Zeros of adjoint polynomials of paths and cycles

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Abstract

The chromatic polynomial of a simple graph G with $n > 0$ vertices is a polynomial $\Sigma_{k=1}^{n}\alpha(G,k)(x)_k$ of degree n, where $(x)_k = x(x-1)\ldots(x-k+1)$ and $\alpha(G,k)$ is real for all k. The adjoint polynomial of G is defined to be $\Sigma_{k=1}^{n}\alpha(\overline{G},k)\mu^k$, where $\overline{G}$ is the complement of G. We find the zeros of the adjoint polynomials of paths and cycles.

1 Introduction

Let G be a simple graph with n vertices. A partition $\{A_1, A_2, \cdots, A_k\}$ of the vertex set of G, where k is a positive integer, is called a *k-independent partition* if each A_i is a nonempty independent set of G. Let $\alpha(G,k)$ denote the number of k-independent partitions of G. Then the *chromatic polynomial* of G is given by

$$P(G,\lambda) = \sum_{k=1}^{n} \alpha(G,k)(\lambda)_k, \tag{1}$$

where $(\lambda)_k = \lambda(\lambda-1)\cdots(\lambda-k+1)$. Two graphs G and H are said to be *chromatically equivalent* if $P(G,\lambda) = P(H,\lambda)$. It is clear that chromatic equivalence defines an equivalence relation on the family of graphs. The determination of chromatic equivalence classes has been an active area of research. (See [6, 7].)

Let $N(G,k)$ be the number of spanning subgraphs of G with exactly k components, each of which is complete. The *adjoint polynomial* of G is defined to be the polynomial

$$h(G,\mu) = \sum_{k=1}^{n} N(G,k)\mu^k. \tag{2}$$

Evidently $N(G,k) = \alpha(\overline{G},k)$, where $\overline{G}$ is the complement of G. Therefore two graphs are chromatically equivalent if and only if their complements have the same adjoint polynomials. Thus adjoint polynomials can be used to determine chromatic equivalence classes of graphs. This idea has been especially fruitful for dense graphs. (See [5, 10, 11, 12, 13, 14, 15, 16, 17].) Moreover the irreducibility of $h(G,\mu)$ over the rational field has been used to determine the chromatic equivalence classes of certain families of graphs ([4, 14]).

Another polynomial that is employed to the same end is the σ-polynomial [8, 9]. This polynomial $\sigma(G,\mu)$ is defined as $h(\overline{G},\mu)/\mu^{\chi(G)}$, where $\chi(G)$ is the chromatic number of G. The question of when all the zeros of this polynomial are real has been studied by Brenti and others [2, 3]. In particular it is shown in [3] that this is the case for K_3 and any graph with triangle-free complement. This result implies that the adjoint polynomial of any such graph has only real zeros.

In this paper we find the zeros of the adjoint polynomials of paths and cycles. In a subsequent paper we use these results to determine the chromatic equivalence classes of some graphs whose complements are disjoint unions of paths and cycles. We will denote by P_n the path, and C_n the cycle with n vertices.

2 A recursive expression for $h(G,\mu)$

By the definition, we have

Lemma 2.1 *$h(K_1,\mu) = \mu$.*

Lemma 2.2 [11] *If $G_1, G_2, \cdots, G_k$ are the components of G, then*

$$h(G,\mu) = \prod_{i=1}^{k} h(G_i,\mu). \tag{3}$$

For a vertex x in G, let $N_G(x)$ (or simply $N(x)$) be the set of vertices adjacent to x, and let $d_G(x)$ (or simply $d(x)$) be the degree of x in G. For $x, y \in V(G)$, let $G \cdot xy$ be the graph obtained from G by identifying x and y and replacing multi-edges by single ones. For $xy \in E(G)$, let

$$E'(xy) = \{xu \in E(G) | u \neq y, yu \notin E(G)\} \cup \{yv \in E(G) | v \neq x, xv \notin E(G)\}.$$

For $S \subseteq E(G)$, let $G - S$ be the spanning subgraph of G with edge set $E(G) - S$. If $xy \in E(G)$, let $G - xy$ simply denote the graph $G - \{xy\}$. For $x \neq y$, let $G \circ xy$ be the graph $(G - E'(xy)) \cdot xy$. (See Figure 1.)

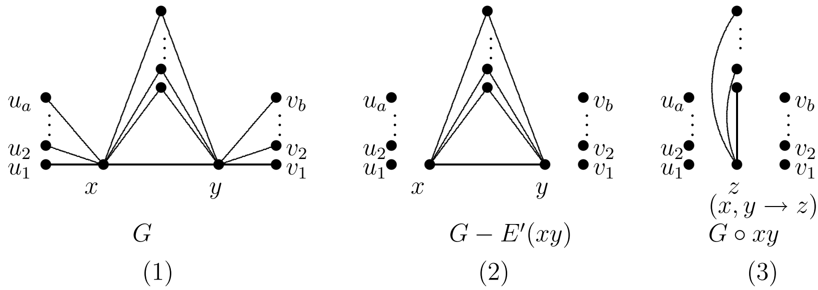

Figure 1

By the definition of $N(G, k)$, the following result is obtained directly.

