G-designs, G-packings and G-coverings of λK_v with a bipartite graph G of six vertices

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Abstract

Let λK_v be the complete multigraph with v vertices, where any two distinct vertices x and y are joined by λ edges (x, y). Let G be a finite simple graph. A G-design (G-packing, G-covering) of λK_v, is denoted by (v, G, λ)-GD ((v, G, λ)-PD, (v, G, λ)-CD). In this paper, we determine the existence spectrum for the G-designs of λK_v, $\lambda > 1$, and construct the maximum packings and the minimum coverings of λK_v with G for any positive integer λ, where the bipartite graph G has six vertices and $e(G) \leq 6$.

1 Introduction

Throughout this paper, graphs are finite, undirected and have no isolated vertices. A complete multigraph of order v and index λ, denoted by λK_v, is a graph with v vertices, where any two distinct vertices x and y are joined by λ edges (x, y). Let G be a finite simple graph. *A G-design* (*G-packing, G-covering*) of λK_v, denoted by (v, G, λ)-GD ((v, G, λ)-PD, (v, G, λ)-CD), is a pair $(X, \mathcal{B})$ where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly (at most, at least) λ blocks of $\mathcal{B}$. A G-packing (G-covering) is said to be *maximum* (*minimum*), denoted by (v, G, λ)-$MPD(MCD)$, if no other such G-packing (G-covering) has more (fewer) blocks. The number of blocks in a maximum G-packing (minimum G-covering), denoted by $p(v, G, \lambda)$ ($c(v, G, \lambda)$), is called the *packing* (*covering*) *number*. It is well known that

$$p(v, G, \lambda) \leq \lfloor \frac{\lambda v(v-1)}{2e(G)} \rfloor \leq \lceil \frac{\lambda v(v-1)}{2e(G)} \rceil \leq c(v, G, \lambda)$$

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where $e(G)$ denotes the number of edges in G, $\lfloor x \rfloor$ denotes the greatest integer y such that $y \le x$ and $\lceil x \rceil$ denotes the least integer y such that $y \ge x$. A (v, G, λ)-PD ((v, G, λ)-CD) is said to be *optimal* and denoted by (v, G, λ)-OPD ((v, G, λ)-OCD) if the left (right) equality holds. Obviously, there exists a (v, G, λ)-GD if and only if $p(v, G, \lambda) = c(v, G, \lambda)$ and a (v, G, λ)-GD can be regarded as (v, G, λ)-OPD or (v, G, λ)-OCD.

By a $L_\lambda(\mathcal{D})$ of a packing $\mathcal{D}$, called the *leave edge graph*, we mean a subgraph of λK_v whose edges are the complement of $\mathcal{D}$ in λK_v. The number of edges in $L_\lambda(\mathcal{D})$ is denoted by $|L_\lambda(\mathcal{D})|$. In particular, when $\mathcal{D}$ is maximum, $|L_\lambda(\mathcal{D})|$ is called the *leave edge number* and is denoted by $l_\lambda(v)$. Similarly, the *repeat edge graph* $R_\lambda(\mathcal{D})$ of a covering $\mathcal{D}$ is a subgraph of λK_v and its edges are the complement of λK_v in $\mathcal{D}$. When $\mathcal{D}$ is minimum, $|R_\lambda(\mathcal{D})|$ is called the *repeat edge number* and is denoted by $r_\lambda(v)$. Generally, the symbols $L_\lambda(\mathcal{D})$, $l_\lambda(v)$, $R_\lambda(\mathcal{D})$ and $r_\lambda(v)$ can be denoted more briefly by L_λ, l_λ, R_λ and r_λ. It is not difficult to show the following result:
If there exists a (v, G, λ)-GD, *then* $p(v, G, \lambda) = c(v, G, \lambda) = \frac{\lambda v(v-1)}{e(G)}$, i.e., $l_\lambda = r_\lambda = 0$. *Else,*

$$l_\lambda = \lambda v(v-1)/2 - e(G) \cdot p(v,\ G,\ \lambda) > 0 \textit{ and}$$
$$r_\lambda = e(G) \cdot c(v,\ G,\ \lambda) - \lambda v(v-1)/2 > 0.$$

Many researchers have been involved in graph design, graph packing and graph covering of λK_v with five vertices or less(see [1–10]). Yin [11] listed the spectrum of graph designs of K_v with six vertices and $e(G) \le 6$. (See Table A.)

For the cycle C_6, there exists a $(v, C_6, 1)$-GD if and only if $v \equiv 1, 9 \pmod{12}$. Furthermore, J.A. Kennedy [12] obtained following theorem:

Theorem For any positive integer λ, the packing number $p(v, C_6, \lambda)$ and covering number $c(v, C_6, \lambda)$ are determined.

When the six-vertex graph G contains an odd cycle and $e(G) \le 6$, Z.Liang [13] gave the G-design, maximum G-packing and minimum G-covering of λK_v.

Let the bipartite graph G have six vertices and its edge number be not greater than 6; for such G, the G-design, maximum G-packing and minimum G-covering of λK_v is solved in this paper.

Subsequently, the following notations ($a, b \in Z$) are used frequently:
$[a, b] = \{x \in Z \mid a \le x \le b\}$, $[a, b]_k = \{x \in Z \mid a \le x \le b,\ x \equiv a \pmod{k}\}$ for $a, b \in Z$, $[a, b, \cdots, c] + i = [a+i, b+i, \cdots, c+i]$ and $(Z_n)_m = \{i_m | \ i \in Z_n\}$.
The edge set $\{(a_1, a_2), (a_2, a_3), \cdots, (a_{n-1}, a_n)\}$ is denoted by $(a_1, a_2, \cdots, a_n)$; the graph G is denoted by $[a, b, c, d, e, f]$.

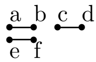
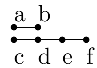
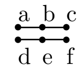
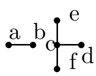
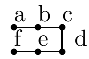
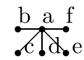
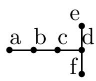
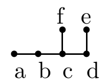
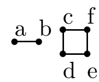
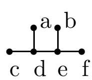
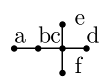
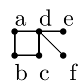
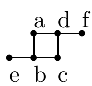
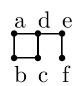
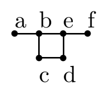
note	G_1	G_2	G_3
graph			
spectrum	$v \equiv 0, 1 \ (mod\ 3)\ v \geq 6$	$v \equiv 0, 1 \ (mod\ 8)\ v \geq 8$	$v \equiv 0, 1 \ (mod\ 8)\ v \geq 8$
note	G_4	G_5	G_6
graph			
spectrum	$v \equiv 0, 1 \ (mod\ 8)\ v \geq 8$	$v \equiv 0, 1 \ (mod\ 5)\ v \geq 6$	$v \equiv 0, 1 \ (mod\ 5)\ v > 6$
note	G_7	G_8	G_9
graph			
spectrum	$v \equiv 0, 1 \ (mod\ 5)\ v > 6$	$v \equiv 0, 1 \ (mod\ 5)\ v \geq 6$	$v \equiv 0, 1 \ (mod\ 5)\ v \geq 6$
note	G_{10}	G_{11}	G_{12}
graph			
spectrum	$v \equiv 0, 1 \ (mod\ 5)\ v \geq 6$	$v \equiv 0, 1 \ (mod\ 5)\ v > 6$	$v \equiv 0, 1, 4, 9 \ (mod\ 12)$
note	G_{13}	G_{14}	G_{15}
graph			
spectrum	$v \equiv 0, 1, 4, 9 \ (mod\ 12)$	$v \equiv 0, 1, 4, 9 \ (mod\ 12)$	$v \equiv 0, 1, 4, 9 \ (mod\ 12)$

Table A

2 Recursion

By $K_{n_1,n_2,\cdots,n_h}$ we mean the complete multipartite graph with h parts of sizes $n_1, n_2, \cdots, n_h$. Let $X = \bigcup_{1 \leq i \leq h} X_i$ be the vertex set of $K_{n_1,n_2,\cdots,n_h}$ where X_i $(1 \leq i \leq h)$ are disjoint sets with $|X_i| = n_i$ and $v = \sum_{1 \leq i \leq h} n_i$. For any fixed graph G, if $K_{n_1,n_2,\cdots,n_h}$ can be decomposed into edge-disjoint subgraphs isomorphic to G, then we call $(X, \mathcal{G}, \mathcal{A})$ a *holey G-design*, where $\mathcal{G}$ =$\{X_1, X_2, \cdots, X_h\}$, and $\mathcal{A}$ is the collection of all subgraphs called *G-blocks* (or simply *blocks*). Each set $X_i (1 \leq i \leq h)$ is said to be a *hole* and the multiset $\{n_1, n_2, \cdots, n_h\}$ is a type of the holey G-design. We denote the design by G-$HGD(n_1^1 n_2^1 \cdots n_h^1)$ (or $K_{n_1,n_2,\cdots,n_h}/G$) and use an "exponential" notation to describe its type in general: a type $1^i 2^j 3^k \cdots$, denotes i occurrences of 1, j occurrences of 2, etc. A G-$HGD(1^{v-w} w^1)$ is called an *incomplete G-design*, denoted by $(v, w, G, 1)$-*IGD*. Obviously, a $(v, G, 1)$-*GD* is a *G*-*HGD*(1^v), which can be thought of as a $(v, w, G, 1)$-*IGD* with $w = 0$ or 1.

Theorem 2.1 If there exist $(n_i, G, 1)$-*GD* for $i \in [1, h]$ and *G*-*HGD*$(n_i^1 n_j^1)$ for $i \neq j$ and $i, j \in [1, h]$, then there exists a $(n, G, 1)$-*GD* for $n = \sum_{1 \leq i \leq h} n_i$.

Corollary 2.2 Suppose that there exist $(n, G, 1)$-*GD* and *G*-*HGD*(n^2); then there exists a $(sn, G, 1)$-*GD* for any positive integer s.

Corollary 2.3 Suppose that there exists a *G*-*HGD*(n^2); then there exist *G*-*HGD*(n^s) for any positive integer s.

Theorem 2.4 If there exist $(n + n', n', G, 1)$-*IGD*, $(n + n', G, 1)$-*GD* (or *CD*, *PD*) and *G*-*HGD*(n^2), then there exists a $(mn+n', G, 1)$-*GD*(or *CD*,*PD*) for any positive integer m and integer $n' \geq 0$.

Theorem 2.5 If there exist $(n, G, 1)$-*GD*, *G*-*HGD*(n^2), *G*-*HGD*(n^1m^1) and $(n + m, G, 1)$-*GD*(or *PD*, *CD*), then there exists a $(tn + m, G, 1)$-*GD* (or *PD*, *CD*) for any positive integer t.

Theorem 2.6 If there exist $(u, w, G, 1)$-*IGD*, *G*-*HGD*$(n_1^1 n_2^1 \cdots n_t^1 u^1)$ and $(n_i, G, 1)$-*GD* for $i \in [1, t]$, then there exists a $(u + \sum_{1 \leq i \leq t} n_i, w, G, 1)$-*IGD*.

Theorem 2.7 If there exist *G*-*HGD*$(n_1^1 n_2^1 \cdots n_t^1)$ and $(n_i + w, w, G, 1)$-*IGD* for $i \in [1, t]$, then there exists a $(w + \sum_{1 \leq i \leq t} n_i, w, G, 1)$-*IGD*.

Theorem 2.8 If there exist $(n, w, G, 1)$-*IGD* and $(w, G, 1)$-*GD* (*PD*, *CD*), then there exists a $(n, G, 1)$-*GD* (*PD*, *CD*).

Theorem 2.9 [6] If there exist *G*-*HGD*$(n^1 m_i^1)$ for $i = 1, 2$, then there exist *G*-*HGD*$((an)^1 (bm_1 + cm_2)^1)$ for integers $a \geq 1$ and b or $c \geq 1$.

Theorem 2.10 If there exist *G*-*HGD*(n^2), *G*-*HGD*$((n + r)^1 n^1)$,$(n, G, 1)$-*GD* and $(n+r, G, 1)$-*GD*(*PD*, *CD*) for $1 \leq r \leq n-1$, then there exist $(v, G, 1)$-*GD*(*PD*, *CD*) for any integer $v \geq n$.

Theorem 2.11 Let l be the leave edge number of the $(n, G, 1)$-*OPD* and $\bar{\lambda}=e(G)/\gcd(e(G), l)$. If there exist (n, G, λ)-*OPD* and (n, G, λ)-*OCD* for $1 \leq \lambda \leq \bar{\lambda}$, then there exist (n, G, λ)-*OPD* and (n, G, λ)-*OCD* for any positive integer λ.

