Cyclic type factorizations of complete bipartite graphs into hypercubes

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Abstract

So far, the smallest complete bipartite graph which was known to have a cyclic type decomposition into cubes Q_d of a given dimension d was $K_{d2^{d-2},d2^{d-2}}$. Using binary Hamming codes we prove in this paper that there exists a cyclic type factorization of $K_{2^{d-1},2^{d-1}}$ into Q_d if and only if d is a power of 2.

1. Introduction

The 1-dimensional cube Q_1 is the graph K_2 while the 2-dimensional cube Q_2 is isomorphic to the cycle C_4. In general, the *d-dimensional hypercube* Q_d is defined recursively as the product $Q_{d-1}\Box K_2$, which is defined as follows: Take two copies, Q and Q', of Q_{d-1} with vertex sets $V(Q)=\{v_1,v_2,\ldots,v_{2^{d-1}}\}$ and $V(Q')=\{v'_1,v'_2,\ldots,v'_{2^{d-1}}\}$ and join each pair of corresponding vertices v_i and v'_i by an edge $v_iv'_i$. The resulting graph is then the hypercube Q_d. Obviously, such a hypercube has 2^d vertices and $d2^{d-1}$ edges. Another useful definition of the hypercube Q_d (often called just a *cube*) can be stated as follows: Take all binary vectors of length d and assign them to vertices $u_1,u_2,\ldots,u_{2^d}$. Then join two vertices by an edge if and only if the corresponding binary vectors differ exactly at one position. We present in Figure 1 bipartite adjacency matrices (BAM) of the cubes Q_d for $d=1,2,3,4$.

One can notice that the $2^{d-1}\times 2^{d-1}$ bipartite adjacency matrix $BAM(Q_d)$ of cube Q_d can be easily recursively constructed from the $2^{d-2}\times 2^{d-2}$ matrix $BAM(Q_{d-1})$ of Q_{d-1} in such a way that we put into both left upper and right lower $2^{d-2}\times 2^{d-2}$ submatrices of $BAM(Q_d)$ a copy of $BAM(Q_{d-1})$. Then we fill the back diagonal with "1"s and all other entries with "0"s.

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$$[1] \qquad \begin{bmatrix}1&1\\1&1\end{bmatrix} \qquad \begin{bmatrix}1&1&0&1\\1&1&1&0\\0&1&1&1\\1&0&1&1\end{bmatrix} \qquad \begin{bmatrix}1&1&0&1&0&0&0&1\\1&1&1&0&0&0&1&0\\0&1&1&1&0&1&0&0\\1&0&1&1&1&0&0&0\\0&0&0&1&1&1&0&1\\0&0&1&0&1&1&1&0\\0&1&0&0&0&1&1&1\\1&0&0&0&1&0&1&1\end{bmatrix}$$

Figure 1

As the hypercubes are bipartite graphs, it is natural to ask which complete bipartite graphs can be decomposed or even factorized into hypercubes. We say that $K_{n,m}$ has a *decomposition* into subgraphs $H_1, H_2, \dots, H_p$, all isomorphic to a given graph H, if $V(H_1) = V(H_2) = \cdots = V(H_p) = V(K_{n,m})$ and the edge sets $E(H_1), E(H_2), \dots, E(H_p)$ form a decomposition of $E(K_{n,m})$. That means that $E(H_1) \cup E(H_2) \cup \cdots \cup E(H_p) = E(K_{n,m})$ and $E(H_i) \cap E(H_j) = \emptyset$ for $i \neq j$. Hence by decomposition of $K_{n,m}$ into hypercubes we in general mean that the graph H has one component isomorphic to a cube Q_d for some d and (possibly) some isolated vertices. By *factorization* we mean that the graph H itself is isomorphic to a hypercube Q_d for an appropriate d and therefore contains no isolated vertices. The subgraphs $H_1, H_2, \dots, H_p$ are then called *factors* of $K_{n,m}$. The necessary condition for factorization of a complete bipartite graph $K_{n,m}$ into d-dimensional hypercubes is that the parts have to be both of the same order 2^{d-1} and d itself must be a power of 2. If it is not so, then the number of edges (or *size*) of the hypercube does not divide the size of $K_{n,m}$. It was proved by El-Zanati and Vanden Eynden [1] that this necessary condition is also sufficient. In fact, they also proved that for other dimensions than powers of 2 the hypercubes Q_d can be packed into $K_{2^{d-1},2^{d-1}}$, the smallest complete bipartite graph that allows embedding of Q_d. Their result follows.

