A new small embedding for partial 8-cycle systems

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Abstract

The upper bound for embedding a partial 8-cycle system of order n is improved from $4n + c\sqrt{n},\ c > 0$, to $4n + 29$.

1 Introduction

An m-cycle system of order n is a pair (S, C), where C is a collection of edge-disjoint m-cycles which partitions the edge set of K_n (the complete undirected graph on n vertices) with vertex set S. A *partial* m-cycle system of order n is a pair (X, P), where P is a collection of edge disjoint m-cycles of the edge set of K_n $(E(K_n))$. The difference between a *partial* m-cycle system and an m-cycle system is that the edges belonging to the m-cycles in a partial m-cycle system do not necessarily include all edges of K_n.

A natural question to ask is the following: given a partial m-cycle system (X, P) or order n, is it always possible to decompose $E(K_n) \setminus E(P)$ into edge disjoint m-cycles? $(E(K_n) \setminus E(P)$ is the complement of the edge set of P in the edge set of K_n.) That is, can a partial m-cycle system always be *completed* to an m-cycle system? The answer to this question is no, since for any m we can construct a partial m-cycle system consisting of one m-cycle of order not satisfying the necessary conditions for the existence of an m-cycle system (see [3] for example).

Given the fact that a partial m-cycle system cannot necessarily be completed, the next question to ask is whether or not a partial m-cycle system can always be *embedded* in an m-cycle system.

The partial m-cycle system (X, P) is said to be *embedded* in the m-cycle system (S, C) provided $X \subseteq S$ and $P \subseteq C$. If the answer to this question is yes, we would like the size of the containing m-cycle system to be as small as possible.

In [5] it is shown that a partial m-cycle system of order n can be embedded in an m-cycle system of order $2mn + 1$ when m is EVEN and embedded in an m-cycle system of order $m(2n + 1)$ when m is ODD [4].

In [1] the following theorem is proved.

Theorem 1.1 (P. Horák and C. C. Lindner [1].) *Let m be even. A partial m-cycle system of order n can be embedded in an m-cycle system of order $\binom{x}{2}(m/2)+x$, where x is the smallest positive integer such that $x \equiv 1 \pmod{4m}$ and $\binom{x}{2} \geq n$.* □

To make a long story short, a partial $(m = 2k)$-cycle system of order n can always be embedded in an m-cycle system of order $\leq (mn)/2 + c\sqrt{n}$, for some positive constant c (depending on m).

In [2] this bound was improved from $3n + c\sqrt{n}$ to $3n + 42$ for 6-cycles.

The object of this note is to give a new construction for 8-cycle systems which improves the upper bound from $4n + c\sqrt{n}$ to $4n + 29$.

2 The 16k + 17 Construction.

Let X and Y be sets of size $4k$ and 17 respectively and set $S = (X \times \{1, 2, 3, 4\}) \cup Y$. Define a collection C of 8-cycles of the edge set of K_{16k+17} with vertex set S as follows:

(1) Let (Y, C^*) be any 8-cycle system of order 17 (see [3]) and place the 8-cycles of C^* in C.

(2) For each pair $x \neq y \in X$, let $C(x, y)$ be a decomposition of $K_{4,4}$ (with parts $\{x\} \times \{1, 2, 3, 4\}$ and $\{y\} \times \{1, 2, 3, 4\}$) into 2 8-cycles and place these 8-cycles in C. Without loss in generality we can assume the 8-cycle $((x, 1), (y, 1), (x, 2), (y, 3), (x, 4), (y, 4), (x, 3), (y, 2))$ belongs to $C(x, y)$.

(3) Let π be a partition of X into subsets of size 2 and for each $\{x, y\} \in \pi$ define 3 8-cycles by $(\infty_1, (x, 1), (x, 2), (x, 3), \infty_5, (y, 3), (y, 2), (y, 1))$, $(\infty_2, (x, 1), (x, 3), (x, 4), \infty_6, (y, 4), (y, 3), (y, 1))$, and $(\infty_3, (x, 1), (x, 4), (x, 2), \infty_4, (y, 2), (y, 4), (y, 1))$, where $\infty_1, \infty_2, \infty_3, \infty_4, \infty_5$, and ∞_6 are 6 distinct elements belonging to Y.

(4) Let C_1 be any partition of $K_{4k,14}$ (with parts $X \times \{1\}$ and $Y \setminus \{\infty_1, \infty_2, \infty_3\}$) into 8-cycles (see [6]) and place these 8-cycles in C.

(5) For each $i \in \{2, 3, 4\}$ let C_i be any partition of $K_{4k,16}$ (with parts $X \times \{i\}$ and $Y \setminus \{\infty_{i+2}\}$) into 8-cycles and place these 8-cycles in C.

Theorem 2.1 *(S, C) is an 8-cycle system of order $16k + 17$.*

Proof: It suffices to show that (i) each edge in K_{16k+17} (with vertex set S) belongs to a cycle of type (1), (2), (3), (4), or (5) and that (ii) the total number of 8-cycles in the $16k + 17$ Construction is $|C| = n(n-1)/16$, $n = 16k + 17$.

(i) Let $\{a, b\} \in E(K_{16k+17})$.

(a) $a, b \in Y$. Then $\{a, b\}$ belongs to a cycle in C^* and therefore to a cycle in C.