The following theorem is a modified version of Theorem 4 in Section 3 of [14].

Theorem 2.12 Given positive integers v, λ and μ. Let X be a v-set.

(1) Suppose that there exists a (v, G, λ)-$MPD = (X, \mathcal{D})$ with leave edge graph $L_\lambda(\mathcal{D})$ and a (v, G, μ)-$MPD = (X, \mathcal{E})$ with leave edge graph $L_\mu(\mathcal{E})$. If $|L_\lambda(\mathcal{D})| + |L_\mu(\mathcal{E})| = l_{\lambda+\mu}(v) < e(G)$, then there exists a $(v, G, \lambda + \mu)$-*MPD* with leave edge graph $L_\lambda(\mathcal{D}) \cup L_\mu(\mathcal{E})$.

(2) Suppose that there exists a (v, G, λ)-$MCD = (X, \mathcal{D})$ with repeat edge graph $R_\lambda(\mathcal{D})$ and a (v, G, μ)-$MCD = (X, \mathcal{E})$ with repeat edge graph $R_\mu(\mathcal{E})$. If $|R_\lambda(\mathcal{D})| + |R_\mu(\mathcal{E})| = r_{\lambda+\mu}(v) < e(G)$, then there exists a $(v, G, \lambda + \mu)$-*MCD* with repeat edge graph $R_\lambda(\mathcal{D}) \bigcup R_\mu(\mathcal{E})$.

(3) Suppose that there exists a (v, G, λ)-$MPD = (X, \mathcal{D})$ with leave edge graph $L_\lambda(\mathcal{D})$ and a (v, G, μ)-$MCD = (X, \mathcal{E})$ with repeat edge graph $R_\mu(\mathcal{E})$. If $R_\mu(\mathcal{E}) \subset L_\lambda(\mathcal{D})$ and $|L_\lambda(\mathcal{D})| - |R_\mu(\mathcal{E})| = l_{\lambda+\mu}(v) < e(G)$, then there exists a $(v, G, \lambda+\mu)$-*MPD* with leave edge graph $L_\lambda(\mathcal{D}) \backslash R_\mu(\mathcal{E})$.

(4) Suppose that there exists a (v, G, λ)-$MCD = (X, \mathcal{D})$ with repeat edge graph $R_\lambda(\mathcal{D})$ and a (v, G, μ)-$MPD = (X, \mathcal{E})$ with leave edge graph $L_\mu(\mathcal{E})$. If $L_\mu(\mathcal{E}) \subset R_\lambda(\mathcal{D})$ and $|R_\lambda(\mathcal{D})| - |L_\mu(\mathcal{E})| = r_{\lambda+\mu}(v) < e(G)$, then there exists a $(v, G, \lambda+\mu)$-MCD with repeat edge graph $R_\lambda(\mathcal{D})\backslash L_\mu(\mathcal{E})$.

If we replace MPD and MCD by OPD and OCD respectively, then the theorem is also true.

Corollary 2.13 If there exist (v, G, λ_1)-GD and (v, G, λ_2)-GD, then there exists $(v, G, \lambda_1 + \lambda_2)$-$GD$.

3 Holey graph designs and incomplete graph designs

Theorem 3.1 There exist $(8 + w, w, G_i, 1)$-IGD for $i \in [2, 4]$, $w \in [2, 7]$.

Proof Let $(8 + w, w, G_i, 1)$-IGD=$(X, \mathcal{A})$, for $i \in [2, 4]$, $w \in [2, 7]$; we construct $\mathcal{A}$ as follows:

<u>$w = 2$</u> On the set $X = Z_8 \cup \{a, b\}$:
For G_2:
$[0, 2, a, 1, 4, b]+i$, $i \in Z_8^*$, $[3, 7, 4, 0, 1, 2]$, $[7, 0, 6, 2, 3, 4]$, $[b, 4, 1, 5, 6, 7]$, $[0, 2, a, 1, 4, 5]$.
For G_3:
$[a, 6, 4, b, 0, 3]+i$, $i \in Z_8^*$, $[1, 2, 6, 4, 0, 7]$, $[2, 3, 7, 4, 5, 6]$, $[0, 1, 5, a, 6, 7]$, $[6, 4, 3, b, 0, 3]$.
For G_4:
$[a, 3, 2, 4, 5, b]+i$, $i \in Z_8^*$, $[2, 6, 4, 0, 3, 5]$, $[2, 5, 7, 0, 3, 6]$, $[a, 3, 1, 0, 2, 5]$, $[5, 6, 2, 4, 3, b]$.

<u>$w = 3$</u> On the set $X = Z_8 \cup \{a, b, c\}$: For G_2: $[a, 2, b, 0, 1, 3]+i$, $i \in Z_8$, $[0, 4, c, 3, 6, 1]+i$, $i \in \{0, 1\}$, $[0, 3, 7, c, 6, 2]$, $[2, 5, 0, c, 1, 4]$, $[3, 7, 2, c, 5, 0]$.
For G_3: $[a, 0, 2, b, 3, 4] + i$, $i \in Z_8$, $[0, 4, 7, c, 2, 5] + i$, $i \in [0, 1]$, $[7, 2, 6, 4, c, 5]$, $[0, 3, 7, 1, c, 6]$, $[6, 1, 4, 0, c, 7]$.
For G_4: $[a, 3, 0, 1, 2, b] + i$, $i \in Z_8$, $[2, 5, 0, 3, 4, c]$, $[3, 7, 1, 4, 5, c]$, $[5, 0, c, 2, 3, 7]$, $[2, 7, c, 4, 5, 6]$, $[4, 7, 6, 1, 2, 3]$.

<u>$w = 4$</u> On the set X=$Z_8 \cup \{a, b, c, d\}$:For G_2: $[a, 2, b, 0, 1, c]+i$, $i \in Z_8^*$, $[3, 7, d, 0, 2, 5]+i$, $i \in [0, 3]$, $[d, 4, b, 0, 1, c]$, $[4, 6, d, 5, 7, 2]$, $[a, 2, d, 7, 1, 6]$, $[1, 4, d, 6, 0, 3]$.
For G_3: $[a, 2, b, 0, 1, d] + i$, $i \in Z_8$, $[c, 1, 5, 2, 4, 7] + i$, $i \in [0, 3]$, $[7, c, 5, 6, 0, 3]$, $[0, c, 6, 7, 1, 4]$, $[0, 2, 5, 1, 3, 6]$.
For G_4: $[a, 3, 1, b, c, d] + i$, $i \in Z_8^*$, $[1, 5, 0, 7, 2, 3] + i$, $i \in [0, 3]$, $[a, 3, 5, 4, 7, 0]$, $[4, 7, 6, 5, 0, 1]$, $[4, 3, 7, 6, 1, 2]$, $[4, 6, 1, b, c, d]$.

<u>$w = 5$</u> On the set $X = Z_8 \cup \{a, b, c, d, e\}$: For G_2: $[a, 2, b, 0, 1, c] + i, i \in Z_8$, $[0, 4, d, 1, 3, e]+i, i \in [0, 3]$, $[0, 3, d, 5, 7, e]$, $[7, 4, d, 6, 0, e]$, $[3, 6, d, 7, 1, e]$, $[6, 1, d, 0, 2, e]$, $[1, 4, 7, 2, 5, 0]$.
For G_3: $[a, 2, b, c, 0, 1] + i$, $i \in Z_8$, $[0, 4, 6, d, 1, e] + i$, $i \in [0, 3]$, $[3, 0, 2, d, 5, e]$, $[7, 2, 4, d, 6, e]$, $[6, 1, 3, d, 7, e]$, $[6, 3, 5, d, 0, e]$, $[1, 4, 7, 2, 5, 0]$.
For G_4: $[a, 3, 1, b, c, d] + i$, $i \in Z_8$, $[1, 5, 0, 7, 2, e] + i$, $i \in [0, 3]$, $[7, 2, 4, 3, 6, e]$, $[3, 6, 5, 4, 7, e]$, $[4, 7, 6, 5, 1, e]$, $[2, 5, 7, 6, 1, e]$, $[1, 4, 0, 3, 5, 6]$.

$\underline{w=6}$ On $X = Z_8 \cup \{a,b,c,d,e,f\}$: For G_2: $[a,2,b,0,1,3]+i,\ i \in Z_8^*$, $[d,2,e,0,3,f]+i,\ i \in Z_8$, $[3,7,b,0,c,4]$, $[2,6,0,1,c,5]$, $[1,5,a,2,c,6]$, $[0,4,1,3,c,7]$.
For G_3: $[a,0,1,b,3,6]+i,\ i \in Z_8^*$, $[d,3,e,0,2,f]+i,\ i \in Z_8$, $[b,3,7,0,c,4]$, $[2,6,3,1,c,5]$, $[0,1,5,2,c,6]$, $[a,0,4,3,c,7]$.
For G_4: $[a,7,0,1,3,b]+i,\ i \in Z_8^*$, $[0,2,1,d,e,f]+i,\ i \in Z_8$, $[c,6,0,1,3,b]$, $[0,4,c,1,2,5]$, $[1,5,c,0,3,4]$, $[2,6,7,3,a,c]$.

$\underline{w=7}$ On the set $X = Z_8 \cup \{a,b,c,d,e,f,g\}$: For G_2: $[a,2,b,0,1,c]+i,\ i \in Z_8$, $[d,1,e,0,2,f]+i,\ i \in Z_8$, $[3,6,7,g,0,4]$, $[4,7,6,g,1,5]$, $[0,3,5,g,2,6]$, $[6,1,4,g,3,7]$, $[1,4,7,2,5,0]$.
For G_3: $[a,2,b,0,1,c]+i,\ i \in Z_8$, $[d,1,e,0,2,f]+i,\ i \in Z_8$, $[3,6,2,7,g,0]$, $[0,4,7,6,g,1]$, $[0,3,7,5,g,2]$, $[6,1,4,7,2,5]$, $[1,5,0,3,g,4]$.
For G_4: $[a,2,0,1,b,c]+i,\ i \in Z_8$, $[d,1,0,2,e,f]+i,\ i \in Z_8$, $[g,1,0,3,4,5]$, $[g,0,2,5,6,7]$, $[3,7,1,4,5,6]$, $[4,7,g,2,3,6]$, $[3,6,g,4,5,7]$. □

Theorem 3.2 There exist G_i-$HGD(8^m)$ for $i \in [2,4],\ m > 1$.

Proof On the set $(Z_4)_1 \cup (Z_4)_2$, we construct
$K_{4,4}/G_2$:
$[1_1,2_2,1_2,2_1,3_2,4_1]$, $[3_1,4_2,1_1,1_2,4_1,2_2]$, $[1_2,3_1,3_2,1_1,4_2,2_1]$, $[4_1,4_2,2_1,2_2,3_1,3_2]$.
$K_{4,4}/G_3$:
$[1_2,2_1,4_2,1_1,2_2,3_1]$, $[2_1,2_2,4_1,1_2,1_1,4_2]$, $[1_2,3_1,4_2,4_1,3_2,1_1]$, $[3_1,3_2,2_1,1_2,4_1,4_2]$.
$K_{4,4}/G_4$:
$[1_1,2_2,1_2,2_1,3_1,4_1]$, $[1_1,1_2,2_2,2_1,3_1,4_1]$, $[4_1,4_2,3_2,1_1,2_1,3_1]$, $[3_2,4_1,4_2,1_1,2_1,3_1]$.
It follows from Theorem 2.9 that there exist G_i-$HGD(8^m)$ for $i \in [2,4],\ m > 1$. □

Theorem 3.3 If there exist $(8+n', G_i, 1)$-$OPD(OCD)$, then there exist $(8m+n', G_i, 1)$-$OPD(OCD)$ for $i \in [2,4]$, $n' \in [2,7]$ and $m > 0$.

Proof By Theorem 2.4, 3.1 and 3.2, we obtain the theorem. □

Theorem 3.4 There exist $(10+w, w, G_i, 1)$-IGD for $i \in [5,11]$, $w = 4,7,8,9,12,13$.