Theorem 1. (El-Zanati, Vanden Eynden [1]) *Let d be a positive integer with $t = 2^{d-1} = dq + r$, $0 \leq r < d$. Then $K_{t,t}$ can be decomposed into q cubes Q_d and an r-factor. If $r \neq 0$ this r-factor itself decomposes into 2^{d-r} cubes Q_r.*

However, in this paper we are interested in cyclic type decompositions and the decompositions used in the proof of Theorem 1 are not of this type. Cyclic type decompositions were studied by Vanden Eynden [6]. We shall follow the notation used in [6]. Let $K_{n,m}$ be a complete bipartite graph and G a bipartite graph such that $nm = q|E(G)|$. We denote edges of $K_{n,m}$ as (i,j), where $i \in \{1,2,\dots,n\}$ and $j \in \{1,2,\dots,m\}$. We say that $K_{n,m}$ has an (r,s)-cyclic decomposition into G if we can assign labels to the vertices of G such that for any edge (i,j) that belongs to $G_0 \cong G$ all edges $(i+lr, j+ls), l = 1,2,\dots,q-1$ belong to different copies $G_1, G_2, \dots, G_{q-1}$ of G, where the set $\{G_0, G_1, \dots, G_{q-1}\}$ forms a decomposition of $K_{n,m}$. Vanden Eynden generalized earlier results of Rosa [5] (concerning decompositions of complete graphs) to prove the following.

Theorem 2. (Vanden Eynden [6]) *Let G be a bipartite graph with parts U, V and edge set E. Suppose that n and m are positive integers and r and s are integers such that $r|m$, $s|n$, and $|E| = \gcd(ms, nr)$. Let $t = \gcd(r, s)$, $R = r/t$, $S = s/t$, and $k = \gcd(Sm, Rn)$. Define $\psi : Z_m \times Z_n \to Z_k \times Z_t$ by $\psi(i, j) = (Si - Rj, \lfloor i/R \rfloor)$. Then there exists an (r, s)-cyclic decomposition of $K_{m,n}$ into copies of G if and only if there exist one-to-one functions M and N from U and V into Z_m and Z_n, respectively, such that the function $\theta : E \to Z_k \times Z_t$ defined by $\theta(u, v) = \psi(M(u), N(v))$ is one-to-one.*

It was proved by Vanden Eynden that for a given $d \geq 2$, graph $K_{d2^{d-2}, d2^{d-1}}$ can be (r, s)-cyclically decomposed into hypercubes Q_d. This result was extended by the author in [3]. It was shown that for any given $d \geq 2$, graph $K_{d2^{d-2}, d2^{d-2}}$ can also be (r, s)-cyclically decomposed into hypercubes Q_d. A $(4, 8)$-cyclic factorization of K_{16} into Q_4 was also presented in the paper. Another (r, s)-cyclic factorization of K_{16} into Q_4 with parameters $r = s = 4$ was found by Flídr [2], who also proved with the help of a computer that $K_{128,128}$ can be $(8, 128)$-cyclically factorized into Q_8.

In this paper we prove that for every d which is a power of 2 there exists an $(d, 2^{d-1})$-cyclic factorization of $K_{2^{d-1}, 2^{d-1}}$ into $2^{d-1}/d$ copies of hypercube Q_d.

2. (r, s)-cyclic factorization of $K_{2^{d-1}, 2^{d-1}}$ into hypercubes Q_d

First we determine suitable values of the parameters r, s. According to Theorem 2 we want to choose r, s such that $r, s | 2^{d-1}$ and $\gcd(r2^{d-1}, s2^{d-1}) = |E(Q_d)| = d2^{d-1}$. Then obviously $\gcd(r, s) = d$. Because we know that r, s and d are powers of 2, we can see that $d = \min\{|r|, |s|\}$. We choose $d = -r$. It follows from Theorem 2 that $t = \gcd(r, s) = d, R = r/t = -d/t = -1$ and $S = s/t = s/d$. Furthermore, $k = \gcd(S2^{d-1}, R2^{d-1}) = \gcd(S2^{d-1}, -2^{d-1}) = 2^{d-1}$. Hence we get $\psi : Z_{2^{d-1}} \times Z_{2^{d-1}} \to Z_{2^{d-1}} \times Z_d$ defined by $\psi(i, j) = (Si + j, -i)$. Similarly, if we set $d = r$, we have $\psi(i, j) = (Si - j, i)$. Thus we get the following necessary condition for the value of the parameter r.

