The properties of self-complementary graphs and new lower bounds for diagonal Ramsey numbers*

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Abstract

Some properties of self-complementary graphs have been studied and 3 new lower bounds for diagonal Ramsey numbers have been obtained. They are: $R(17,17) \geq 8917$, $R(18,18) \geq 11005$, $R(19,19) \geq 17885$.

1 Introduction

In 1955 Greenwood and Gleason ([2]) utilized quadratic residues modulo prime numbers $p = 5$ and $p = 17$ to construct self-complementary graphs G_5 and G_{17}. Afterwards, Kalbfleisch([3]), Burling & Reyner ([1]), Mathon([5]) and Shearer([7]) extended the discussion of the clique numbers $c(G_p)$ of self-complementary graphs G_p to the range $p < 3000$ and proved the following result:

Lemma 1 ([5, 7]) *$c(G_p) = k$ implies $R(k+2, k+2) > 2p+2$.*

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The survey [6] summarizes their work and keeps a record of the best lower bounds for diagonal Ramsey numbers up to date. They are:

$R(6,6) \geq 102,\ R(7,7) \geq 205,\ R(8,8) \geq 282,\ R(9,9) \geq 565,$
$R(10,10) \geq 798,\ R(11,11) \geq 1597,\ R(13,13) \geq 2557,\ R(14,14) \geq 2989,$
$R(15,15) \geq 5485,\ R(16,16) \geq 5605$ and $R(12,12) \geq 1597 + 11$.

As far as we know, there has been no significant progress in the study of $c(G_p)$ and $R(k,k)$ in the last ten years. In [4, 8, 9, 10, 11, 12] we studied some properties of cyclic graphs of prime order and obtained some new lower bounds for Ramsey numbers. This paper investigates further properties of self-complementary graphs and introduces a new algorithm to estimate lower bounds for diagonal Ramsey numbers from which three new results have been obtained:

$$R(17,17) \geq 8917, R(18,18) \geq 11005, R(19,19) \geq 17885.$$

2 Basic properties of self-complementary graphs

Let $p = 4m+1 \geq 5$ be a prime number and let A denote the set of quadratic residues modulo p. Let $\mathbb{Z}_p$ denote $\{-2m, \ldots, -1, 0, 1, \ldots, 2m\}$. Then $\mathbb{Z}_p$ is a complete system of residues of integers modulo p. An integer n shall be understood to be an element $\overline{n} \in \mathbb{Z}_p$ such that $p|n - \overline{n}$ if the context makes it clear. When two integers a and b have the same residue modulo p we often write $a = b$ instead of $a \equiv b \pmod p$.

Definition 1 *For a prime $p = 4m + 1 \geq 5$ the graph G_p is defined as follows:*

1. *The vertex set V of G_p is $\mathbb{Z}_p$.*
2. *The edge set is $E = \{\{x, y\}|x - y \in A\}$.*

The clique number of G_p is denoted by $c(G_p)$.

Since $p \equiv 1 \pmod 4$, we have $(\frac{-1}{p}) = 1$, where $(\frac{-1}{p})$ is the Legendre symbol of -1. This means that $-1 \in A$. So $x - y \in A$ if and only if $y - x \in A$, which implies that the edge set in Definition 1 is well-defined.

If $a \in A$, then $x - y \in A$ if and only if $a(x - y) \in A$. This implies the following result:

Lemma 2 *Let $a \in A$, $b \in \mathbb{Z}_p$. Then the affine transform $f : x \mapsto ax + b$ is an automorphism of G_p.*

Definition 2 *Let $B = \{x \in A|x - 1 \in A\}$. Let $G[B]$ be the subgraph of G_p defined as follows:*

1. *The vertex set of $G[B]$ is B.*
2. *The edge set of $G[B]$ is $\{\{x, y\}|x, y \in B, x - y \in A\}$.*

The clique number of B is denoted by $[B]$. We make a convention that $[B] = 0$ if $B = \emptyset$.

Lemma 3

$$c(G_p) = [B] + 2.$$

Proof. First note that $a \in A$ if and only if $a^{-1} \in A$ for any $a \neq 0$, because $(\frac{a}{p})(\frac{a^{-1}}{p}) = (\frac{1}{p}) = 1$.

Next we show an important property of the graph G_p. That is: G_p is edge-transitive. Let $\{x_1, x_2\}$ be an edge of G_p, i.e., $x_2 - x_1 \in A$. Then $(x_2 - x_1)^{-1} \in A$. By Lemma 2 $f(x) = (x_2 - x_1)^{-1}(x - x_1)$ is an automorphism of G_p. Obviously f carries the edge $\{x_1, x_2\}$ into $\{0, 1\}$.

