t - 15), \text{ and}\end{aligned}$$

$$\deg_{G_t}(X) \geq 6\cdot 24\cdot 2^t - 72t - 90 \text{ for each vertex } X \text{ of } T. \tag{7}$$

The length of a longest $V_0'V_0''$-path p^* is $\rho(p^*) = 2l(O_t)$. By construction each longest path of the generalized 3-star S_t has this length. Assertion (4) implies that each longest path of S_t has

$$\rho(p^*) + 1 = 2l(O_t) + 1 = 24\cdot 2^t - 1 \text{ vertices.} \tag{8}$$

We put $k - 1 = \rho(p^*) + 1 = 24\cdot 2^t - 1$ and $t = \log_2 k - \log 24$. Then each path with k vertices contains a vertex Y of T. By (7) this vertex has a degree

$$\begin{aligned}\deg_{G_t}(X) &\geq 6\cdot 24\cdot 2^t - 72t - 90\\ &= 6k - 72(\log_2 k - \log_2 24) - 90\\ &> 6k - 72\log_2 k + 240.\end{aligned} \tag{9}$$

Each path of G_t with k vertices contains a vertex of degree $> 6k - 72\log_2 k + 240$ for $k = 24\cdot 2^t$, $t = 1, 2, \ldots$. Next let k lie in between $12\cdot 2^t = 24\cdot 2^{t-1} < k \leq 24\cdot 2^t$, $t \geq 2$. Hence $\log_2 k - \log_2 24 \leq t < \log_2 k - \log_2 24 + 1$. We consider two cases.

Case 1. Let k be an even integer. We put $2r := 24\cdot 2^t - k$, where $0 \leq r \leq 12\cdot 2^{t-1} - 1$. In S_t we change the blocks near the "center" Z (see Fig. 7). Now this is described in more details. In S_t the blocks B_{2t}', B_{2t}'', and B_{2t}''' are pairwise isomorphic and $l(B_{2t}') = l(B_{2t}'') = l(B_{2t}''') = 12\cdot 2^{t-1} - 1$.

If $t \geq 2$ and $0 \leq r \leq (12\cdot 2^{t-1} - 1) - 10$ then replace B_{2t}', B_{2t}'', and B_{2t}''' by $\widetilde{B}_{2t}', \widetilde{B}_{2t}''$, and $\widetilde{B}_{2t}'''$, respectively, with $\widetilde{B}_{2t}' \cong \widetilde{B}_{2t}'' \cong \widetilde{B}_{2t}''' \cong H_i$, where $l(\widetilde{B}_{2t}') = l(H_i) = l(B_{2t}') - r$.

If $t \geq 3$ and $(12\cdot 2^{t-1} - 1) - 10 < r \leq 12\cdot 2^{t-1} - 1$ then let $s := (12\cdot 2^{t-1} - 1) - r$, where $s \leq 10$. Replace B_{2t}', B_{2t}'', and B_{2t}''' by $\widetilde{B}_{2t}', \widetilde{B}_{2t}''$, and $\widetilde{B}_{2t}'''$, respectively, with $\widetilde{B}_{2t}' \cong \widetilde{B}_{2t}'' \cong \widetilde{B}_{2t}''' \cong H_i$, where $l(\widetilde{B}_{2t}') = l(H_i) = 10$, i.e. $H_i \cong B_0$, and replace B_{2t-2}', B_{2t-2}'', and B_{2t-2}''' by $\widetilde{B}_{2t-2}', \widetilde{B}_{2t-2}''$, and $\widetilde{B}_{2t-2}'''$, respectively, with $\widetilde{B}_{2t-2}' \cong \widetilde{B}_{2t-2}'' \cong \widetilde{B}_{2t-2}''' \cong H_j$ and $l(\widetilde{B}_{2t-2}') = l(H_j) = l(B_{2t-2}') - s$. The construction is possible for $k \geq 66$. The new generalized 3-star obtained from S_t by these replacements is denoted by $\widetilde{S}_t$. The same replacements applied to the chain of blocks O_t result in a chain of blocks $\widetilde{Q}_t$.