Proposition 3. *If for a given $d, d = 2^c \geq 2$, there exists an (r, s)-cyclic factorization of $K_{n,m}$ into Q_d with $|r| \leq |s|$, then $n = m = 2^{d-1}, |r| = t = d, k = 2^{d-1}, |R| = 1, S = s/d$ and the function $\psi : Z_{2^{d-1}} \times Z_{2^{d-1}} \to Z_{2^{d-1}} \times Z_d$ is defined as $\psi(i, j) = (Si + j, -i)$ for $r = -d$ or $\psi(i, j) = (Si - j, i)$ for $r = d$.*

Before we proceed to a general construction, we present here an (r, s)-cyclic factorization of $K_{8,8}$ into Q_4. We define the cube Q_4 by the bipartite adjacency matrix presented in Figure 1 and label the vertices of each partite set of the cube Q_4 (that means, define the functions θ, M, N) with labels from the set $\{0, 1, \ldots, 7\}$. From now on we always assign vertices $u_1, u_2, \ldots, u_{2^{d-1}}$ to the rows $1, 2, \ldots, 2^{d-1}$, respectively, and $v_1, v_2, \ldots, v_{2^{d-1}}$ to the columns $1, 2, \ldots, 2^{d-1}$, respectively.

According to Proposition 3, we set $r = -4, s = 8$. This yields $t = 4, R = -1, S = 2$ and $k = 8$. The function $\psi : Z_8 \times Z_8 \to Z_8 \times Z_4$ is then $\psi(i, j) =$

$(2i+j,-i)$. We define the functions M and N from $U=\{u_1,u_2,\dots,u_8\}$ and $V=\{v_1,v_2,\dots,v_8\}$ both into Z_8 as $M(u_a)=a-1$ for $a=1,2,3,4$, $M(u_5)=5, M(u_6)=4, M(u_7)=7, M(u_8)=6$ and $N(v_b)=b-1$ for $b=1,2,\dots,8$. The function θ defined in Theorem 2 is one-to-one, as can be observed from the "labeling array" shown in Figure 2.

Notice that the asterisks correspond to "0"s in the bipartite adjacency matrix of Q_4. A non-blank entry in a row i and a column j denotes the value of the first entry of the function $\theta(u_i,v_j)=(2M(u_i)+N(v_j),-M(u_i))$, that means, the sum $2M(u_i)+N(v_j)$ taken modulo 8 (because $k=8$). The second entry, $-M(u_i)$, is taken modulo 4, as the parameter t equals 4.

	N	0	1	2	3	4	5	6	7
M		v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
0	u_1	0	1	*	3	*	*	*	7
1	u_2	2	3	4	*	*	*	0	*
2	u_3	*	5	6	7	*	1	*	*
3	u_4	6	*	0	1	2	*	*	*
5	u_5	*	*	*	5	6	7	*	1
4	u_6	*	*	2	*	4	5	6	*
7	u_7	*	7	*	*	*	3	4	5
6	u_8	4	*	*	*	0	*	2	3

Figure 2

From now on, if we view a row i of a matrix as a vector, we denote it by $\bar{\imath}$. We can observe that the sums of row vectors $\bar{1}+\bar{6}, \bar{2}+\bar{5}, \bar{3}+\bar{8}$ and $\bar{4}+\bar{7}$ of the bipartite adjacency matrix $BAM(Q_4)$ always give the 8-dimensional vector $(1,1,\dots,1)$. Moreover, if we consider also the rows of the labeling array of cube Q_4 as vectors (with asterisks replaced by "0"s), we can see that the set of entries of each of four resulting vectors $\bar{1}+\bar{6}, \bar{2}+\bar{5}, \bar{3}+\bar{8}$ and $\bar{4}+\bar{7}$ is precisely the set $\{0,1,2,\dots,7\}$. Why is it so? Notice that $M(u_1)\equiv M(u_6)\equiv 0 \pmod 4, M(u_2)\equiv M(u_5)\equiv 1 \pmod 4, M(u_3)\equiv M(u_8)\equiv 2 \pmod 4, M(u_4)\equiv M(u_7)\equiv 3 \pmod 4$ and $d=4$.