Therefore $c(G_p)$ is equal to the number of vertices of a maximal clique of G_p that contains both 0 and 1. This implies that $c(G_p) = [B] + 2$. □

This lemma tells us that the computation of the clique number of G_p can be reduced to that of its subgraph $G[B]$, which is much simpler.

3 Basic properties of $G[B]$

Let $|B|$ denote the number of elements in B.

Lemma 4

$$|B| = (p - 5)/4.$$

Proof. It follows from the definition of B that $x \in B$ if and only if $x, x - 1 \in A$. Thus

$$\begin{aligned} |B| &= \frac{1}{4}\sum_{x=2}^{p-1}(1 + (\frac{x}{p}))(1 + (\frac{x-1}{p})) \\ &= \frac{1}{4}\sum_{x=2}^{p-1}(1 + (\frac{x}{p}) + (\frac{x-1}{p}) + (\frac{x(x-1)}{p})). \end{aligned}$$

Note that $(\frac{1}{p}) = (\frac{p-1}{p}) = 1$ and $\sum_{x=1}^{p-1}(\frac{x}{p}) = 0$. Let x' be an element in $\mathbb{Z}_p$ such that $x'x \equiv 1 \pmod{p}$, Then we have

$$\begin{aligned} 4|B| &= \sum_{x=2}^{p-1} 1 + (\sum_{x=1}^{p-1}(\frac{x}{p}) - (\frac{1}{p})) + (\sum_{x=1}^{p-1}(\frac{x}{p}) - (\frac{p-1}{p})) \\ &\quad + \sum_{x=1}^{p-2}(\frac{x'^2 x(x+1)}{p}) \\ &= (p-2) - 1 - 1 + \sum_{x'=1}^{p-2}(\frac{x'+1}{p}) \\ &= p - 4 + \sum_{x'=1}^{p-1}(\frac{x'}{p}) - (\frac{1}{p}). \end{aligned}$$

Hence $4|B| = p - 5$. □

Now we study the structure of B. We assume that $B \neq \emptyset$ in the following discussion.

Definition 3 *Assume that $x_1, x_2 \in B$. If there is an affine transform $f : x \mapsto ax + b$ with $a \in A, b \in \mathbb{Z}_p$ carrying the set $\{0, 1, x_1\}$ into $\{0, 1, x_2\}$ then x_1 and x_2 are defined to be linearly related and we denote this by $x_1 \sim x_2$.*

Lemma 5 *The relation of being linearly related in B is an equivalence relation. Moreover, every equivalence class is a subset of six elements in the form*

$$\{a, a^{-1}, 1 - a^{-1}, a(a - 1)^{-1}, (1 - a)^{-1}, 1 - a\} \tag{1}$$

with the following two exceptions:

1) When $2 \in B$, there is a unique class $\{2, 2^{-1}, -1\}$ with three elements.

2) When $a(1 - a) = 1$, there is a unique class $\{a, 1 - a\}$ with two elements.

Proof. It is easy to verify that $\sim$ is an equivalence relation. Note that for any $a \in B$, there are only 6 affine transformations that carry the set $\{0, 1, a\}$ to $\{0, 1, b\}$ for some $b \in B$. They are

$$f_0(x) = x, f_1(x) = a^{-1}x, f_2(x) = 1 - a^{-1}x,$$

$$f_3(x) = (1 - a)^{-1}(x - a), f_4(x) = (a - 1)^{-1}(x - 1), f_5(x) = 1 - x.$$

For fixed a let $\{f_j(0), f_j(1), f_j(a)\} = \{0, 1, b_j\}, 0 \leq j \leq 5$. If $b_0, \ldots, b_5$ are mutually distinct then the set $\{b_0, \ldots, b_5\}$ of six elements is in the form of (1), otherwise one of the 15 equalities

$$a = a^{-1}, a = 1 - a^{-1}, \ldots, (1 - a)^{-1} = 1 - a$$

must hold. This implies that either $a \in \{2, 2^{-1}, -1\}$ or $a(1 - a) = 1$. The proof of the lemma is concluded. □

Let $b \equiv |B| \pmod 6$ with $0 \leq b \leq 5$. Lemma 5 implies that

1. If $b = 0$ then every equivalence class in B has 6 elements.
2. If $b = 2$ or $b = 5$ then there is an equivalence class with 2 elements in B.
3. If $b = 3$ or $b = 5$ then there is an equivalence class with 3 elements in B.

The following lemma follows from Lemma 4 and Lemma 5 immediately.