The assertions (5), (6) and (8) imply that by this method, a chain of blocks $\widetilde{O}_t$ and a graph $\widetilde{G}_t$ is obtained with

$$\rho(\widetilde{p}^*) + 1 = \rho(p^*) + 1 - 2r = 24 \cdot 2^t - 1 - 2r = k - 1, \text{ and} \tag{10}$$

$$l(\widetilde{O}_t) = l(O_t) - r = 12 \cdot 2^t - 1 - r = \frac{k}{2} - 1, \text{ and} \tag{11}$$

$$\rho(\widetilde{p}) + 1 = \rho(p) + 1 - 2r = 24 \cdot 2^t - 12t - 16 - 2r, \text{ and} \tag{12}$$

$$a(\widetilde{O}_t) = a(B_0) + 12 \cdot 2^t - 6t - 11 - 2r, \tag{13}$$

where $a(B_0) \in \{2, 3\}$. Hence $k = 24 \cdot 2^t - 2r$, and

$$\begin{aligned} \deg_{\widetilde{G}_t}(X) &\geq \deg_T(X) + \mathrm{de}_T(X)(a(\widetilde{p}) + 1) \\ &= \deg_T(X)(a(\widetilde{p}) + 2) \\ &\geq 6(24 \cdot 2^t - 2r - 12t - 15) \\ &= 6(24 \cdot 2^t - 2r) - 72t - 90 \\ &\geq 6k - 72 \log_2 k + 72 \log_2 24 - 162. \end{aligned}$$

Consequently, each path of $\widetilde{G}_t$ with k vertices contains a vertex Y of degree

$$\deg_{\widetilde{G}_t}(Y) > 6k - 72 \log_2 k + 118, \ k \geq 66.$$

Case 2. Let k be an odd integer. With $k > k - 1, \ k - 1$ even, we obtain

$$deg_{\widetilde{G}_t}(Y) > 6(k-1) - 72 \log_2(k-1) + 118 = 6k - 72 \log_2(k-1) + 112, \ k \geq 66.$$

This completes the proof of the lower bound in Theorem 8.

Note that (13) implies

$$a(\widetilde{O}_t) \geq \frac{k}{2} - \log_2 k + 18. \tag{14}$$

6. Proof of Theorem 7 - the lower bound

We use R_j and H_j as defined in section 5. Let $k \geq 66, k \equiv 2(\mod 4)$, be an integer. Let $E_k = V_0 B_0 V_1 B_1 V_2 B_2 V_3 B_3 V_4$ be a chain of blocks with the following properties:

1. $B_0 \cong R_j$ with $j = \frac{k-2}{4} - 9$, i.e., $l(B_0) = l(R_j) = \frac{k-2}{4} - 1$,
2. B_1 and B_3 are one–edge blocks, and
3. $B_2 \cong H_j$ with $j = \frac{k-2}{4} - 9$, i.e., $l(B_2) = l(H_j) = \frac{k-2}{4} - 1$.

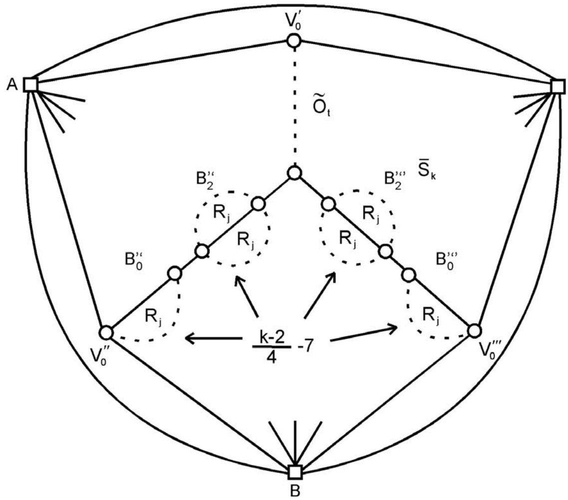

FIG. 8. $l(\widetilde{O}_t) = \frac{k-2}{2}, l(H_j) = l(R_j) = \frac{k-2}{2} - 1$.

The length of E_k is $l(E_k) = 2(\frac{k-2}{4} - 1) + 2 = \frac{k-2}{2}$.