Let $M(u_{i_1})\equiv M(u_{i_2})\equiv b \pmod d$, say $M(u_{i_1})=gd+b, M(u_{i_2})=hd+b$. Then for $S=s/d=2$ we get $SM(u_{i_1})=\frac{s}{d}(gd+b)=sg+\frac{s}{d}b=8g+2b$. Similarly, $SM(u_{i_2})=\frac{s}{d}(hd+b)=sh+\frac{s}{d}b=8h+2b$. Because the first entry of function θ, $2M(u_i)+N(v_j)$, is taken modulo 8, we can see that $2M(u_{i_1})+N(v_{j_1})=2b+j_1-1$ and $2M(u_{i_2})+N(v_{j_2})=2b+j_2-1$. On the other hand, the second entry, $-M(u_i)$, is taken modulo 4. Thus we have shown that θ is really one-to-one: If $M(u_{i_1})\not\equiv M(u_{i_2}) \pmod 4$, then $\theta(u_{i_1},v_{j_1})$ differs from $\theta(u_{i_2},v_{j_2})$ in the second entry. If $M(u_{i_1})\equiv M(u_{i_2}) \pmod 4$, then $\theta(u_{i_1},v_{j_1})$ differs from $\theta(u_{i_2},v_{j_2})$ in the first entry as long as $j_1\neq j_2$. But for every choice of i_1 and i_2 such that $M(u_{i_1})\equiv M(u_{i_2}) \pmod 4$ we can see that in each column of $BAM(Q_4)$ exactly one of the rows i_1,i_2 contains "1" while the other one has "0" there. Or, as we said equivalently above, the sum of the row vectors $\bar{\imath}_1$ and $\bar{\imath}_2$ is equal to $(1,1,\dots,1)$.

One can also check that in $BAM(Q_8)$ there are eight classes of sixteen rows each with the property that the sum of the sixteen corresponding vectors always gives the 128-dimensional vector $(1, 1, \ldots, 1)$. For instance, one such a class consists of rows $1, 10, 19, 28, 34, 41, 52, 59, 72, 79, 86, 93, 103, 112, 117, 126$, another one of rows $6, 13, 24, 31, 37, 46, 55, 64, 67, 76, 81, 90, 100, 107, 114, 121$. Therefore, it is reasonable to expect that we could use similar approach as in the previous case and obtain an (r, s)-cyclic factorization of $K_{128,128}$ into Q_8. We thus formalize the method used above for the general case for any $d = 2^c > 4$. Later we shall guarantee the existence of the classes of rows giving always as their sum the 2^{d-1}-dimensional vector $(1, 1, \ldots, 1)$.

Suppose that for a given $d = 2^c \geq 4$ and for $p = 2^{d-1}/d$ there exists a set of p row vectors of $BAM(Q_d)$, $\{\bar{\imath}_1, \bar{\imath}_2, \ldots, \bar{\imath}_p\}$, such that their sum is equal to the 2^{d-1}-dimensional vector $(1, 1, \ldots, 1)$. We will say that this set is the *summing class* of these rows and/or of the corresponding vertices $u_{i_1}, u_{i_2}, \ldots, u_{i_p}$. It is obvious that no two rows of the same summing class SC can have "1" in the same column. We will also say that two edges of Q_d, (u_1, v_1) and (u_2, v_2), belong to the same *labeling class* if $\theta(u_1, v_1) = (x, z)$ and $\theta(u_2, v_2) = (y, z)$ for some $z \in \{0, 1, \ldots, d-1\}$ and $x, y \in \{0, 1, 2, \ldots, 2^{d-1} - 1\}$. Each "1" appearing in one of rows $\bar{\imath}_1, \bar{\imath}_2, \ldots, \bar{\imath}_p$ (and consequently in the vector $(1, 1, \ldots, 1)$) corresponds to one edge of Q_d. We want to define the one-to-one functions M and N in such a way that the 2^{d-1} edges corresponding to the 2^{d-1} "1"s of a summing class SC will form a labeling class. Moreover, we want to guarantee that the function θ restricted to the vertices of the summing class SC will be one-to-one.