Proof $K_{1,5}/G_6$ is trivial. By Theorem 2.9, we have $K_{10,w}/G_6$ for $w = 4,7,8,9,10,12,13$. On the set $Z_5 \cup \{a,b\}$, $K_{5,2}/G_7$: $[1,a,3,b,4,0]$, $[1,b,2,a,4,0]$.
$K_{5,2}/G_{11}$: $[a,2,b,1,3,4]$, $[b,0,a,1,3,4]$.
On the set $(Z_4)_1 \cup (Z_5)_2$, $K_{4,5}/G_5$: $[0_2,0_1,1_2,3_1,3_2,2_1]+i,\ i \in [0,1]$, $[0_2,3_1,2_2,2_1,4_2,1_1]$, $[1_2,2_1,0_2,1_1,3_2,0_1]$.
$K_{4,5}/G_8$:
$[0_2,0_1,1_2,3_1,3_2,2_1]$, $[1_2,1_1,2_2,0_1,4_2,2_1]$, $[2_1,0_2,3_1,4_2,1_1,2_2]$, $[0_2,1_1,3_2,2_1,4_2,0_1]$.
$K_{4,5}/G_{10}$:
$[0_1,1_2,2_1,0_2,1_1,3_2]$, $[1_1,1_2,3_1,4_2,2_1,2_2]$, $[2_1,1_2,0_1,3_2,3_1,0_2]$, $[3_1,1_2,1_1,2_2,0_1,4_2]$.
On the set $Z_5 \cup \{a,b,c,d,e\}$, $K_{5,5}/G_5$: $[4,a,1,b,2,c]$, $[1,c,3,d,4,e]$, $[0,b,3,e,2,a]$, $[2,d,0,c,4,b]$, $[d,1,e,0,a,3]$.
$K_{5,5}/G_8$: $[0,a,2,b,3,c]$, $[2,e,4,b,0,c]$, $[b,1,a,4,d,3]$, $[c,0,d,1,e,2]$, $[1,c,3,e,0,d]$.
$K_{5,5}/G_{10}$: $[4,e,3,b,2,d]$, $[c,0,a,3,e,4]$, $[3,c,1,d,4,a]$, $[2,b,0,a,1,e]$, $[1,d,2,c,0,b]$.
On the set $(Z_5)_1 \cup (Z_5)_2$, $K_{5,5}/G_9$: $[3_1,4_2,0_1,0_2,1_1,3_2]$ (mod 5).
On the set $\{(Z_4)_1 \cup \{\infty\}\} \cup (Z_4)_2$, $K_{5,4}/G_9$: $[3_2,\infty,0_1,0_2,1_1,2_2]$ (mod 4).
From $K_{5,4}/G_i$, $i = 5,8,9,10$, we can obtain $K_{10,4}/G_i$, $K_{10,8}/G_i$, $K_{5,8}/G_i$ and $K_{10,12}/G_i$ for $i = 5,8,9,10$. From $K_{5,5}/G_i$, $i = 5,8,9,10$, we can obtain $K_{10,5}/G_i$

and $K_{10,10}/G_i$ for $i = 5, 8, 9, 10$. By $K_{5,2}/G_i,\ i = 7, 11$, we can obtain $K_{10,5}/G_i$ and $K_{10,j}/G_i$ for $j = 2, 4, 8, 10, 12, i = 7, 11$. By $K_{10,4}/G_i$ and $K_{10,5}/G_i,\ i \in [5, 11]$, we can obtain $K_{10,9}/G_i$ for $i \in [5, 11]$. By $K_{10,8}/G_i$ and $K_{10,5}/G_i,\ i \in [5, 11]$, we can obtain $K_{10,13}/G_i$ for $i \in [5, 11]$.

On the set $X = Z_5 \cup \{a, b\}$:

$(7, 2, G_5, 1)$-IGD=$(X, \mathcal{A})$, $\mathcal{A}$: $[0, 1, 2, a, 4, b]$, $[a, 1, b, 2, 3, 0]$, $[a, 0, 2, 4, 3, b]$, $[a, 3, 1, 4, 0, b]$.

$(7, 2, G_8, 1)$-IGD=$(X, \mathcal{A})$, $\mathcal{A}$: $[3, 0, 4, 1, b, 2]$, $[0, 2, 3, 4, b, a]$, $[2, 1, 0, b, 3, a]$, $[b, 2, a, 1, 3, 4]$.

$(7, 2, G_{10}, 1)$-IGD=$(X, \mathcal{A})$, $\mathcal{A}$: $[1, b, 3, 4, 0, 2]$, $[b, a, 0, 1, 2, 3]$, $[a, 0, 2, b, 3, 1]$, $[2, 3, b, 4, a, 0]$.

When $i = 5, 8, 10$, by $K_{5,5}/G_i$ and $(7, 2, G_i, 1)$-IGD, we obtain $(12, 2, G_i, 1)$-IGD.

When $i = 6, 7, 11$, by $K_{10,2}/G_i$ and $(10, G_i, 1)$-GD, we obtain $(12, 2, G_i, 1)$-IGD.

On the set $X = Z_{10} \cup \{a, b\}$, $(12, 2, G_9, 1)$-IGD=$(X, \mathcal{A})$, $\mathcal{A}$: $[a, 2, 0, 1, 3, 6] + i$, $[b, 2, 5, 6, 8, 1] + i$, $i \in [1, 4]$ and $[0, 5, a, 7, b, 8]$, $[1, 6, a, 9, b, 0]$, $[3, 8, a, 1, b, 2]$, $[2, 7, 0, 1, 3, 6]$, $[4, 9, 5, 6, 8, 1]$. By $(12, 2, G_i, 1)$-IGD and $K_{10,5}/G_i$, we have $(10 + 7, 7, G_i, 1)$-IGD. Again since there exist $(10, G_i, 1)$-GD and G_i-$HGD(10^1 w^1)$ for $w = 4, 8, 9, 12, 13$, there exist $(10 + w, w, G_i, 1)$-IGD for $w = 4, 7, 8, 9, 12, 13$ and $i \in [5, 11]$. □

Theorem 3.5 If there exist $(10 + n', G_i, 1)$-$OPD(OCD)$, then there exist $(10m + n', G_i, 1)$-$OPD(OCD)$ for $i \in [5, 11]$, $n' = 2, 3, 4, 7, 8, 9$ and $m > 0$.

Proof By Theorem 2.4 and 3.4, we can obtain the theorem. □

Theorem 3.6 When $m \not\equiv 1, 4, 9 \pmod{12}$ and $6 \le m \le 17$, there exist G_i-$HGD((12)^n m^1)$ for $i \in [12, 15]$.

Proof On the set X=$\{1, 2, 3, 4, 5, 6\} \cup \{a, b\}$:

$K_{6,2}/G_{12}$: $[1, a, 3, b, 4, 5]$, $[2, b, 6, a, 4, 5]$. $K_{6,2}/G_{13}$: $[1, a, 3, b, 4, 5]$, $[2, b, 6, a, 4, 5]$.

By Theorem 2.9, there exist G_i-$HGD((12)^1 m^1)$ for $i = 12, 13$ and $m = 6, 8, 10, 12, 14$.

On the set X=$\{1, 2, 3, 4\} \cup \{a, b, c\}$:

$K_{3,4}/G_{14}$: $[b, 1, a, 2, c, 3]$, $[a, 3, b, 4, c, 1]$. $K_{3,4}/G_{15}$: $[1, c, 3, a, 4, b]$, $[c, 2, a, 1, b, 3]$.

By Theorem 2.9, there exist G_i-$HGD((12)^1 m^1)$ for $i = 14, 15$ and $m = 6, 8, 12, 15$.

On the set X=$(Z_6)_0 \cup (Z_7)_1$

$K_{6,7}/G_{12}$: $[0_0, 0_1, 1_0, 3_1, 2_0, 5_0] + i,\ i \in [0, 3]$, $[4_0, 4_1, 5_0, 0_1, 2_0, 3_0] + i,\ i \in [0, 2]$.

$K_{6,7}/G_{13}$: $[4_1, 1_0, 1_1, 2_0, 5_1, 0_1] + i,\ i = 0, 1, 3, 4$, $[6_1, 3_0, 3_1, 4_0, 4_1, 5_1]$, $[0_0, 0_1, 1_0, 2_1, 3_0, 4_0]$, $[0_0, 6_1, 1_0, 3_1, 5_0, 2_0]$.

$K_{6,7}/G_{14}$: $[0_1, 1_0, 3_1, 0_0, 4_1, 3_0] + i,\ i = 0, 2$, $[3_1, 4_0, 6_1, 3_0, 0_1, 2_0] + i,\ i \in [0, 2]$, $[2_1, 0_0, 6_1, 1_0, 5_1, 4_0]$, $[4_1, 1_0, 1_1, 2_0, 3_1, 5_0]$.

$K_{6,7}/G_{15}$: $[5_1, 1_0, 0_1, 0_0, 3_1, 2_0] + i,\ i \in [0, 3]$, $[3_1, 5_0, 4_1, 4_0, 0_1, 2_0]$, $[2_1, 5_0, 5_1, 0_0, 1_1, 3_0]$, $[4_1, 0_0, 6_1, 1_0, 2_1, 4_0]$.

By $K_{6,7}/G_i,\ i \in [12, 15]$, we obtain $K_{12,7}/G_i,\ i \in [12, 15]$. By $K_{6,2}/G_i,\ i = 12, 13$ and $K_{3,4}/G_i,\ i = 14, 15$, we obtain $K_{12,4}/G_i,\ i \in [12, 15]$. Therefore, $K_{12,11}/G_i,\ i \in [12, 15]$ can be obtained. By $K_{3,4}/G_i$ and $K_{6,7}/G_i, i = 14, 15$, we obtain $K_{12,10}/G_i,\ i = 14, 15$ and $K_{12,14}/G_i,\ i = 14, 15$. By $K_{6,2}/G_i,\ i = 12, 13$ and $K_{3,4}/G_i,\ i = 14, 15$, we obtain $K_{12,8}/G_i,\ i \in [12, 15]$. Furthermore, by $K_{12,7}/G_i,\ i \in [12, 15]$, we can obtain $K_{12,15}/G_i,\ i \in [12, 15]$. By $K_{12,7}/G_i,\ i \in [12, 15]$ and $K_{12,10}/G_i,\ i \in [12, 15]$, there are $K_{12,17}/G_i$ for $i \in [12, 15]$.

It follows from Theorem 2.1 that the theorem is true. □

Theorem 3.7 When $i \in [12, 15]$, if there exist $(m, G_i, 1)$-$OPD(OCD)$ for $m = 6, 7, 8, 10, 11$ and $(12+m, G_i, 1)$-$OPD(OCD)$ for $m = 2, 3, 5$, then there exist $(12k + m, G_i, 1)$-$OPD(OCD)$ for $k \geq 1,\ m = 2, 3, 5, 6, 7, 8, 10, 11$.

Proof Since there exist $(12, G_i, 1)$-GD, it follows from Theorem 3.6, Theorem 2.5 and Theorem 2.10 that the theorem is true. □

4 Packings and coverings for $\lambda = 1$

Let P be the necessary and sufficient condition for the existence of $(v, G, 1)$-GD. When v does not satisfy P, we discuss (v, G, λ)-PD and (v, G, λ)-CD. We easily obtain the following lemma:

Lemma 4.1 If there exists $(v, G, 1)$-OPD with leave-edge number $l_1 = 1$, then there exists $(v, G, 1)$-OCD.

Lemma 4.2 For any positive integer n, there exists a G_1-$HGD(6^n)$.
Proof Since $K_{3,3}$ is 1-factorable, the lemma is true. □

Theorem 4.3 There exist $(v, G_1, 1)$-OPD (or OCD) for $v \equiv 2 (\mathrm{mod}\ 3)$.
Proof 1) Both $(8, 2, G_1, 1)$-IGD and $(8, G_1, 1)$-OPD are the same. On the vertex set $X = Z_6 \cup \{a, b\}$, let $(8, G_1, 1)$-$OPD = (X, \mathcal{B})$.
$\mathcal{B}$: $[a, 3, b, 1, 2, 4]$ (mod 6),$[1, 4, 2, 3, 0, 5]$, $[2, 5, 3, 4, 0, 1]$, $[0, 3, 1, 2, 4, 5]$. $(8, G_1, 1)$-OCD=$(X, \mathcal{A})$, where $\mathcal{A} = \mathcal{B} \cup \{[\dashv, \lfloor, \infty, \in, \ni, \triangle]\}$.
By Lemma 4.2, there exists a G_1-$HGD(6^2)$. Therefore, there exist $(6m + 2, G_1, 1)$-OPD (or OCD) for all $m \geq 1$.
2) On the vertex set $X = Z_6 \cup \{a, b, c, d, e\}$, we construct
A: $[a, 0, b, 1, c, 2]$ and $[d, 0, e, 1, 2, 4]$ (mod 6);
B: $[1, 4, 2, 3, 0, 5]$, $[2, 5, 3, 4, 0, 1]$, $[0, 3, 1, 2, 4, 5]$;
C: $[a, c, b, d, 1, 4]$, $[a, d, e, c, 2, 3]$, $[a, e, c, d, 0, 5]$, $[b, c, e, d, 2, 5]$, $[b, e, 3, 4, 0, 1]$, $[0, 3, 1, 2, 4, 5]$; D: $[a, b, 1, 2, 3, 4]$.
It is easy to verify that $(X, A \cup B)$ is a $(11, 5, G_1, 1)$-IGD, $(X, A \cup C)$ is a $(11, G_1, 1)$-OPD and $(X, A \cup C \cup D)$ is a $(11, G_1, 1)$-OCD. Therefore, there exist $(6m+5, G_1, 1)$-OPD (or OCD) for all $m \geq 1$. It follows from 1) and 2) that the theorem is true. □

Lemma 4.4 There is no $(6, G_i, 1)$-OCD for $i = 3, 4$.