Lemma 2.3 *For any graph G with $xy \in E(G)$ and any integer $k \geq 1$,*

$$N(G, k) = N(G - xy, k) + N(G \circ xy, k). \tag{4}$$

By (2) and (4), we have

Theorem 2.1 *For any graph G and $xy \in E(G)$,*

$$h(G, \mu) = h(G - xy, \mu) + h(G \circ xy, \mu). \tag{5}$$

Let $G - S$ be the graph $G[V(G) - S]$, where $S \subseteq V(G)$. By Theorem 2.1 and Lemmas 2.1 and 2.2, we have the following corollary.

Corollary *For $xy \in E(G)$ not contained in any triangle of G, we have*

$$h(G, \mu) = h(G - xy, \mu) + \mu h(G - \{x, y\}, \mu). \tag{6}$$

3 Zeros of $h(P_n, \mu)$ and $h(C_n, \mu)$

In this section, we shall find the zeros of $h(P_n, \mu)$ for $n \geq 2$ and $h(C_n, \mu)$ for $n \geq 4$.

Lemma 3.1 *$h(P_1, \mu) = \mu$, $h(P_2, \mu) = \mu^2 + \mu$ and for $n \geq 3$,*

$$h(P_n, \mu) = \mu(h(P_{n-1}, \mu) + h(P_{n-2}, \mu)).$$

Proof. By the definition of adjoint polynomials, it is easy to find that $h(P_1, \mu) = \mu$, $h(P_2, \mu) = \mu^2 + \mu$. The recursive expression follows from the Corollary to Theorem 2.1. □

Lemma 3.2 [10] *For any integer $n \geq 1$,*

$$h(P_n, \mu) = \sum_{k \leq n} \binom{k}{n-k} \mu^k.$$

For any integer $n \geq 1$ and real number x, define

$$g_n(x) = \begin{cases} x^n h(P_n, 1/x), & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases} \tag{7}$$

By Lemma 3.1, we have

Lemma 3.3 *For any real number x, $g_1(x) = 1$, $g_2(x) = x + 1$ and for $n \geq 3$,*

$$g_n(x) = g_{n-1}(x) + x g_{n-2}(x).$$

Lemma 3.4 *For any real number u,*

$$g_n(u^2 + u) = \sum_{i=0}^{n} (1+u)^i (-u)^{n-i}.$$

Proof. It is clear that the result holds when $n = 1, 2$. Now let $n \geq 3$. By induction, we have

$$\begin{aligned} g_n(u^2+u) &= g_{n-1}(u^2+u) + (u^2+u) g_{n-2}(u^2+u) \\ &= \sum_{i=0}^{n-1} (1+u)^i (-u)^{n-1-i} + (u^2+u) \sum_{i=0}^{n-2} (1+u)^i (-u)^{n-2-i} \\ &= \sum_{i=0}^{n-1} (1+u)^i (-u)^{n-1-i} - \sum_{i=0}^{n-2} (1+u)^{i+1} (-u)^{n-1-i} \\ &= (1+u)^{n-1} + \sum_{i=0}^{n-2} (1+u)^i (-u)^{n-i} \\ &= \sum_{i=0}^{n} (1+u)^i (-u)^{n-i}. \end{aligned}$$

□

Corollary *For any real number u,*

$$(2u+1) g_n(u^2+u) = (1+u)^{n+1} - (-u)^{n+1}.$$

Lemma 3.5 [1, p.64] *For real numbers a, b and positive integer n,*

(i) if n is odd,

$$a^n - b^n = (a-b) \prod_{s=1}^{(n-1)/2} \left(a^2 + b^2 - 2ab \cos \frac{2s\pi}{n} \right);$$

(ii) if n is even,

$$a^n - b^n = (a-b)(a+b) \prod_{s=1}^{(n-2)/2} \left(a^2 + b^2 - 2ab \cos \frac{2s\pi}{n} \right).$$

Lemma 3.6 *For positive integer n,*

$$g_n(x) = \prod_{s=1}^{\lfloor n/2 \rfloor} \left(2x + 1 + 2x \cos \frac{2s\pi}{n+1}\right). \tag{8}$$

Proof. By Lemma 3.5 and the corollary to Lemma 3.4, for any real number $u \neq -1/2$,

$$\begin{aligned} g_n(u^2+u) &= \prod_{s=1}^{\lfloor n/2 \rfloor} \left((u+1)^2 + u^2 + 2(u^2+u) \cos \frac{2s\pi}{n+1}\right) \\ &= \prod_{s=1}^{\lfloor n/2 \rfloor} \left(2u^2 + 2u + 1 + 2(u^2+u) \cos \frac{2s\pi}{n+1}\right). \end{aligned}$$