Let (u_1, v_1) and (u_2, v_2) be edges such that vertices u_1 and u_2 belong to the same summing class SC. Because $\theta(u_i, v_j) = (SM(u_i) + N(v_j), -M(u_i))$ and we want (u_1, v_1) and (u_2, v_2) to fall to one labeling class, it must hold that $M(u_1) \equiv M(u_2) \equiv b \pmod{d}$ for some b. Furthermore, in summing class SC there is exactly one "1" in each column and therefore $v_1 \neq v_2$. Because N has to be one-to-one, it must hold that $N(v_1) \not\equiv N(v_2) \pmod{2^{d-1}}$, since $d | 2^{d-1}$. But then $\theta(u_1, v_1)$ can be equal to $\theta(u_2, v_2)$ only if $SM(u_1) \not\equiv SM(u_2) \pmod{2^{d-1}}$. Hence it remains to find the parameter S and the function M such that $SM(u_1) \equiv SM(u_2) \pmod{2^{d-1}}$ for every pair of vertices u_1, u_2 that belong to the same summing class SC. Because we already know that $M(u_1) \equiv M(u_2) \equiv b \pmod{d}$ for some b, we can write $M(u_1) = a_1 d + b$ and $M(u_2) = a_2 d + b$. To guarantee that $SM(u_1) \equiv SM(u_2) \pmod{2^{d-1}}$ it suffices to choose S such that $Sd \equiv 0 \pmod{2^{d-1}}$. Because $S = s/d$, we can see that $S = 2^{d-1}/d$ will do for any choice of a_1, a_2. This yields $s = 2^{d-1}$.

Let us briefly repeat what we have done so far. We have summing class SC of rows $\{\bar{\imath}_1, \bar{\imath}_2, \ldots, \bar{\imath}_p\}$, and the corresponding set of vertices $\{u_{i_1}, u_{i_2}, \ldots, u_{i_p}\}$ of partite set U of Q_d. We have an arbitrary one-to-one function N and the function M defined as $M(u_z) = a_z d + b$ for each $z \in \{i_1, i_2, \ldots, i_p\}$. We have chosen $s = 2^{d-1}$. When we set $a_z = z - 1$ for each $z = 1, 2, \ldots, p$, we can see that M restricted to summing class $SC = \{u_{i_1}, u_{i_2}, \ldots, u_{i_p}\}$ is one-to-one.

Therefore, if we now find a way how to split the rows of $BAM(Q_d)$ into d summing classes $SC^0, SC^1, \ldots, SC^{d-1}$ forming a decomposition of the set of all rows of $BAM(Q_d)$, we are done. Each summing class SC^b will then determine

2^{d-1} edges that will form one of the labeling classes. Before we do that, we first formalize our previous thoughts as follows.

Lemma 4. *For a given $d, d = 2^c \geq 4$, set $r = -d, s = 2^{d-1}$ and $p = 2^{d-1}/d$. Choose $b \in \{0, 1, \ldots, d-1\}$. Let $SC^b = \{\bar{\imath}_1, \bar{\imath}_2, \ldots, \bar{\imath}_p\}$ be a summing class of row vectors of $BAM(Q_d)$ and $U^b = \{u_{i_1}, u_{i_2}, \ldots, u_{i_p}\}$ be the corresponding set of vertices of partite set U of Q_d. Denote E^b the set of all edges having one of its endvertices in U^b. Let $N : V \to Z_{2^{d-1}}$ be defined as $N(v_q) = q - 1$ for each $q \in \{1, 2, \ldots, 2^{d-1}\}$ and $M^b : U^b \to Z_{2^{d-1}}$ be defined as $M^b(u_{i_z}) = (z-1)d + b$ for each $z \in \{1, 2, \ldots, p\}$. Then the function $\theta^b : E^b \to Z_{2^{d-1}} \times Z_d$ defined by $\theta^b(u, v) = \left(\frac{2^{d-1}}{d} M^b(u) + N(v), -M^b(u)\right)$ is one-to-one.*

Proof. For any fixed b_0 and any $u_{i_z} \in U^{b_0}$ we have $M^{b_0}(u_{i_z}) = (z-1)d + b_0 \equiv b_0 \pmod{d}$. This yields $\theta^{b_0}(u_{i_z}, v_q) = \left(\frac{2^{d-1}}{d}\left((z-1)d + b_0\right) + q - 1, -b_0\right) = \left(2^{d-1}(z-1) + \frac{2^{d-1}}{d} b_0 + q - 1, -b_0\right) = \left(\frac{2^{d-1}}{d} b_0 + q - 1, -b_0\right)$ for every $q \in \{1, 2, \ldots, 2^{d-1}\}$. Therefore from $\theta^{b_0}(u_{i_z}, v_q) = \theta^{b_0}(u_{i_x}, v_y)$ it follows that $(\frac{2^{d-1}}{d} b_0 + q - 1, -b_0) = (\frac{2^{d-1}}{d} b_0 + y - 1, -b_0)$ which yields $q = y$. Thus we have $v_q = v_y$ which implies immediately that $u_{i_z} = u_{i_x}$, because among all rows $\bar{\imath}_1, \bar{\imath}_2, \ldots, \bar{\imath}_p$ there is exactly one "1" in the column corresponding to v_q. This proves that θ^{b_0} is one-to-one. $\square$