Proof If there exists a $(6, G_3, 1)$-OCD, then $c(6, G_3, 1) = 4$ and there is one edge repeated; let the edge be $(0, 1)$. The 0 and 1 must appear as a 2-degree vertex of two blocks, but 0 and 1 cannot appear as two 2-degree vertices of the same block. Four other vertices occupy one 2-degree vertex of four blocks, respectively. In this case, each edge of K_6 cannot appear only once in four blocks, except the edge $(0, 1)$. This is a contradiction.

If there exists a $(6, G_4, 1)$-OCD, then $c(6, G_4, 1) = 4$ and four 3-degree vertices in four blocks are distinctly labelled. Suppose a and b do not appear in any 3-degree vertex of the four blocks. Then the degree of vertex a is 1 in every block. In the five edges incident with vertex a in K_6, one edge is not contained in any block: this is contrary to the definition of covering. □

Theorem 4.5 There exist $(v, G_i, 1)$-*OPD* (or *OCD*) for $i = 2, 3, 4$ and $v \not\equiv 0$ or 1 (mod 8), except for $(6, G_i, 1)$-*OCD* for $i = 3$ and 4.

Proof $\underline{v = 6}$: On the set $X = Z_6$, $(6, G_2, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[1, 4, 0, 2, 3, 5] + i$, $i \in [0, 2]$. Leave edges: $(5, 0, 1, 2)$.
$(6, G_2, 1)$-OCD=$(X, \mathcal{A} \cup \{[3, 4, 5, 0, 1, 2]\})$.
$(6, G_3, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[0, 1, 4, 3, 5, 2]$, $[0, 3, 1, 2, 4, 5]$, $[1, 2, 3, 5, 0, 4]$.
Leave edges: $02, 15, 34$.
$(X, \mathcal{A} \cup \{[0, 2, 3, 1, 4, 5], [0, 1, 5, 2, 3, 4]\})$ is a $(6, G_3, 1)$-CD. By Lemma 4.4, we have $c(6, G_3, 1) = 5$.
$(6, G_4, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[4, 5, 0, 1, 2, 3] + i = 0, 1, [1, 5, 4, 0, 2, 3]$ Leave edges: $(5, 3, 2, 5)$.
$(X, \mathcal{A} \cup \{[2, 3, 0, 1, 4, 5], [0, 1, 5, 2, 3, 4]\})$ is a $(6, G_4, 1)$-CD. By Lemma 4.4, we have $c(6, G_4, 1) = 5$.

$\underline{v = 7}$: On the set $X = Z_7$, $(7, G_2, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[1, 4, 0, 2, 3, 5] + i$, $i \in [0, 2]$, $[2, 6, 5, 1, 0, 3]$, $[1, 2, 5, 6, 0, 4]$.
$(7, G_3, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[2, 3, 6, 4, 5, 0]$, $[2, 4, 0, 5, 6, 1]$, $[3, 0, 6, 1, 2, 5]$, $[0, 1, 5, 3, 4, 6]$, $[1, 3, 5, 0, 2, 6]$.
$(7, G_4, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[4, 5, 0, 1, 2, 3]+i$, $i \in [0, 2]$, $[1, 5, 4, 0, 3, 6]$, $[0, 5, 6, 1, 2, 3]$.
In this case $l_1 = 1$. Apply Lemma 4.1; there exist $(7, G_4, 1)$-OCD for $i \in [2, 4]$.

$\underline{v = 10}$: Both $(10, 2, G_i, 1)$-IGD and $(10, G_i, 1)$-OPD are the same for $i = 2, 3$ and 4. Since $l_1 = 1$, there exist $(10, G_4, 1)$-OCD for $i \in [2, 4]$.

$\underline{v = 11}$: On the set X=Z_{11}, $(11, G_2, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[0, 2, 1, 4, 8, 3] + i$, $i \in Z_{11}$, $[7, 8, 3, 4, 5, 6]$, $[2, 3, 8, 9, 10, 0]$. Leave edges: $(0, 1, 2)$ and 67.
$(11, G_2, 1)$-OCD=$(X, \mathcal{A} \cup \{[6, 7, 0, 1, 2, 4]\})$. Repeat edge: 24.
$(11, G_3, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[5, 0, 2, 1, 4, 8] + i$, $i \in Z_{11}$,
$[2, 3, 4, 7, 8, 9]$, $[4, 5, 6, 9, 10, 0]$. Leave edges: 01, 12 and 67.
$(11, G_3, 1)$-OCD=$(X, \mathcal{A} \cup \{[5, 6, 7, 0, 1, 2]\})$. Repeat edge: 56.
$(11, G_4, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[3, 8, 4, 7, 0, 6] + i$, $i \in Z_{11}^* \backslash \{1\}, [0, 10, 4, 3, 5, 9]$,
$[5, 6, 8, 3, 7, 9]$, $[2, 3, 4, 7, 0, 6]$, $[9, 10, 5, 8, 1, 7]$. Leave edges: 01, 12 and 67.
$(11, G_4, 1)$-OCD=$(X, \mathcal{A} \cup \{[6, 7, 1, 0, 2, 3]\})$. Repeat edge: 13.

$\underline{v = 12}$: On the set $Z_8 \cup \{a, b, c, d\}$
$(12, G_2, 1)$-OPD $=$ $(X, \mathcal{A})$, $\mathcal{A}$: $[a, 2, b, 0, 1, c] + i$, $i \in Z_8^*$, $[3, 7, d, 0, 2, 5] + i$, $i \in [0, 3]$, $[a, d, b, 0, 1, c]$, $[a, b, d, 4, 6, 1]$, $[a, c, d, 5, 7, 2]$, $[a, 2, d, 7, 1, 4]$, $[b, c, d, 6, 0, 3]$. Leave edges: (b, d, c).
$(12, G_2, 1)$-OCD $= (X, \mathcal{A} \cup \{[a, 1, b, d, c, 2]\})$. Repeat edges: $a1$, $c2$.
$(12, G_3, 1)$-OPD $= (X, \mathcal{A})$, $\mathcal{A}$: $[a, 2, b, 0, 1, d] + i$, $i \in Z_8$, $[c, 1, 5, 2, 4, 7] + i$, $i \in [0, 3]$, $[a, c, 5, 6, 0, 3]$, $[b, c, 6, 7, 1, 4]$, $[b, a, d, 0, 2, 5]$, $[7, c, 0, 1, 3, 6]$. Leave edges: bd and cd.
$(12, G_3, 1)$-OCD $= (X, \mathcal{A} \cup \{[b, d, c, 0, 1, 2]\})$. Repeat edges: $(0, 1, 2)$.
$(12, G_4, 1)$-OPD $= (X, \mathcal{A})$, $\mathcal{A}$: $[a, 0, 1, b, c, d] + i$, $i \in Z_8$, $[1, 5, 0, 7, 2, 3] + i$, $i \in [0, 3]$, $[a, b, 4, 3, 6, 7]$, $[a, c, 5, 4, 7, 0]$, $[a, d, 6, 5, 0, 1]$, $[b, c, 7, 6, 1, 2]$. Leave edges: bd and cd.
$(12, G_4, 1)$-OCD $= (X, \mathcal{A} \cup \{[0, 1, d, b, c, 2]\})$. Repeat edges: $(0, 1)$, $(d, 2)$.

$\underline{v = 13}$: On the set $Z_9 \cup \{a, b, c, d\}$
$(13, G_2, 1)$-OPD $= (X, \mathcal{A})$, $\mathcal{A}$: $[a, 2, b, 0, 1, 5] + i$, $i \in Z_9$, $[c, 4, d, 0, 2, 5] + i$, $i \in Z_9^*$, $[0, 2, a, b, c, 4]$, $[2, 5, a, c, d, 0]$. Leave edges: (a, d, b).

$(13, G_2, 1)$-$OCD = (X, \mathcal{A} \cup \{[0, 1, 2, a, d, b]\})$. Repeat edges: $(0, 1)$, $(a, 2)$.
$(13, G_3, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$: $[a, 1, 3, c, 2, 6] + i$, $i \in Z_9$, $[b, 2, 3, d, 1, 4] + i$, $i \in Z_9^*$, $[a, b, c, d, 1, 4]$, $[b, 2, 3, a, c, d]$. Leave edges: (a, d, b).
$(13, G_3, 1)$-$OCD = (X, \mathcal{A} \cup \{[0, 1, 2, a, d, b]\})$. Repeat edges: $(0, 1, 2)$.
$(13, G_4, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$: $[a, 3, 1, b, c, d] + i$, $i \in Z_9$, $[1, 5, 0, 8, 2, 3] + i$, $i \in Z_9^*$, $[1, 5, c, a, b, d]$, $[a, b, 0, 8, 2, 3]$. Leave edges: (a, d, b).
$(13, G_4, 1)$-$OCD = (X, \mathcal{A} \cup \{[0, 1, d, b, a, 2]\})$. Repeat edges: $(0, 1)$, $(d, 2)$.

$\underline{v = 14}$: On the $X = Z_{11} \cup \{a, b, c\}$,
$(14, G_2, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$: $[a, 2, b, 0, 4, c] + i$, $i \in Z_{11}$, $[2, 5, 0, 1, 6, 8] + i$, $i \in Z_{11}^*$, $[a, c, 0, 1, 6, 8]$. Leave edges: (a, b, c) and 25.
$(14, G_2, 1)$-$OCD = (X, \mathcal{A} \cup \{[2, 5, a, b, c, 0]\})$. Repeat edge: $0c$.
$(14, G_3, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$: $[c, 3, 5, 0, 1, 6] + i$, $i \in Z_{11}$, $[a, 2, 5, b, 0, 4] + i$, $i \in Z_{11}^*$, $[c, a, 2, b, 0, 4]$. Leave edges: (a, b, c) and 25 .
$(14, G_3, 1)$-$OCD = (X, \mathcal{A} \cup \{[2, 5, 0, a, b, c]\})$. Repeat edge: 05.
$(14, G_4, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$: $[1, 3, 0, a, 5, c] + i$, $i \in Z_{11}$, $[2, 5, 0, 1, b, 4] + i$, $i \in Z_{11}^*$, $[a, c, 0, 1, b, 4]$. Leave edges: (a, b, c) and 25 .
$(14, G_4, 1)$-$OCD = (X, \mathcal{A} \cup \{[2, 5, b, a, c, 0]\})$.Repeat edge: $0b$.
Since there exist $(15, 7, G_i, 1)$-IGD and $(7, G_i, 1)$-$OPD(OCD)$, there exist $(15, G_i, 1)$-$OPD(OCD)$ for $i \in [2, 4]$. It follows from Theorem 3.3 that there exist $(v, G_i, 1)$-$OPD(OCD)$ for $i \in [2, 4]$, $v \not\equiv 0,\ 1 \pmod 8$ and $v > 6$. □

Lemma 4.6 $c(8, G_6, 1) = 7$, $c(7, G_6, 1) = 6$, $p(7, G_6, 1) = 3$, $c(7, G_9, 1) = 5$ and $p(7, G_9, 1) = 3$.

Proof If there exists a $(8, G_6, 1)$-OCD, then it contains 6 blocks. In eight vertices on K_8, there are two vertices that cannot occur on the center of the 6 blocks. Let the two vertices be a and b. The edge ab cannot occur on any block. This is a contradiction.

On $X = Z_5 \cup \{a, b, c\}$, set A: $[0, 1, 2, a, b, c] + i$, $i \in Z_5$, $[a, b, c, 1, 2, 3]$, $[c, b, 0, 1, 2, 3]$. Then (X, A) is a $(8, G_6, 1)$-CD, repeat edges: $(1, a, 2), (a, 3), (0, c, 1), (2, c, 3)$. Similarly, we can show there is no $(7, G_6, 1)$-OCD.

If there exists a $(7, G_6, 1)$-OPD, then $p(7, G_6, 1) = 4$ and $l_1 = 1$. Let the other 3 vertices except the center of the 4 blocks be a, b and c; then edges ab, ac and bc cannot appear in the 4 blocks. This is a contradiction.