A generalized 3-star $\overline{S}_k$ is constructed in the following way: three disjoint chains of blocks O', O'', O''' are embedded into the plane, where $O' \cong \widetilde{O}_t$ with $l(\widetilde{O}_t) = \frac{k-2}{2}$ and $O'' \cong O''' \cong E_k$ with $l(E_k) = \frac{k-2}{2}$. The vertices $Z := V'_{2t+2} = V''_4 = V'''_4$ are identified so that the outer $V''_0 V'''_0$-path p'' contains the outer path of B''_0 and B'''_0 of length $\frac{k-2}{4} - 7 > 1$; and the outer $V'''_0 V'_0$-path p''' and the outer $V'_0 V''_0$-path p' contains the outer path of B'''_0 or B''_0 of length 1, respectively (see the embedding of $\overline{S}_k$ into a triangular face $[A, B, C]$ in Fig. 8).

Let $p^{*\prime}, p^{*\prime\prime}$, and $p^{*\prime\prime\prime}$ denote the longest $V'_0 V''_0$-path, $V''_0 V'''_0$ -path, and $V'''_0 V'_0$-path of $\overline{S}_k$. Obviously, $\rho(p^{*\prime}) = \rho(p^{*\prime\prime}) = \rho(p^{*\prime\prime\prime}) = k - 2$, and $\rho(p'') = 4 + 4(\frac{k-2}{4} - 7) = k - 26$, and $\rho(p''') = \rho(p') \geq 1 + 1 + (\frac{k-2}{4} - 7) + 1 + a(\widetilde{O}_t) > (\frac{k-2}{4} - 4) + \frac{k}{2} - 6 \log_2 k + 18 = \frac{3k-2}{4} - 6 \log_2 k + 14$.

Next let T be a triangulation of the plane having only vertices of degrees 5 and 6, where any two vertices of degree 5 have a distance ≥ 4. In each triangle $[A, B, C]$ of T with all vertices of degree 6 we insert a generalized 3-star $\widetilde{S}_t$ of length $l(\widetilde{S}_t) = k - 2, k \geq 66$, (defined in section 5) so that A, B, C and V'_0, V''_0, V'''_0 appear in the same order around Z. We join all vertices of the outer $V'_0 V''_0$-path of $\widetilde{S}_t$ (not containing V'''_0) with A, all vertices of the outer $V''_0 V'''_0$-path with B and all vertices of the outer $V'''_0 V'_0$-path with C (see Fig. 8). In the same way in each triangle $[A, B, C]$ of T with degree $\deg_T(A) = \deg_T(C) = 6$ and $\deg_T(B) = 5$ a generalized 3-star $\overline{S}_k$ of length $l(\overline{S}_k) = k - 2$ is inserted. The obtained polyhedral plane graph G has the following properties. If X is a degree–5 vertex of T then

$$\begin{aligned}\deg_G(X) &> 5 + 5(\rho(p'') + 1) = 5 + 5(k - 26 + 1) = 5(k - 27) \\ &= 5k - 120.\end{aligned}$$

If X is a degree–6 vertex of T which is adjacent to a degree–5 vertex of T then

$$\deg_G(X) > 6 + 2(\rho(p') + 1) + 4(\rho(\widetilde{p}) + 1),$$

where p' is an outer $V_0'V_0''$-path of $\overline{S}_k$ of length $\rho(p') = \frac{3k-2}{4} - 6\log_2 k + 14$ and by (5) the path $\widetilde{p}$ is an outer $V_0'V_0''$-path of $\widetilde{S}_t$ of length $\rho(\widetilde{p}) = a(\widetilde{O}_t) \geq k - 12\log_2 k + 18$. Hence

$$\begin{aligned}\deg_G(X) &> 6 + 2(\frac{3k-2}{4} - 6\log_2 k + 15) + 4(k - 12\log_2 k + 19) \\ &= 5k - 220 + (\frac{k}{2} - 60\log_2 k + 331), \text{ and} \\ \deg_G(X) &> 5k - 220.\end{aligned}$$

If X is a degree–6 vertex of T which is *not* adjacent to a degree–5 vertex of T then assertion (9) of section 4 implies

$$\begin{aligned}\deg_G(X) &> 6k - 72\log_2 k + 240 \\ &= (5k - 220) + (k - 72\log_2 k + 460), \text{ and} \\ \deg_G(X) &> 5k - 220 \text{ for all } k \geq 66, \quad k \equiv 2 (\bmod\ 4).\end{aligned}$$

Hence

$$deg_G(X) > 5(k - 3) - 220 = 5k - 235$$

for all $k \geq 66$. This completes the proof of the lower bound of Theorem 7.