Now we are going to present a method of decomposition of rows of $BAM(Q_d)$ into $2^{d-1}/d$ disjoint summing classes. No two rows $\bar{\imath}_1, \bar{\imath}_2$ belonging to the same class SC can have "1" in the same column. But this means that the corresponding vertices u_{i_1}, u_{i_2} of partite set U of Q_d have no common neighbor and hence their distance in Q_d is greater than 2. As Q_d is bipartite, it follows that for any two vertices u_{i_1}, u_{i_2} that belong to the same summing class it must hold that their distance in Q_d is at least 4. To achieve our goal, we use the extended Hamming binary code $H\hat{a}m(2, d)$.

The *binary linear code* C of length n and dimension k (2-$[n, k]$-code for short) is a k-dimensional subspace of the n-dimensional vector space over the field $GF(2)$ and is defined by its *generating matrix* $G(C) = G_{k \times n}$ whose rows form a basis of the subspace (i.e., the code) C. The *parity check matrix* $H(C) = H_{(n-k) \times n}$ of the code C is the matrix satisfying $GH^T = O$, where $O = O_{k \times (n-k)}$ is the zero matrix. The vectors of the subspace C are called *codewords* and the *Hamming distance* of two codewords $\bar{x}, \bar{y}$ of C, denoted $\mathrm{dist}_{Ham}(\bar{x}, \bar{y})$, is the number of positions where $\bar{x}$ and $\bar{y}$ differ. The *minimum distance* of code C, denoted $\mathrm{d}(C)$, is then the smallest distance among all pairs $\bar{x}, \bar{y}$ of C.

The *Hamming binary code* $Ham(2, 2^c - 1)$ is the 2-$[2^c - 1, 2^c - c - 1]$-code defined by its parity check matrix $H_{c \times (2^c - 1)}$. Recall that $d = 2^c$. The i-th column of matrix H is just the number i in binary form. Again, the rows of generating matrix G form the basis of the $(2^c - c - 1)$-dimensional subspace. The following property of Hamming codes is a classical result and can be found in every coding theory book.

Theorem 5. (Hamming [4]) *The minimum distance of any binary Hamming code, $\mathrm{d}\left(Ham(2, 2^c - 1)\right)$, is equal to* 3.

The *extended* Hamming binary code $H\hat{a}m(2, 2^c)$ is the 2-$[2^c, 2^c - c - 1]$-code defined again by its parity check matrix $\hat{H}_{(c+1)\times 2^c}$ which arises from matrix $H_{c\times(2^c-1)}$ by first adding a new last column consisting of c "0"s and then adding a new last row consisting of 2^c "1"s. This is known in coding theory as *adding the overall parity check*. The new row guarantees that the sum of all entries of every codeword of extended Hamming code $H\hat{a}m(2, 2^c)$ will be zero and therefore the number of "1"s in every codeword $\bar{x} \in H\hat{a}m(2, 2^c)$ (called the *weight* of $\bar{x}$ and denoted $w(\bar{x})$) will be even. The following property of extended Hamming codes appears to be fundamental in our construction.

Theorem 6. (Hamming [4]) *The minimum distance of any extended binary Hamming code,* $\mathrm{d}\big(H\hat{a}m(2, 2^c)\big)$*, is equal to* 4.

The assertion is obvious. By adding the averall parity check, we cannot lower the distance between two codewords. But because the weight of any codeword of $(H\hat{a}m(2, 2^c)$ is even, the distance between any two codewords must be also even and therefore at least four.