Let $X = Z_5 \cup \{a, b\}$, construction A: $[0, 1, 2, 3, a, b]$, $[4, 0, 1, 2, a, b]$, $[3, 1, 2, 4, a, b]$; B: $[a, 1, 2, 3, 4, 0]$, $[b, 1, 2, 3, 4, a]$, $[1, 2, 3, 4, 0, a]$. Then (X, A) is a $(7, G_6, 1)$-PD, leave edges: $(b, 1, a, 2, b, a)$, $(1, 2)$. $(X, A \cup B)$ is a $(7, G_6, 1)$-CD, repeat edges: $(4, b, 3, a, 4, 1, 0, a, 1, 3)$.

The degree of every vertex on K_7 is 6. Since $G_9 = P_2 \cup C_4$, an OPD contains 4 cycles C_4. Using enumeration, we know that at least an edge on K_7 cannot match with the 4 cycles C_4. Therefore, there does not exist $(7, G_9, 1)$-OPD. On the set $X = Z_7$, let A: $[0, 5, 1, 2, 3, 6]$, $[0, 3, 2, 4, 1, 5]$, $[2, 6, 0, 1, 3, 4]$; B: $[1, 3, 0, 5, 4, 6]$, $[0, 2, 3, 4, 6, 5]$. Then (X, A) is a $(7, G_9, 1)$-PD and leave edges are $(5, 4, 6, 0), (0, 2), (6, 5, 3)$. And $(X, A \cup B)$ is a $(7, G_9, 1)$-OCD and repeat edges are $(1, 3, 4, 6), (0, 5)$. □

Theorem 4.7 There exist $(v, G_i, 1)$-OPD (or OCD) for $i \in [5, 11]$ and $v \not\equiv 0, 1 \pmod 5$, for packing except for $v = 7, i = 6$ and 9; for covering except for $(i, v) =$

$(6,7),(6,8)$ and $(9,7)$.

Proof $\underline{v=7}$: The $(7,G_i,1)$-OPD with $(7,2,G_i,1)$-IGD are the same when $i=5,8,10$ (see the proof of Theorem 3.4).

On the set $X=Z_5\cup\{a,b\}$, $(7,G_7,1)$-$OPD=(X,\mathcal{A})$,
$\mathcal{A}$: $[a,3,4,b,0,2]$,$[4,2,1,0,3,a]$, $[b,3,2,a,1,4]$, $[2,0,4,1,3,b]$.

$(7,G_9,1)$-$PD(CD)$; see Lemma 4.6.

$(7,G_{11},1)$-$OPD=(X,\mathcal{A})$,
$\mathcal{A}$: $[4,b,2,1,3,a]$, $[1,a,3,0,4,b]$, $[b,1,4,0,2,a]$,$[3,1,0,2,a,b]$.

There exist $(7,G_i,1)$-OCD for $i=5,7,8,10,11$ by Lemma 4.1.

$\underline{v=8}$: On the set $X=Z_5\cup\{a,b,c\}$,
A: $[3,0,b,1,c,2]$, $[a,2,b,3,c,4]$, $[0,1,2,3,4,a]$,$[c,0,2,4,1,a]$; B: $[b,4,0,a,3,1]$; C: $[c,b,4,0,a,3]$, $[c,a,b,0,3,1]$.
$(8,G_5,1)$-$OPD=(X,A\cup B)$, leave edges: (a,b,c,a). $(8,G_5,1)$-$OCD=(X,A\cup C)$, repeat edges: $(b,0,3)$.
A: $[0,1,2,a,b,c]+i,\ i\in Z_5$. $(8,G_6,1)$-$OPD=(X,A)$, leave edges: (a,b,c,a).
A: $[a,0,1,3,b,c]+i,\ i\in[0,2]$; B: $[a,0,1,3,b,c]+i,\ i\in[3,4]$; C: $[3,4,1,b,c,2]$, $[2,0,4,a,b,c]$, $[a,3,0,c,1,2]$.
$(8,G_7,1)$-$OPD=(X,A\cup B)$, leave edges: (a,b,c,a). $(8,G_7,1)$-$OCD=(X,A\cup C)$, repeat edges: $(c,0,3)$.
A: $[a,0,1,3,b,c]+i,\ i\in[0,3]$; B: $[a,4,0,2,b,c]$; C: $[2,b,a,c,0,4]$, $[4,0,2,b,c,1]$.
$(8,G_8,1)$-$OPD=(X,A\cup B)$, leave edges: (a,b,c,a). $(8,G_8,1)$-$OCD=(X,A\cup C)$, repeat edges: $(1,2,b)$.
A: $[b,c,a,0,1,2]$, $[c,0,b,2,3,4]$, $[a,b,c,3,0,4]$, $[0,b,c,2,4,1]$, $[0,2,a,1,b,3]$; B: $[1,3,c,a,4,0]$. $(8,G_9,1)$-$OPD=(X,A)$, leave edges: $(1,3),(c,a,4)$. $(8,G_9,1)$-$OCD=(X,A\cup B)$, repeat edges: $(c,0,4)$.
A: $[a,b,c,0,1,3]+i,\ i=0,2,3$; B: $[a,b,c,0,1,3]+i,\ i=1,4$; C: $[1,a,b,c,4,0]$, $[1,2,c,a,b,0]$, $[3,0,a,1,2,4]$.
$(8,G_{10},1)$-$OPD=(X,A\cup B)$, leave edges: (a,b,c,a). $(8,G_{10},1)$-OCD=$(X,A\cup C)$, repeat edges: $(a,1,3)$.
A: $[a,0,1,3,b,c]+i,\ i=1,2,4$; B: $[a,0,1,3,b,c]+i,\ i=0,3$; C: $[4,3,a,b,c,0]$, $[c,b,1,0,3,4]$, $[1,c,4,b,2,3]$.
$(8,G_{11},1)$-$OPD=(X,A\cup B)$, leave edges: (a,b,c,a). $(8,G_{11},1)$-$OCD=(X,A\cup C)$, repeat edges: $(2,4,3)$.

$\underline{v=9}$: On $X=Z_7\cup\{a,b\}$, $(9,G_5,1)$-$OPD=(X,\mathcal{A})$, $\mathcal{A}$: $[a,0,1,3,6,b]$ (mod 7).
$(9,G_6,1)$-OPD=$(X,\mathcal{A})$, $\mathcal{A}$: $[0,1,2,3,a,b]$ (mod 7). $(9,G_7,1)$-$OPD=(X,\mathcal{A})$, $\mathcal{A}$: $[0,1,3,6,a,b]$ (mod 7).
$(9,G_8,1)$-$OPD=(X,\mathcal{A})$, $\mathcal{A}$: $[0,1,3,6,a,b]$ (mod 7).
$(9,G_9,1)$-$OPD=(X,\mathcal{A})$, $\mathcal{A}$: $[2,5,a,0,6,1]$, $[1,4,a,2,b,3]$, $[0,1,b,4,6,5]$, $[0,2,a,4,3,5]$, $[4,5,6,3,1,2]$, $[a,6,b,0,5,1]$, $[6,b,2,4,0,3]$.
$(9,G_{10},1)$-$OPD=(X,\mathcal{A})$, $\mathcal{A}$: $[a,b,0,1,3,6]$ (mod 7). $(9,G_{11},1)$-$OPD=(X,\mathcal{A})$, $\mathcal{A}$: $[0,1,3,6,a,b]$ (mod 7).
There exist $(9,G_i,1)$-OCD for $i\in[5,11]$ by Lemma 4.1.

$\underline{v=12}$: By the proof of Theorem 3.4, there exist $(12,G_i,1)$-OPD for $i\in[5,11]$. By Lemma 4.1, there exist $(12,G_i,1)$-OCD for $i\in[5,11]$.

$\underline{v=13}$: For G_i, $i=5,8,9,10$, since the $(8,G_i,1)$-OPD is a $(8,3,G_i,1)$-IGD (see $v=8$), by Theorem 2.4 and $K_{5,5}/G_i$ we obtain $(13,G_i,1)$-OPD. By Theorem 2.9, $K_{5,1}/G_6 \Rightarrow K_{5,3}/G_6 \Rightarrow K_{10,3}/G_6$. From $K_{10,3}/G_6$ and $(10,G_6,1)$-GD, we can obtain $(13,G_6,1)$-OPD.
On the set $Z_5 \cup \{a,b,c\}$, $K_{5,3}/G_7$: $[0,b,1,a,2,3]$, $[0,c,3,b,2,4]$, $[0,a,4,c,1,2]$.
$K_{5,3}/G_{11}$: $[c,3,a,0,2,4]$, $[a,1,b,2,3,4]$, $[b,0,c,1,2,4]$.
By Theorem 2.9, $K_{5,3}/G_i \Rightarrow K_{10,3}/G_i$. From $K_{10,3}/G_i$ and $(10,G_i,1)$-GD, we obtain $(13,G_i,1)$-OPD for $i=7,11$. Leave edges: (a,b,c,a)

In the same way, we can obtain $(13,G_i,1)$-OCD for $i \in [5,11]$.

$\underline{v=14}$: On the set $X=Z_{12}\cup\{a,b\}$, $(14,G_5,1)$-$OPD=(X,\mathcal{A})$, $\mathcal{A}$: $[b,0,1,3,6,10]+i$, $i\in Z_{12}$, $[7,1,6,11,a,5]+i$, $i\in[0,3]$, $[6,0,5,10,a,9]$, $[10,3,a,4,11,5]$.
$(14,G_6,1)$-$OPD=(X,\mathcal{A})$, $\mathcal{A}$: $[6,7,8,9,1,2]+i$, $i\in[0,3]$, $[6,a,b,0,10,11]+i$, $i\in[0,3]$, $[2,3,4,5,a,b]+i$, $i\in[0,2]$, $[0,1,2,3,4,10]$, $[1,2,3,4,10,11]$, $[5,0,1,6,7,8]$, $[10,2,3,4,5,11]$, $[11,0,2,3,4,5]$, $[a,0,1,5,10,11]$, $[b,0,1,5,10,11]$.
$(14,G_7,1)$-$OPD=(X,\mathcal{A})$, $\mathcal{A}$: $[0,6,9,1,a,b]+i$, $i\in[0,5]$, $[2,0,3,7,a,b]+i$, $i\in[0,5]$, $[1,0,5,6,8,11]+i$, $i\in[0,3]$, $[11,4,9,10,0,3]$, $[4,5,10,11,0,1]$.
$(14,G_8,1)$-$OPD=(X,\mathcal{A})$, $\mathcal{A}$: $[0,6,9,1,a,b]+i$, $[2,0,3,7,a,b]+i$, $i\in[0,5]$, $[11,0,5,6,8,10]+i$, $i\in[0,4]$, $[3,10,11,4,5,1]$.
$(14,G_9,1)$-$OPD=(X,\mathcal{A})$, $\mathcal{A}$: $[3,9,0,1,6,a]+i$, $i\in[0,5]$, $[2,4,0,1,6,b]+i$, $i\in[6,10]$,
$[8,0,1,4,7,10]$, $[0,4,2,5,8,11]$, $[6,8,11,0,5,b]$, $[5,9,2,4,6,10]$, $[2,6,3,5,7,11]$, $[4,8,1,3,7,9]$, $[1,5,0,3,6,9]$.
$(14,G_{10},1)$-$OPD=(X,\mathcal{A})$, $\mathcal{A}$: $[a,b,0,2,5,9]+i$, $i\in Z_{12}$,
$[7,11,0,1,6,5]+i$, $i\in[0,4]$, $[4,5,10,11,0,6]$.
$(14,G_{11},1)$-$OPD=(X,\mathcal{A})$, $\mathcal{A}$: $[0,6,9,1,a,b]+i$, $[2,0,3,7,a,b]+i$, $i\in[0,5]$,
$[0,1,6,7,8,11]+i$, $i\in[1,4]$, $[0,1,6,7,8,5]$, $[5,0,11,1,4,6]$.
By Lemma 4.1, we can obtain $(14,G_i,1)$-OCD for $i\in[5,11]$.

$\underline{v=17}$: On the set $X=Z_{15}\cup\{a,b\}$, $(17,G_6,1)$-$OPD=(X,\mathcal{A})$,
$\mathcal{A}$: $[0,3,4,5,6,7]$ (mod 15), $[0,2,13,14,a,b]+3i$, $i\in[0,4]$, $[1,0,2,14,a,b]+3i$, $i\in[0,4]$, $[a,2,5,8,11,14]$, $[b,2,5,8,11,14]$. By Lemma 4.1, there exist $(17,G_6,1)$-OCD.