Observe that for any real number x with $x > -1/4$, there is a real number $u \neq -1/2$ such that $u^2 + u = x$. Thus for each real number x with $x > -1/4$,

$$g_n(x) = \prod_{s=1}^{\lfloor n/2 \rfloor} \left(2x + 1 + 2x \cos \frac{2s\pi}{n+1}\right).$$

Since $g_n(x)$ is a polynomial with degree less than n, the above equality also holds for any real number x such that $x \leq -1/4$. Thus (8) is obtained. □

Theorem 3.1 *For any positive integer n,*

$$h(P_n, \mu) = \mu^{\lceil n/2 \rceil} \prod_{s=1}^{\lfloor n/2 \rfloor} \left(\mu + 2 + 2 \cos \frac{2s\pi}{n+1}\right). \tag{9}$$

Proof. By (7), $h(P_n, \mu) = \mu^n g_n(1/\mu)$ when $\mu \neq 0$. Thus when $\mu \neq 0$, the result follows from (8). Since $h(P_n, 0) = 0$ for any $n \geq 1$, the result also holds for $\mu = 0$. □

Corollary *For any positive integer n, $h(P_n, \mu)$ has the following zeros:*

$$\underbrace{0, 0, \cdots, 0}_{\lceil n/2 \rceil}, \quad -2 - 2\cos \frac{2s\pi}{n+1}, \quad s = 1, 2, \cdots, \lfloor n/2 \rfloor.$$

We now consider $P(C_n, \mu)$. By the corollary to Theorem 2.1, we have

Lemma 3.7 *For any positive integer $n \geq 4$,*

$$h(C_n, \mu) = h(P_n, \mu) + \mu h(P_{n-2}, \mu).$$

For any integer $n \geq 4$ and real number x, define

$$f_n(x) = \begin{cases} x^n h(C_n, 1/x), & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases} \tag{10}$$

By (7) and Lemma 3.7, we have

Lemma 3.8 *For any positive integer $n \geq 4$ and real number x,*

$$f_n(x) = g_n(x) + x g_{n-2}(x).$$

Lemma 3.9 *For any positive integer $n \geq 4$ and real number u,*

$$f_n(u^2 + u) = (1 + u)^n + (-u)^n.$$

Proof. For any real number u, by Lemma 3.8 and the corollary to Lemma 3.4,

$$\begin{aligned} & (2u+1) f_n(u^2+u) \\ = \; & (2u+1) g_n(u^2+u) + (u^2+u)(2u+1) g_{n-2}(u^2+u) \\ = \; & (1+u)^{n+1} - (-u)^{n+1} + (u^2+u)\left((1+u)^{n-1} - (-u)^{n-1}\right) \\ = \; & (2u+1)\left((1+u)^n + (-u)^n\right). \end{aligned}$$

Thus the result holds for $u \neq -1/2$. Since $f_n(u^2+u)$ is a polynomial in u, the result also holds for $u = -1/2$. □

The next result follows from Lemma 3.5.

Lemma 3.10 ([1, p.65] *For real numbers a, b and positive integer n,*
(i) if n is odd,

$$a^n + b^n = (a+b) \prod_{s=1}^{(n-1)/2} \left(a^2 + b^2 - 2ab \cos \frac{(2s-1)\pi}{n} \right);$$

(ii) if n is even,

$$a^n + b^n = \prod_{s=1}^{n/2} \left(a^2 + b^2 - 2ab \cos \frac{(2s-1)\pi}{n} \right).$$

By a proof similar to that of Lemma 3.6, the next result follows from Lemmas 3.9 and 3.10.

Lemma 3.11 *For any integer $n \geq 4$,*

$$f_n(x) = \prod_{s=1}^{\lfloor n/2 \rfloor} \left(2x + 1 + 2x \cos \frac{(2s-1)\pi}{n} \right). \tag{11}$$

Theorem 3.2 *For any integer $n \geq 4$,*

$$h(C_n, \mu) = \mu^{\lceil n/2 \rceil} \prod_{s=1}^{\lfloor n/2 \rfloor} \left(\mu + 2 + 2 \cos \frac{(2s-1)\pi}{n} \right). \tag{12}$$

Proof. Since $h(C_n, 0) = 0$, the result holds for $\mu = 0$. When $\mu \neq 0$, $h(C_n, \mu) = \mu^n f_n(1/\mu)$ by (10), and thus the result follows from (11). □

Corollary *For any integer $n \geq 4$, $h(C_n, \mu)$ has the following zeros:*

$$\underbrace{0, 0, \cdots, 0}_{\lceil n/2 \rceil}, \quad -2 - 2\cos\frac{(2s-1)\pi}{n}, \quad s = 1, 2, \cdots, \lfloor n/2 \rfloor.$$

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(Received 2 Jan 2001)