Let us now turn our attention back to the definition of Q_d using binary vectors. We can see that one partite set consists precisely of all vertices whose corresponding vectors have an even weight while the other one consists precisely of all vertices whose corresponding vectors have an odd weight. It is not difficult to observe that the even-weight vectors form a $(d-1)$-dimensional subspace W of the space $V(2, d)$ of all binary vectors of length d (where $d = 2^c$) and that $H\hat{a}m(2, 2^c)$ is a $(d - c - 1)$-dimensional subspace of W. Therefore, we can now decompose W into $2^{d-1}/2^{d-c-1} = 2^c = d$ cosets $\bar{x} + H\hat{a}m(2, 2^c)$, where $\bar{x} \in W$. Indeed, the order of each coset is also $2^{d-c-1} = 2^{d-1}/d$. It is also well known from coding theory that the minimum distance in each such a coset is again 4. We summarize these well-known facts as a corollary.

Corollary 7. *Let $H\hat{a}m(2, 2^c)$ be the extended binary Hamming code of length $d = 2^c$. Let W be the $(d-1)$-dimensional subspace of $V(2, d)$ consisting of all vectors of even weight. Then there exists a decomposition of W into $2^{d-1}/2^{d-c-1} = 2^c = d$ cosets $\bar{x} + H\hat{a}m(2, 2^c)$, where $\bar{x} \in W$. Moreover, for each such coset $\bar{x} + H\hat{a}m(2, 2^c)$ it holds that* $\mathrm{d}(\bar{x} + H\hat{a}m(2, 2^c)) = 4$.

To arrive at the desired conclusion, we have to make one more observation. It is another well known fact that the graph distance of two vertices in a cube is equal to the Hamming distance of their binary vector labels. The proof of this observation is not difficult and can be left to the reader.

Proposition 8. *Let Q_d be a d-dimensional hypercube and let ϕ be a one-to-one function from the vertex set of Q_d into $V(2, d)$ defined in such a way that two vertices $x, y \in Q_d$ are adjacent if and only if their respective labels, $\bar{x} = \phi(x)$ and $\bar{y} = \phi(y)$, differ exactly at one position. Then*

$$\mathrm{dist}_{Q_d}(x, y) = \mathrm{dist}_{Ham}(\bar{x}, \bar{y}),$$

where $\mathrm{dist}_{Ham}(\bar{x}, \bar{y})$ is the Hamming distance of the vectors $\bar{x}$ and $\bar{y}$.

But now we are done. We have partite set U of Q_d corresponding to subspace W. We can decompose U into d subsets corresponding to the cosets of W according to the vector labels of the vertices of U. In each of these subsets there are $2^{d-1}/d$ vertices and the distance between any two vertices in each subset is at least 4. Therefore no two vertices of the same subset (which is in fact our summing class) have a common neighbor and therefore they do not have "1" in the same column of $BAM(Q_d)$. Because every vertex of Q_d is indeed of degree d, there are exactly 2^{d-1} "1"s in the $2^{d-1}/d$ rows of $BAM(Q_d)$ corresponding to the vertices of the same subset. But it then follows that the sum of the row vectors of the same subset of $BAM(Q_d)$ is equal to the 2^{d-1}-dimensional vector $(1, 1, \ldots, 1)$ and the vertices really form a summing class. We summarize our observations in the following theorem.

Theorem 9. *Let Q_d with $d = 2^c \geq 4$ be a d-dimensional hypercube with vertex bipartition U, V defined by bipartite adjacency matrix $BAM(Q_d)$ whose rows correspond to partite set U and columns correspond to partite set V. Let every vertex of Q_d be labeled by a vector of d-dimensional binary vector space $V(2, d)$ in such a way that*

(1) *the labeling function $\phi : U \cup V \to V(2, d)$ is one-to-one,*
(2) *U consists of the vertices whose vector labels have an even weight, and*
(3) *two vertices u and v are adjacent if and only if their vector labels $\phi(u)$ and $\phi(v)$ differ exactly in one position.*

Let $H\hat{a}m(2, d)$ be the extended binary Hamming code and $\bar{x}$ be an arbitrary vector of $V(2, d)$ of even weight. Then the vectors of the coset $\bar{x} + H\hat{a}m(2, d)$ determine the vertices of one summing class of U, and summing classes $SC^0, SC^1, \ldots, SC^{d-1}$ form a decomposition of partite set U.