$\underline{v=18}$: On the set $X=Z_{15}\cup\{a,b,c\}$, let A: $[0,1,2,3,4,5]$ (mod 15);
B: $[0,6,7,a,b,c]+i$, $i\in Z_{15}$; C: $[0,6,7,a,b,c]+i$, $i\in Z_{15}\backslash\{0,1,6,7,8\}$, $[a,b,c,0,1,6]$, $[b,c,0,1,2,7]$, $[c,0,1,2,7,8]$, $[6,12,13,0,b,c]$, $[7,13,14,0,1,a]$, $[8,14,0,1,a,b]$. It is easy to verify that $(X,A\cup B)$ is a $(18,G_6,1)$-OPD and $(X,A\cup C)$ is a $(18,G_6,1)$-OCD.

It follows from Theorem 3.5 and Lemma 4.6 that the theorem is true. □

Lemma 4.8 $p(6,G_{12},1)=1$, $c(6,G_{12},1)=4$ and $c(6,G_{13},1)=4$.

Proof Since $v(K_6)=v(G_{12})=6$, $V(K_6)=V(G_{12})$ and $d(v)=5$ for every $v\in V(K_6)$. If $p(6,G_{12},1)=2$, then there are two C_4 and two vertices whose degree is four on two G_{12}. In eight vertices of the two C_4, there are two vertices on the K_6 which are used twice, and the degree of these two vertices is not four. Let x_1 and x_2 be the two 4-degree vertices; then $x_1(x_2)$ only appears in the pendant vertices of the other G_{12}, and the edge x_1x_2 is repeated once. This is contrary to the definition of packing.

Let Z_6 be the vertex set of K_6. Since $(Z_6, \{[0,1,2,3,4,5]\})$ is a $(6, G_{12}, 1)$-PD, the packing number $p(6, G_{12}, 1) = 1$, leave edges: $(0,2,4,5,2), (1,4,0,5,1,3)$.
If there exists a $(6, G_{12}, 1)$-OCD, then it contains three blocks and $r_1 = 3$. The three 4-degree vertices in the 3 blocks are different. Since $V(G_{12}) = V(K_6)$, the degree set of the three 4-degree vertices all are $\{4, 1, 1\}$ in the three blocks. In this case, there is a edge on K_6 that cannot appear in any block. This is a contradiction. Similarly, we can obtain $c(6, G_{13}, 1) = 4$. □

Theorem 4.9 There exist $(v, G_i, 1)$-OPD (or OCD) for $i \in [12, 15]$ $v \equiv 2, 3, 5, 6, 7, 8, 10, 11 \pmod{12}$, for packing except for $v = 6, i = 12$, for covering except for $v = 6, i = 12$ and 13.

Proof $\underline{v = 6}$: By the above lemma, $(6, G_{12}, 1)$-OPDdoes not exist. On $X = Z_6$,
$(6, G_{13}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[0,3,1,2,4,5]$, $[0,5,1,4,3,2]$. Leave edges: 01, 23, 45.
$(6, G_{14}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[0,3,1,2,4,5]$, $[0,5,1,4,3,2]$. Leave edges: 01, 25, 35.
$(6, G_{14}, 1)$-OCD=$(X, \mathcal{A} \cup \{[2,5,3,0,1,4]\})$. Repeat edges: 20, 03, 14.
$(6, G_{15}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[4,1,0,3,2,5]$, $[2,0,4,3,5,1]$. Leave edges: 13, 24, 45.
$(6, G_{15}, 1)$-OCD=$(X, \mathcal{A} \cup \{[2,4,5,1,3,0]\})$. Repeat edges: 51, 43, 30.

$\underline{v = 7}$: On $X = Z_7$. $(7, G_{12}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$:
$[3,0,1,2,5,6]$, $[0,5,3,4,2,6]$, $[1,4,5,6,0,3]$. Leave edges: 02, 15, 13.
$(7, G_{12}, 1)$-OCD=$(X, \mathcal{A} \cup \{[0,2,3,1,5,4]\})$. Repeat edges: 01, 14, 23.
$(7, G_{13}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[1,2,3,0,4,6]$, $[3,4,0,5,6,1]$, $[6,1,4,5,3,2]$. Leave edges: $(0,2,6,3)$.
$(7, G_{13}, 1)$-OCD=$(X, \mathcal{A} \cup \{[1,2,6,3,0,5]\})$ Repeat edges: $(2,1,3,5)$.
$(7, G_{14}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[0,1,2,3,6,4]$, $[3,4,0,5,2,6]$, $[1,4,5,6,0,2]$.
Leave edges: 24, 15, 13.
$(7, G_{14}, 1)$-OCD=$(X, \mathcal{A} \cup \{[5,0,3,1,2,4]\})$. Repeat edges: 50, 03, 12.
$(7, G_{15}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[5,1,0,3,2,6]$, $[2,4,0,5,3,1]$, $[0,6,1,4,5,2]$.
Leave edges: 02, 46, 63.
$(7, G_{15}, 1)$-OCD=$(X, \mathcal{A} \cup \{[4,6,0,2,3,5]\})$. Repeat edges: 60, 23, 35.

$\underline{v = 8}$: On the set $X = Z_8$, $(8, G_{12}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$:
$[0,2,4,6,3,5]$, $[2,3,0,1,4,6]$, $[5,1,3,4,7,0]$, $[2,6,7,5,0,3]$. Leave edges:
72, 17, 70, 37.
$(8, G_{12}, 1)$-$OCD = (X, \mathcal{A} \cup \{[1,4,0,7,2,3]\})$. Repeat edges: 14, 40.
$(8, G_{13}, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$: $[0,1,2,3,6,5]$, $[6,0,2,4,7,1]$, $[1,3,4,5,7,6]$,
$[2,6,7,5,3,0]$. Leave edges: 04, 47, 71, 27.
$(8, G_{13}, 1)$-$OCD = (X, \mathcal{A} \cup \{[4,7,2,0,1,3]\})$. Repeat edges: 20, 03.
$(8, G_{14}, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$: $[0,1,2,3,5,6]$, $[0,2,4,6,1,7]$, $[5,1,3,4,0,7]$,
$[5,2,6,7,4,1]$. Leave edges: 27, 73, 36, 05.
$(8, G_{14}, 1)$-$OCD = (X, \mathcal{A} \cup \{[6,5,0,3,7,2]\})$. Repeat edges: 65, 30.
$(8, G_{15}, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$: $[5,3,0,1,2,7]$, $[7,0,2,4,6,1]$, $[6,5,1,3,4,0]$,
$[0,5,2,6,7,4]$. Leave edges: 63, 37, 71, 14.
$(8, G_{15}, 1)$-$OCD = (X, \mathcal{A} \cup \{[6,3,7,1,4,0]\})$. Repeat edges: 34, 40.
$\underline{v = 10}$: On the $X = Z_8 \cup \{a, b\}$, $(10, G_{12}, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$:
$[0,2,1,4,a,b]+i$, $i \in [0,3]$, $[5,0,7,6,1,4]$, $[b,3,a,1,0,7]$, $[b,2,a,0,3,6]$. Leave edges:
72, 57, ab.

$(10, G_{12}, 1)$-$OCD = (X, \mathcal{A} \cup \{[a, b, 5, 7, 2, 0]\})$. Repeat edges: $b5$, $a7$, 70.
$(10, G_{13}, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$:
$[0, 2, 1, 4, a, b]+i$, $i \in [0, 3]$, $[1, b, 2, 7, 0, 5]$, $[0, 6, 7, a, 4, 1]$, $[5, 0, 1, 6, 3, a]$. Leave edges: 07, ab, $b3$.
$(10, G_{13}, 1)$-$OCD = (X, \mathcal{A} \cup \{[3, 7, 0, b, 5, a]\})$. Repeat edges: $b0$, 57, 73.
$(10, G_{14}, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$:
$[0, 2, 1, 4, 6, a]+i$, $i \in [0, 3]$, $[7, 6, 5, 0, b, 4]$, $[b, 1, 0, 3, a, 4]$, $[2, a, 5, b, 6, 1]$. Leave edges: 27, $7b$, ba.
$(10, G_{14}, 1)$-$OCD = (X, \mathcal{A} \cup \{[a, 4, 3, b, 7, 2]\})$. Repeat edges: $b3$, 34, $4a$.
$(10, G_{15}, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$:
$[a, 0, 2, 1, 4, b]+i$, $i \in [0, 3]$, $[0, 5, a, 4, 6, 7]$, $[5, 7, a, b, 1, 6]$, $[3, b, 2, 7, 0, 1]$. Leave edges: $(a, 6, 0, 3)$.
$(10, G_{15}, 1)$-$OCD = (X, \mathcal{A} \cup \{[1, a, 6, 0, 3, 2]\})$. Repeat edges: $(1, a, 3, 2)$.

$\underline{v = 11}$: On the set $X = Z_9 \cup \{a, b\}$, $(11, G_{12}, 1)$-$OPD = (X, \mathcal{A})$,
$\mathcal{A}$: $[0, 2, 1, 4, a, b]$ (mod 9).
$(11, G_{13}, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$: $[0, 2, 1, 4, a, b]$ (mod 9).
$(11, G_{14}, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$: $[a, 0, 1, 4, 8, 6] + i$, $i \in [0, 3]$, $[b, 4, 5, 8, 3, 1] + i$, $[0, 1]$, $[1, b, 6, 7, 3, 0]$, $[5, 3, b, 7, 8, 0]$, $[b, a, 8, 2, 6, 4]$.
$(11, G_{15}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[a, 0, 2, 1, 4, b]$ (mod 9).
Since $l_1 = 1$, there exist $(11, G_i, 1)$-OCD for $i \in [12, 15]$.

$\underline{v = 14}$: On the set $X = Z_{14}$,
$(14, G_{12}, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$: $[5, 1, 3, 0, 6, 7]+i$, $i \in [1, 6]$, $[12, 8, 10, 7, 13, 6]+i$, $i \in [0, 6]$, $[1, 3, 4, 5, 6, 0]$, $[1, 2, 3, 0, 6, 7]$. $(14, G_{12}, 1)$-$OCD = (X, \mathcal{A} \cup \{[13, 0, 1, 2, 3, 4]\})$.
$(14, G_{13}, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$: $[5, 1, 3, 0, 9, 7]+i$, $i \in \{1, 3, 4, 5, 6\}$, $[12, 8, 10, 7, 2, 6]+i$, $i \in [0, 6]$, $[1, 2, 3, 0, 9, 13]$, $[1, 3, 4, 5, 11, 0]$, $[2, 7, 3, 5, 0, 6]$. $(14, G_{13}, 1)$-$OCD = (X, \mathcal{A} \cup \{[1, 9, 2, 3, 4, 5]\})$.
$(14, G_{14}, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$: $[5, 1, 3, 0, 7, 13]+i$, $i \in [1, 6]$, $[12, 8, 10, 7, 6, 0]+i$, $i \in [0, 6]$, $[1, 2, 3, 0, 7, 13]$, $[1, 3, 4, 5, 0, 13]$. $(14, G_{14}, 1)$-$OCD = (X, \mathcal{A} \cup \{[1, 2, 3, 4, 5, 6]\})$.
$(14, G_{15}, 1)$-$OPD = (X, \mathcal{A})$, $\mathcal{A}$: $[9, 3, 1, 5, 0, 7] + i$, $i \in [1, 6]$, $[2, 10, 8, 12, 7, 6] + i$, $i \in \{0, 1, 3, 4, 5, 6\}$, $[9, 3, 2, 1, 0, 7]$, $[12, 4, 3, 1, 5, 0]$, $[13, 0, 10, 12, 9, 8]$. $(14, G_{15}, 1)$-$OCD = (X, \mathcal{A} \cup \{[1, 2, 3, 4, 5, 6]\})$.