(b) $a = (z, 1), b \in \{\infty_1, \infty_2, \infty_3\}$. Then $\{a, b\}$ belongs to a cycle of type (3).

(c) $a = (z, 1), b \in Y \setminus \{\infty_1, \infty_2, \infty_3\}$. Then $\{a, b\}$ belongs to a cycle of type (4).

(d) $a = (z, i), i \in \{2, 3, 4\}, b = \infty_{i+2}$. Then $\{a, b\}$ belongs to a cycle of type (3).

(e) $a = (z, i), i \in \{2, 3, 4\}, b \in Y \setminus \{\infty_{i+2}\}$. Then $\{a, b\}$ belongs to a cycle of type (5).

(f) $a = (x, i), b = (y, i)$. Then $\{a, b\}$ belongs to a cycle of type (2).

(g) $a = (x, i), b = (y, j), i \neq j$. If $x = y$, then $\{a, b\}$ belongs to a type (3) 8-cycle. If $x \neq y$, then $\{a, b\}$ belongs to a type (2) 8-cycle.

Combining the above cases shows that each edge of K_{16k+17} belongs to an 8-cycle of type (1), (2), (3), (4), or (5) in the $16k + 17$ Construction.

(ii) Counting the 8-cycles in the $16k+17$ Constuction gives: 34 type (1), $2\binom{4k}{2} = 16k^2 - 4k$ type (2), $6k$ type (3), $7k$ type (4), and $24k$ type (5) 8-cycles. Adding these numbers gives $n(n - 1)/16$ (remember that $n = 16k + 17$).

Combining parts (i) and (ii) completes the proof. □

3 The $16\mathbf{k} + 17$ embedding.

Let (Z, P) be a partial 4-cycle system of order n and X a set of size $4k \geq n$, where $4k$ is as small as possible; so $4k = n, n + 1, n + 2$, or $n + 3$. Let X be a set of size $4k$ such that $Z \subseteq X$ and use the $16k + 17$ Construction to construct an 8-cycle system (S, C) of order $16k + 17$. If the edge $\{x, y\}$ belongs to the cycle c in the partial 8-cycle system (Z, P) denote by $c(x, y)$ the type (2) 8-cycle $((x, 1), (y, 1), (x, 2), (y, 3), (x, 4), (y, 4), (x, 3), (y, 2))$ in the $16k + 17$ Construction. For each 8-cycle $c = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in P$ denote by $8c$ the collection of eight 8-cycles $c(x_i, x_{i+1})$. We define a *balanced set* of 8-cycles $8c^*$ on the edges belonging to $8c$ as follows: $((x_1, i), (x_2, i), (x_3, i), (x_4, i), (x_5, i), (x_6, i), (x_7, i), (x_8, i))$, $i \in \{1, 4\}$; and for each $(i, i{+}1)$, $i = 1, 2, 3$, the two 8-cycles $((x_1, i), (x_2, i{+}1), (x_3, i), (x_4, i{+}1), (x_5, i), (x_6, i{+}1), (x_7, i), (x_8, i{+}1))$ and $((x_1, i{+}1), (x_2, i), (x_3, i{+}1), (x_4, i), (x_5, i{+}1), (x_6, i), (x_7, i{+}1), (x_8, i))$. Since $8c$ and $8c^*$ are balanced (contain the same edges) $(C \setminus 8c) \cup 8c^*$ is an 8-cycle system. If $c_i \neq c_j \in P$, then $8c_i$ and $8c_j$ are edge disjoint. Hence $(C \setminus \{8c \mid c \in P\}) \cup \{8c^* \mid c \in P\}$ is an 8-cycle system containing two disjoint copies of P; namely the cycles having the *same* second coordinate (1 and 4) in each collection $8c^*$.

Theorem 3.1 *A partial 8-cycle system of order n can be embedded in an 8-cycle system of order $16k + 17$, where $4k$ is the smallest positive integer such that $4k \geq n$.*

Corollary 3.2 *A partial 8-cycle system of order n can be embedded in an 8-cycle system of order* at most $4n + 29$.

Proof: Since $4k \geq n$ is as small as possible, $4k = n, n + 1, n + 2$, or $n + 3$. Hence $16k + 17 \leq 4n + 29$. □

4 Concluding remarks.

Some comments about size are appropriate! The results in [1] (Theorem 1.1 in this paper), [2], and in this note all involve estimating the size of n. The $16n+1$ embedding in [5] does this *exactly* whereas the estimation in Theorem 1.1 uses the smallest $x \equiv 1 \pmod{32}$ such that $\binom{x}{2} \geq n$. For small n this can be a very bad estimate. For example, if $n = 23$, then $x = 33$, $\binom{33}{2} = 528$, and the $4\binom{x}{2} + x$ embedding gives a containing 8-cycle system of order 2145 which is a lot worse than the bound of 361 given by the $16n+1$ embedding. However, the $4\binom{x}{2} + x$ embedding is eventually better than the $16n+1$ embedding and is *asymptotic* to $4n$. However, in every case the $16k+17$ embedding is better, particularly for small n, since the estimation of n is off by at most 3. For $n = 23$, $4k = 24$, and the $16k+17$ embedding gives a containing system of order 113.

Unfortunately, the technique used in the $16k+17$ Construction (to use CATCH-22 vernacular) *always never* works for $2k \neq 8$.

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