Proof. Let $U = \{u_1, u_2, \ldots, u_{2^{d-1}}\}$ and for $i = 1, 2, \ldots, 2^{d-1}$ let $\bar{\imath}$ be the row vector of $BAM(Q_d)$ corresponding to vertex u_i. Let $\phi : U \cup V \to V(2, d)$ be a bijection defined in such a way that two vertices u_i and v_j are adjacent in Q_d if and only if vectors $\bar{u}_i = \phi(u_j)$ and $\bar{v}_j = \phi(v_j)$ differ exactly in one position. If $\bar{x}$ is an arbitrary vector of an even weight, then according to Corollary 7 for every two vectors $\bar{u}_{i_1}$ and $\bar{u}_{i_2}$ that belong to coset $\bar{x} + H\hat{a}m(2, d)$ it holds that $\mathrm{dist}_{Ham}(\bar{u}_{i_1}, \bar{u}_{i_2}) \geq \mathrm{d}\big(\bar{x} + H\hat{a}m(2, d)\big) = 4$. But from Proposition 8 it follows that $\mathrm{dist}_{Q_d}(u_{i_1}, u_{i_2}) = \mathrm{dist}_{Ham}(\bar{u}_{i_1}, \bar{u}_{i_2}) \geq 4$. Therefore, vertices u_{i_1} and u_{i_2} do not have a common neighbor in Q_d. Hence for the corresponding row vectors $\bar{\imath}_1$ and $\bar{\imath}_2$ it holds that $w(\bar{\imath}_1 + \bar{\imath}_2) = w(\bar{\imath}_1) + w(\bar{\imath}_2) = 2d$. It immediately follows that if $\bar{x} + H\hat{a}m(2, d) = \{\bar{u}_{i_1}, \bar{u}_{i_2}, \ldots, \bar{u}_{i_p}\}$, where $p = 2^{d-c-1} = 2^{d-1}/d$, then $w(\bar{\imath}_1 + \bar{\imath}_2 + \cdots + \bar{\imath}_p) = w(\bar{\imath}_1) + w(\bar{\imath}_2) + \cdots + w(\bar{\imath}_p) = pd = 2^{d-c-1}d = 2^{d-1}$. Therefore, row vectors $\bar{\imath}_1, \bar{\imath}_2, \ldots, \bar{\imath}_p$ form a summing class of $BAM(Q_d)$ and the corresponding vertices $u_{i_1}, u_{i_2}, \ldots, u_{i_p}$ form a summing class of U. Moreover, because ϕ is one-to-one, the summing classes $SC^0, SC^1, \ldots, SC^{d-1}$ clearly form a decomposition of the partite set U. $\square$

Our main result now follows instantly.

Theorem 10. *The complete bipartite graph $K_{m,n}$ has an (r,s)-cyclic factorization into hypercubes Q_d if and only if $m = n = 2^{d-1}$ and $d = 2^c$, where c is a non-negative integer.*

Proof. The necessary condition is evident: the size of $K_{m,n}$ is divisible by the size of Q_d only if $m = n = 2^{d-1}$ and $d = 2^c$.

For $c = 0$ and $c = 1$ we observe that $K_{1,1} = Q_1$ and $K_{2,2} = Q_2$. For $c \geq 2$ it follows from Theorem 9 that one partite set of graph $K_{2^{d-1},2^{d-1}}$ can be decomposed into d summing classes $SC^0, SC^1, \ldots, SC^{d-1}$. From Lemma 4 it follows that there exists one-to-one function $N : V \to Z_{2^{d-1}}$ defined as $N(v_q) = q - 1$ for $q \in \{1, 2, \ldots, 2^{d-1}\}$ and for each SC^b there is one-to-one function $M^b : U^b \to Z_{2^{d-1}}$ defined as $M^b(u_{i_z}) = (z-1)d + b$ for $z \in \{1, 2, \ldots, p\}$. It also follows from the Lemma that functions $\theta^b : E^b \to Z_{2^{d-1}} \times Z_d$ are one-to-one. Therefore we can define functions $M : U \to Z_{2^{d-1}}$ and $\theta : E \to Z_{2^{d-1}} \times Z_d$ as "joins" of the respective partial functions M^b and θ^b in the obvious way: $M(u) = M^b(u)$ and $\theta(u,v) = \theta^b(u,v)$ if and only if the vertex u belongs to summing class SC^b. Functions M, N and θ are clearly one-to-one. This according to Theorem 2 guarantees the existence of the $(-d, 2^{d-1})$-cyclic factorization of $K_{2^{d-1},2^{d-1}}$ into hypercubes Q_d. $\square$

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