$\underline{v = 15}$: On the set $X = Z_{15}$, $(15, G_{12}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[7, 1, 5, 0, 2, 3] + i$, $i \in Z_{15} \backslash \{0, 4, 7\}$, $[7, 6, 5, 0, 2, 14]$, $[11, 5, 9, 4, 3, 6]$, $[14, 13, 12, 7, 4, 9]$, $[2, 3, 0, 1, 5, 7]$, $[9, 10, 7, 8, 12, 14]$. Leave edges: $(4, 5), (10, 11, 12)$.
$(15, G_{12}, 1)$-OCD=$(X, \mathcal{A} \cup \{[5, 4, 10, 11, 12, 13]\})$. Repeat edges: $(4, 10), (5, 11, 13)$.
$(15, G_{13}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[7, 1, 5, 0, 4, 2] + i$, $i \in Z_{15} \backslash \{0, 1, 7\}$, $[0, 7, 1, 5, 6, 2]$, $[7, 14, 8, 12, 0, 13]$, $[1, 8, 2, 6, 7, 5]$, $[1, 2, 3, 4, 0, 5]$, $[8, 9, 10, 11, 7, 12]$.
Leave edges: $(13, 14), (0, 1, 3)$.
$(15, G_{13}, 1)$-OCD=$(X, \mathcal{A} \cup \{[3, 1, 14, 13, 0, 2]\})$. Repeat edges: $(14, 1), (3, 13, 2)$.
$(15, G_{14}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[5, 0, 7, 1, 3, 6] + i$, $i \in Z_{15} \backslash \{0, 7\}$, $[5, 0, 7, 1, 3, 2]$, $[12, 7, 14, 8, 10, 9]$, $[3, 4, 5, 6, 7, 8]$, $[10, 11, 12, 13, 14, 0]$. Leave edges: $(8, 9), (0, 1, 2)$.
$(15, G_{14}, 1)$-OCD=$(X, \mathcal{A} \cup \{[0, 3, 2, 1, 8, 9]\})$. Repeat edges: $(8, 1), (0, 3, 2)$.
$(15, G_{15}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[10, 7, 0, 5, 1, 3] + i$, $i \in Z_{15} \backslash \{0, 7\}$, $[3, 1, 7, 0, 5, 6]$, $[10, 8, 14, 7, 12, 13]$, $[6, 7, 8, 9, 10, 11]$, $[13, 14, 0, 1, 2, 3]$. Leave edges: $(3, 4, 5), (11, 12)$.

$(15, G_{15}, 1)$-OCD=$(X, \mathcal{A} \cup \{[3,4,11,12,5,1]\})$. Repeat edges: $(4,11), (12,5,1)$.
$\underline{v=17}$: On the $X = Z_{17}$, $(17, G_{12}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[2,6,12,0,7,8]+i$, $i \in Z_{17}$, $[2,3,0,1,15,4]+6i$, $i \in [0,2]$, $[6,3,4,5,2,8]+6i$, $i \in [0,1]$. Leave edges: $(2,16), (15,16,0,14)$.
$(17, G_{12}, 1)$-OCD=$(X, \mathcal{A} \cup \{[5,14,0,16,15,2]\})$. Repeat edges: $(14,5,16)$.
$(17, G_{13}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[2,6,12,0,14,7]+i$, $i \in Z_{17}^*$, $[0,2,6,12,16,9]$, $[1,2,3,0,5,7]$, $[5,6,7,4,9,1]$, $[7,8,9,10,5,11]$, $[12,13,14,11,10,8]$, $[15,16,0,14,13,6]$. Leave edges: $(6,3,4)$,$(12,15,1)$.
$(17, G_{13}, 1)$-OCD=$(X, \mathcal{A} \cup \{[6,15,1,3,12,4]\})$. Repeat edges: $(15,6), (1,3)$.
$(17, G_{14}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[2,6,12,0,7,15]+i$, $i \in Z_{17}$, $[2,3,0,1,4,7]+6i$, $i \in [0,2]$, $[6,3,4,5,8,11]+6i$, $i \in [0,1]$. Leave edges: $(2,5), (1,15,16,0)$.
$(17, G_{14}, 1)$-OCD=$(X, \mathcal{A} \cup \{[1,2,5,15,16,0]\})$. Repeat edges: $(5,15), (1,2)$.
$(17, G_{15}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[10,2,6,12,0,7]+i$, $i \in Z_{17}$, $[5,2,3,0,1,4]+3i$, $i \in [0,4]$. Leave edges: $(2,16), (1,15,16,0)$.
$(17, G_{15}, 1)$-OCD=$(X, \mathcal{A} \cup \{[3,2,1,15,16,0]\})$. Repeat edges: $(3,2,1)$.
$\underline{v=18}$: On the $X = Z_{17} \cup \{a\}$, $(18, G_{12}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[9,6,11,5,14,a]+i$, $i \in Z_{17}^*$, $[15,0,1,8,9,6]+i$, $i \in [0,6]$, $[9,6,11,5,7,a]$, $[14,5,15,16,0,6]$. Leave edges: $(13,15), (0,7,8)$.
$(18, G_{12}, 1)$-OCD=$(X, \mathcal{A} \cup \{[13,15,0,7,8,9]\})$. Repeat edges: $(13,7,9), (15,0)$.
$(18, G_{13}, 1)$-OPD=$(X, \mathcal{A})$, $\mathcal{A}$: $[6,0,4,1,8,a]+i$, $i \in Z_{17}$, $[1,8,10,0,9,2]+i$, $i \in [0,6]$, $[0,7,9,16,8,15]$. Leave edges: $(16,1), (8,15,0)$.
$(18, G_{13}, 1)$-OCD=$(X, \mathcal{A} \cup \{[8,16,1,15,2,0]\})$. Repeat edges: $(2,16,8), (1,15)$.
By $K_{12,6}/G_i$, K_{12}/G_i and $(6, G_i, 1)$-$OPD(OCD)$ for i=14,15, we can obtain $(18, G_{12}, 1)$-$OPD(OCD)$. By $K_{12,j}/G_i$, K_{12}/G_i and $(j, G_i, 1)$-$OPD(OCD)$ for $j = 7,8,10,11$, $i \in [12,15]$, we have $(12+j, G_i, 1)$-$OPD(OCD)$ for $j = 7,8,10,11$, $i \in [12,15]$.

From Theorem 2.5, Theorem 2.10 and Lemma 4.8, it follows that the theorem is true. □

5 Coverings and packings for $\lambda > 1$

Theorem 5.1 If there exist $(v, G, 1)$-OPD and $(v, G, 1)$-OCD, then when $r_1 = 1$ (or $l_1 = 1$), there exist (v, G, λ)-$OPD(OCD)$ for any $\lambda \geq 1$.

Proof If $r_1 = 1$, then $l_1 = e(G) - 1$. For $1 \leq \lambda \leq e(G)$, we have $l_\lambda = e(G) - \lambda$ and $r_\lambda = \lambda$. When $\lambda = 1$, from the assumptions of the theorem, there exist $(v, G, 1)$-OPD and $(v, G, 1)$-OCD. We proceed by induction on λ for $1 \leq \lambda < e(G)$. Suppose that there is (v, G, λ)-$OPD = (X, D')$ and its leave edge graph is $L_\lambda(D')$. We can construct an isomorphic mapping f of the $(v, G, 1)$-OCD, such that the isomorphic image of the mapping f is (X, D) and its repeat edge graph $R_1(D)$ is a subgraph of $L_\lambda(D')$. It is easy to see that $(X, D \cup D')$ is a $(v, G, \lambda+1)$-OPD and its leave edge graph is $L_\lambda(D') \backslash R_1(D)$. It follows from Theorem 2.11 that there exist (v, G, λ)-OPD for any positive integer λ.
When $1 \leq \lambda \leq e(G)$, we take the $(v, G, 1)$-$OCD = (X, D)$, and construct $\lambda - 1$ isomorphic mappings of the $(v, G, 1)$-OCD, f_i, $i = 1, 2, \cdots, \lambda-1$, such that the repeat

edge graph of every f_i's image is a subgraph of G, and these subgraphs are different. Let f_i's isomorphic image be (X, D_i), $i = 1, 2, \cdots, \lambda - 1$; then $(X, D \cup (\bigcup_{1 \le i \le \lambda-1} D_i))$ is a (v, G, λ)-OCD. It follows from Theorem 2.11 that there exist (v, G, λ)-OCD for any positive integer λ.

When $l_1 = 1$, the theorem is true also. □

Theorem 5.2 Let $l_1 = e(G)/2$ be an integer. If there exist $(v, G, 1)$-OPD $= (X, \mathcal{A})$ and $(v, G, 1)$-OCD $= (X, \mathcal{B})$, and $L_1(\mathcal{A}) \cong R_1(\mathcal{B})$, then there exist$(v, G, \lambda)$-$OPD(OCD)$ for any positive integer λ.

Proof When $\lambda = 1$, this is well-known. When $\lambda = 2$, we can construct an isomorphic mapping, which transforms $\mathcal{B}$ to $\mathcal{B}'$, and $R_1(\mathcal{B}) \cong R_1(\mathcal{B}')$ and $L_1(\mathcal{A}) = R_1(\mathcal{B}')$ are satisfied. We take $(X, \mathcal{A})$ and $(X, \mathcal{B}')$; then $(X, \mathcal{A} \cup \mathcal{B}')$ is a $(v, G, 2)$-GD. It follows from Theorem 2.11 that there exist (v, G, λ)-$OPD(OCD)$ for any positive integer λ.
□

Example Let $X = Z_7$, $(7, G_{15}, 1)$-OPD $= (X, \mathcal{A})$, $\mathcal{A}$:
$[5, 1, 0, 3, 2, 6]$, $[2, 4, 0, 5, 3, 1]$, $[0, 6, 1, 4, 5, 2]$, leave edges: 02, 46, 63.
$(7, G_{15}, 1)$-OCD $= (X, \mathcal{B})$, $\mathcal{B} = \mathcal{A} \cup \{[4, 6, 0, 2, 3, 5]\}$, repeat edges: 60, 23, 35.
Transforming $\mathcal{B}$ to $\mathcal{B}'$ under the mapping $2 \to 4$, $4 \to 5$, $5 \to 3$, $3 \to 6$, $6 \to 2$ and $x \to x$ for other x. Then $(X, \mathcal{A} \cup \mathcal{B}')$ is a $(v, G, 2)$-GD.

Theorem 5.3 There exists a (v, G_1, λ)-OPD (or OCD) for $v \equiv 2 (\text{mod } 3)$ and integer $\lambda \ge 1$.

Proof It immediately follows from Theorem 2.11 and Theorem 2.12. □

Theorem 5.4 There exist (v, G_i, λ)-OPD (or OCD) for $i \in [2, 4]$, $v \not\equiv 0, 1$ (mod 8) and $\lambda \ge 1$, for covering except $(v, i, \lambda) = (6, 3, 1)$ and $(6, 4, 1)$.

Proof Since $l_1 = 1$ when $v \equiv 2, 7$ (mod 8) and $r_1 = 1$ when $v \equiv 3, 6$ (mod 8), there exist (v, G_i, λ)-OPD (or OCD) for $i \in [2, 4]$ and $\lambda \ge 1$. When $v \equiv 4, 5$ (mod 8), $l_1 = 2$ $r_1 = 2$ and $\bar{\lambda} = 2$. By Theorem 2.12, we can list the following table to get (v, G_i, λ)-OPD and (v, G_i, λ)-OCD for $1 \le \lambda$, $i \in [2, 4]$.

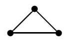
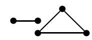
for	G_2	for	G_3	for	G_4
L_1		L_1		L_1	
R_1		R_1		R_1	

Lemma 5.5 [15] There exists $(v, K_{1,5}, \lambda)$-GD if and only if $\lambda v(v-1) \equiv 0$ (mod 10), when $\lambda = 1$, $v \ge 10$; when λ is an even number, $v \ge 6$; when $\lambda > 1$ and λ is an odd number, $v \ge 6 + 5/\lambda$.

Lemma 5.6 (1). When $\lambda \ge 2$, there exist (n, G_6, λ)-$OPD(OCD)$ for $n = 7, 8$, except for $n = 7$ and $\lambda = 3$. (2). When $\lambda \ge 2$, there exist $(7, G_9, \lambda)$-$OPD(OCD)$.

Proof (1). $\underline{n = 8}$ On the set $X = Z_5 \cup \{a, b, c\}$, let $A = \{[3, 4, a, b, c, 0], [a, b, c, 0, 1, 2],$ $[b, c, 0, 1, 2, 3]$, $[1, 2, 3, 4, a, c]$, $[2, 1, 3, 4, a, c]$, $[4, 1, 2, a, b, c]$, $[0, 1, 2, 4, b, c]$, $[3, 1, 2, 4,$ $a, c]$, $[b, 1, 2, 4, a, c]\}$, $B = \{[4, a, b, c, 0, 1]\}$, $C = \{[0, 1, 2, 3, 4, a]$, $[c, a, 0, 1, 2, 4]\}$, $D = \{[0, 1, 2, a, b, c] + i | i \in Z_5\} \cup \{[0, 1, 2, 3, 4, c]$, $[a, b, c, 4, 0, 1]$, $[c, a, b, 1, 2, 4]\}$; then

$(X, A\cup C)$ is a $(8, G_6, 2)$-OPD. Leave edge: $4a$. $(X, A\cup B\cup C)$ is a $(8, G_6, 2)$-OCD. Repeat edges: $4b, 4c, 40, 41$. $(X, A\cup D)$ is a $(8, G_6, 3)$-OCD. Repeat edge: $a1$. When $\lambda \geq 2$, from the following table and Theorem 2.12, we find that the theorem is true.