7. Proof of Theorem 6 - the lower bound

Each compact 2-manifold $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M}) \leq 0$ has a triangulation T of $\mathbb{M}$ of minimum degree with the property: in every triangle T of $\mathbb{M}$ a root vertex is labelled so that each vertex X of T is no root vertex of at least four triangles incident with X. Such a triangulation has been constructed in [9].

Into every triangular face $O = [A_1, A_2, A_3]$ of T we insert a generalized 3-star consisting of a central vertex Z and three paths starting in Z, one of length $\lceil \frac{k}{2} \rceil$ and the others of length $\lfloor \frac{k}{2} \rfloor$, where w.l.o.g. A_1 is the root vertex of T. To each path $P_1P_2P_3P_4 \ldots Z$ the edges P_1P_3 and P_2P_4 are added. Let the paths be denoted by p_1, p_2, and p_3 so that p_1 and p_2 have length $\lfloor \frac{k}{2} \rfloor$ and p_3 has length $\lceil \frac{k}{2} \rceil$. In O the vertex A_i is joined with all vertices of p_i and p_{i+1} which can be reached from A_i (note that in such a path $P_1P_2P_3P_4 \ldots Z$ either the vertex P_2 or the vertex P_3 cannot be reached from A_i). The obtained triangulation is denoted by G.

The root vertex A_1 of O is joint with $\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{2} \rfloor - 3$ vertices of the inserted 3-star, and the two other vertices are joint with $\lfloor \frac{k}{2} \rfloor + \lceil \frac{k}{2} \rceil - 3$ of its vertices. Since each vertex X of T is incident with at least 6 triangles, and X is no root vertex of at least 4 of them, the vertex X has a degree

$$\begin{aligned}\deg_G(X) &\geq \deg_T(X) + 4\left(\left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil - 3\right) + 2\left(\left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor - 3\right) \\ &\geq \begin{cases} 6k - 12, \text{ for even } k \geq 8 \\ 6k - 14, \text{ for odd } k \geq 9. \end{cases}\end{aligned}$$

Each path with k vertices contains a vertex of T. This completes the proof of the lower bound of $\phi_N(4, P_k, \mathbb{M}), \chi(\mathbb{M}) \leq 0, k \geq 8$.

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References

[1]. I. Fabrici and S. Jendroľ, *Subgraphs with restricted degrees of their vertices in planar 3–connected graphs*, Graphs and Combinatorics **13** (1997), 245–250.

[2]. I. Fabrici, E. Hexel, S. Jendroľ, H. Walther, *On Vertex-degree Restricted Paths in Polyhedral Graphs*, Discrete Math. **212** (2000), 61–73.

[3]. B. Grünbaum, *Convex Polytopes*, Interscience, New York (1967).

[4]. B. Grünbaum and G. C. Shephard, *Analogues for tiling of Kotzig's theorem on minimal weights of edges*, Ann. Discrete Math. **12** (1982), 129–140.

[5]. J. Ivančo, *The weight of a graph*, Ann. Discrete Math. **51** (1992), 113–116.

[6]. S. Jendroľ, *On face vectors of trivalent maps*, Math. Slovaca **36** (1986), 367–386.

[7]. S. Jendroľ and H.-J. Voss, *A local property of polyhedral maps on compact 2-dimensional manifolds*, Discrete Math. **212** (2000), 111-120.

[8]. S. Jendroľ and H.-J. Voss, *A local property of large polyhedral maps on compact 2-dimensional manifolds*, Graphs and Combinatorics **15** (1999), 303–313.

[9]. S. Jendroľ and H.-J. Voss, *Light paths with an odd number of vertices in large polyhedral maps*, Annals of Combinatorics **2** (1998), 313–324.

[10]. S. Jendroľ and H.-J. Voss, *Light paths with an odd number of vertices in polyhedral maps*, Czechoslovak Math. J. **50(125)**, 555–564.

[11]. A. Kotzig, *Contribution to the theory of Eulerian polyhedra*, Math. Čas. SAV (Math. Slovaca) **5** (1955), 111–113.

[12]. A. Kotzig, *Extremal polyhedral graphs*, Ann. New York Acad. Sci. **319** (1979), 569–570.

[13]. J. Zaks, *Extending Kotzig's theorem*, Israel J. Math. **45** (1983), 281–296.

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