λ	1	2	3	4
L_λ			 L_1+L_2	L_2+L_2
R_λ	$K_{1,4}\cup K_{1,3}$	$K_{1,4}$		$K_{1,3}$ R_2-L_2

$\underline{n=7}$ On the set $X = Z_7$, let A=$\{[4, 0, 1, 2, 5, 6] + i|\ i = 0, 1, 2, 3\} \cup \{[1, 0, 2, 3, 4, 5],$ $[2, 0, 1, 3, 4, 5],\ [3, 0, 1, 2, 4, 5],\ [6, 0, 1, 2, 3, 5]\}$, B=$\{[4, 0, 1, 2, 3, 5]\}$; then (X, A) is a $(7, G_6, 2)$-OPD, leave edges: $34, 45$. $(X, A\cup B)$ is a $(7, G_6, 2)$-OCD, repeat edges: $42, 40, 41$.

In a $(7, G_6, 3)$-CD, for every vertex on K_7, sum of its degree number is not less than 18. Suppose that there exists $(7, G_6, 3)$-OCD which contains 13 blocks. There is a vertex on the K_7 which appears in the center of the 13 blocks at most once, and the sum of its degree number is at most $5 + 12 = 17$. This is a contradiction. We easily get $c(7, G_6, 3) = 14$.

When $\lambda \geq 2$, from the following table, we find that the theorem is true.

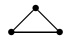
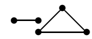
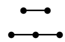
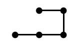
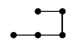
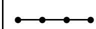
λ	1	2	3	4	5	6	7	8	9
L_λ	$K_{1,3}\cup K_3$	P_3	K_3 L_1-R_2	$K_{1,4}$ L_2+L_2	GD	P_2 L_2-R_4	P_3 L_3-R_4	$K_{1,3}$ L_4-R_4	$K_{1,4}$ L_2+L_7
R_λ	$K_{1,4}\cup$ $K_{1,3}\cup P_3$	$K_{1,3}$	$K_{1,4}\cup K_{1,3}$	P_2 R_2-L_2	GD	$K_{1,4}$ R_2+R_4	$K_{1,3}$ R_3-L_4	P_3 R_4+R_4	P_2 R_7-L_2

(2). On the set $X = Z_7$, A: $[2, 4, 0, 1, 3, 6] + i, i = 0, 1, 2, 5$, $[1, 5, 2, 3, 4, 6]$, $[1, 6, 4, 5, 2, 0]$, $[1, 3, 6, 0, 5, 2]$, $[1, 4, 5, 6, 3, 0]$; B: $[2, 4, 0, 1, 3, 6] + i, i = 0, 3, 4, 5, 6$, $[3, 6, 1, 2, 4, 0]$, $[0, 4, 2, 3, 5, 1]$, $[1, 4, 0, 2, 5, 3]$, $[1, 5, 3, 2, 6, 4]$. The (X, A) is a $(7, G_9, 2)$-OPD and leave edges are $(0, 3, 4)$. The (X, B) is a $(7, G_9, 2)$-OCD and repeat edges are $(4, 3, 2, 0)$. From the following table, we find that the theorem is true.

λ	1	2	3	4
L_λ	$\bigcup_{2\leq i\leq 4} P_i$	P_3	$P_\cup P_3$ L_1-R_2	P_4 L_2+L_2
R_λ	$P_2\cup P_4$	P_4	$P_2\cup P_2$ R_1-L_2	P_2 R_2-L_2

Theorem 5.7 There exist (v, G_i, λ)-OPD (or OCD) for $i \in [5, 11]$, $v \not\equiv 0, 1$ (mod 5) and $\lambda \geq 1$, for covering except $(i, v, \lambda) = (6, 8, 1)$, $(6, 7, 1)$ and $(6, 7, 3)$, for packing except $(i, v, \lambda) = (6, 7, 1)$ and $(9, 7, 1)$.

Proof When $v \equiv 2, 4, 7, 9$ (mod 10), by Theorem 5.1 and Lemma 5.6, we find that the theorem is true. When $v \equiv 3, 8$ (mod 10), $\bar{\lambda} = 5$. By Theorem 2.12, we can list the following table to get (v, G_i, λ)-OPD and (v, G_i, λ)-OCD for $\lambda > 1$, $i \in [5, 11]$.

$G_i, i \in$	λ	1	2	3	4
$[5,8] \cup [10,11]$	L_λ		$L_1 - R_1$	 $L_2 + L_1$	$L_2 + L_2$
$\{9\}$	L_λ		$L_1 - R_1$	 $L_2 + L_1$	$L_2 + L_2$
$[5,11]$	R_λ		 $R_1 + R_1$	$R_1 - L_2$	 $R_2 - L_2$

Lemma 5.8 When $\lambda \geq 2$, there exist $(6, G_i, \lambda)$-$OPD(OCD)$ for $i = 12, 13$.

Proof On the set $X = Z_6$, let $A = \{[0,3,2,1,4,5],\ [0,5,3,1,4,2],\ [0,4,5,2,1,3],\ [0,5,1,4,3,2]\}$, $B = \{[0,3,4,2,1,5]\}$; then $(X, A \cup B)$ is a $(6, G_{13}, 2)$-GD. It is also $(6, G_{13}, 2)$-$OCD(OPD)$. Let $C = \{[0,3,1,2,4,5],\ [0,5,1,4,3,2],\ [0,2,3,1,5,4],\ [2,4,3,0,5,1]\}$; then $(X, A \cup C)$ is a $(6, G_{13}, 3)$-OCD. The union of a $(6, G_{13}, 1)$-OPD and a $(6, G_{13}, 2)$-OPD is a $(6, G_{13}, 3)$-OPD. Since there exists $(6, G_{13}, 2)$-GD, there exist $(6, G_{13}, 2n)$-GD for $n \geq 1$. Again by $(6, G_{13}, 3)$-$OPD(OCD)$, we find that there exist $(6, G_{13}, \lambda)$-OCD for $\lambda \geq 2$.

On the set $X = Z_6$, let $A = \{[0,2,3,1,4,5],\ [0,3,4,2,1,5],\ [0,4,5,3,1,2],\ [0,5,1,4,3,2]\}$, $B = \{[0,1,2,5,3,4]\}$; then $(X, A \cup B)$ is a $(6, G_{12}, 2)$-GD. It is also a $(6, G_{12}, 2)$-OCD or (OPD).

Let $C = \{[0,1,2,3,4,5],\ [2,0,4,5,1,3],\ [0,5,2,1,3,4]\}$, $D = \{[4,5,0,2,1,3]\}$; then $(X, A \cup C)$ is a $(6, G_{12}, 3)$-OPD, and $(X, A \cup C \cup D)$ is a $(6, G_{12}, 3)$-OCD. Using the same as proof as G_{13}, we find that $(6, G_{12}, \lambda)$-$OPD(OCD)$ exists for $\lambda \geq 2$. □

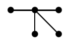
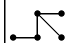
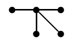
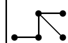
Theorem 5.9 There exist (v, G_i, λ)-OPD (or OCD) for $i \in [12, 15]$, $v \equiv 2, 3, 5, 6, 7, 8, 10, 11 \pmod{12}$ and $\lambda \geq 1$, for covering except $(v, i, \lambda) = (6, 12, 1)$ and $(6, 13, 1)$, for packing except $(v, i, \lambda) = (6, 12, 1)$.

Proof When $v \not\equiv 0, 1, 4, 9 \pmod{12}$, it is easy to see that l_1 takes three values 1, 3, 4, and $r_1 = 6 - l_1$. When $v \equiv 2, 11 \pmod{12}$, $l_1 = 1$, it follows from Theorem 5.1 that the theorem is true. When $v \equiv 3, 6, 7, 10 \pmod{12}$, $l_1 = 3$, it follows from Theorem 5.2 that the theorem is true.

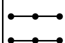
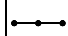
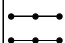
When $v \equiv 5, 8 \pmod{12}$, $l_1 = 4$ and $\bar{\lambda} = 3$. Let $(X, \mathcal{A})$ and $(X, \mathcal{B})$ be $(v, G_i, 1)$-OPD and $(v, G_i, 1)$-OCD,$i \in [12, 15]$ in Theorem 4.9, and L_1 and R_1 be leave edge graph of the $\mathcal{A}$ and repeat edge graph of $\mathcal{B}$, respectively.

By the proof of Theorem 4.9, L_1 and R_1 is the special graph listed in under table. By Theorem 2.12, we can list the following table to get (v, G_i, λ)-OPD and (v, G_i, λ)-OCD for $\lambda \geq 1$, $i \in [12, 15]$.

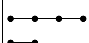
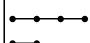
For G_{12}

λ	1	2	or	λ	1	2
L_λ			or	L_λ		
R_λ			or	R_λ		

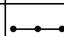
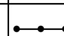
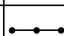
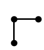
For G_{13}

λ	1	2	or	λ	1	2
L_λ			or	L_λ		
R_λ			or	R_λ		

For G_{14}

λ	1	2
L_λ		
R_λ		

For G_{15}

λ	1	2	or	λ	1	2
L_λ			or	L_λ		
R_λ			or	R_λ		

6 Graph designs for $\lambda \geq 1$

Lemma 6.1 The necessary conditions for (v, G, λ)-GD to exist are (1) $\lambda v(v-1) \equiv 0 \pmod{2e(G)}$; (2) $\lambda(v-1) \equiv 0$ (mod n), where $n = \gcd(\{d(u)|u \in V(G)\})$.

By Corollary 2.13, Section 5 and Table A, we easily obtain the following theorem:

Theorem 6.2 If v satisfies the conditions in Lemma 6.1 and $v > 6$, then there exist (v, G_i, λ)-GD for $i \in [1, 15]$ and $\lambda \geq 1$.

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References

[1] A.M. Assaf, *On covering designs with block size* 5 *and index* 5, Designs, Codes and Cryptography **5** (1995), 91–107.

[2] A.M. Assaf, —it On packing designs with block size 5 and index $7 \leq \lambda \leq 21$, JCMCC **26** (1998), 3–71.

[3] A.M. Assaf, N. Shalaby and L.P.S. Singh, *On packing designs with block size* 5 *and index* 3 *and* 6, Ars Combinatoria **45** (1997), 143–156.

[4] A.M. Assaf, N. Shalaby and J. Yin, *Directed packings with block size* 5, Australasian Journal of Combinatorics **17** (1998), 235–256.

[5] F.E. Bennett, J. Yin, H. Zhang and R.J.R. Abel, *Perfect Mendelsohn packing designs with block size* 5, Designs, Codes and Cryptography **14** (1998), 5–22.

[6] J.C. Bermond, C. Huang, A. Rosa and D. Sotteau, *Decomposition of complete graphs into isomorphic subgraphs with five vertices*, Ars Combinatoria **10** (1980), 211–254.

[7] Y. Caro and R. Yuster, *Covering graphs: the covering problem solved*, Journal of Combinatorial Theory, series A **83** (1988), 273–282.

[8] H.L. Fu and C.C. Lindner, —it The Doyen-Wilson theorem for maximum packing of K_n with 4-cycles, Discrete Mathematics **183** (1998), 103–117.

[9] D.G. Hoffman, Kimberly S. Kirkpatrick, *G-designs of order* n *and index* λ *where* G *has* 5 *vertices or less*, Australasian Journal of Combinatorics **18** (1998), 13–37.

[10] N. Shalaby and J. Yin, *Nested optimal* λ*-packings and* λ*-coverings of pairs with triples*, Designs, Codes and Cryptography **15** (1998), 271–278.

[11] J.X. Yin and B.S. Gong, *Existence of* G*-designs with* $|V(G)| = 6$, in Combinatorial designs and applications, Volume 126 (1988), 201–218.

[12] J.A. Kennedy, *Minimum coverings of* K_n *with hexagons*, Australasian Journal of Combinatorics **16** (1997), 295–303.

[13] Z. Liang, *On* G*-designs (packing, covering) of* λK_v *with six vertices, nonbipartite* G, manuscript.

[14] Q. Kang and Z. Liang, *Optimal packings and coverings of* λDK_v *with* k*-circuits*, JCMCC, to appear.

[15] C.J. Colbourn and J.H. Dinitz, *The CRC handbook of combinatorial designs*, CRC Press, 1996.

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