Lemma 6 *If $p = 24k + 5$ then B has k equivalence classes. If $p = 24k + 1, 24k + 13$ or $24k + 17$ then B has $k + 1$ classes.*

4 Method for computing $[B]$

If $B = \emptyset$ then $[B] = 0$ by our convention. Hence we may assume that $B \neq \emptyset$ throughout this section. It is easy to see from Lemma 5 that every equivalence class in B contains a positive integer.

Definition 4 *The minimal positive integer a in an equivalence class in B is called the representative of that class and that class is denoted by $\langle a \rangle$. Let N denote the set of all representatives in B.*

Lemma 7 *For every $a \in B$, let $D(a) = \{x \in B | x - a \in A\}$ and let $d(a) = |D(a)|$. Then the condition*

$$\max\{d(a)|a \in N\} = 0$$

implies that $[B] = 1$.

Proof. First we point out a property of equivalent elements in B. Assume that $a \in B$ and $x \in D(a)$. By the definitions of B and $D(a)$ we know that $x \in B$ and $x-a \in A$. Moreover, $\{0, 1, a, x\}$ is a 4-clique in G_p. If $a \sim b$ then there exists an affine transformation f carrying $\{0, 1, a\}$ to $\{0, 1, b\}$ for some b. Apply f to G_p then Lemma 2 implies that $\{0, 1, b, f(x)\}$ is still a 4-clique of G_p. By the definition of $D(b)$ we have $f(x) \in D(b)$. Thus $x \in D(a)$ if and only if $f(x) \in D(b)$. Therefore $d(a) = d(b)$ if $a \sim b$. It follows that $\max\{d(a)|a \in B\} = 0$ whenever $\max\{d(a)|a \in N\} = 0$, which amounts to saying that $D(a) = \emptyset$ for every $a \in B$. Hence $x - a \notin A$ for any $a, x \in B$. The clique $\{0, 1, a\}$ is the largest clique of G_p and $c(G_p) = 3$. It follows from Lemma 3 that $[B] = 1$. □

Next we consider the case $[B] \geq 2$. Let us introduce a total order $\prec$ in B as follows.

Definition 5 *(i) The order inside an equivalence class in B is defined as:*

1. *If $\langle a \rangle$ contains 6 elements, then*

$$a \prec a^{-1} \prec 1 - a^{-1} \prec a(a-1)^{-1} \prec (1-a)^{-1} \prec 1 - a;$$

2. *If $\langle a \rangle$ contains 2 elements, then*

$$a \prec 1 - a;$$

3. *If $\langle a \rangle$ contains 3 elements, which means $a = 2$, then*

$$2 \prec 2^{-1} \prec -1.$$

(ii) If $x, y \in B$ belong to different classes,say $x \in \langle a \rangle$ and $y \in \langle b \rangle$, then $x \prec y$ if and only if either $d(a) < d(b)$ or $d(a) = d(b)$ and $a < b$.

Obviously this makes $(B, \prec)$ a totally-ordered set.

Definition 6 *A chain $x_0 \prec x_1 \prec \cdots \prec x_k$ of length k in $(B, \prec)$ is called an A-chain if $x_i - x_j \in A$ for all i, j satisfying $0 \le i < j \le k$. Let $l(x_0)$ denote the maximal length of all A-chains starting with x_0.*

Theorem 1

$$[B] = 1 + \max\{l(a)|a \in N\}. \tag{2}$$

Proof. It is immediate by the definition that the $k+1$ elements in an A-chain $a \prec x_1 \prec \cdots \prec x_k$ form a clique in $G[B]$. Hence $[B] \ge k+1$. It remains to show that $[B] \le k+1$.

Suppose that $[B] = k+1 \ge 2$. Then there exists a $k+1$ clique $D = \{b, x_1, \ldots, x_k\}$ in $G[B]$. Arrange these vertices in ascending order to obtain an A-chain of length k in $(B, \prec)$. We may assume that b is the starting point of this chain. If $b \in N$ then the right hand side of (2) is greater than or equal to $k+1 = [B]$, as desired. If $b \notin B$ assume that $b \in \langle a \rangle$. Then by Definition 3 there exists an affine transformation f carrying $\{0, 1, b\}$ into $\{0, 1, a\}$. Lemma 2 implies that f is an automorphism of G_p. It is easy to see that f is also an automorphism of $G[B]$. Hence f maps the clique D onto a $k+1$ clique $D^* = \{a, f(x_1), \ldots, f(x_k)\}$ in $G[B]$. Thus we get an A-chain of length k in $(B, \prec)$. From the rule of ordering the start point of this chain must be a. Since $a \in N$, the right hand side of (2) is greater than or equal to $k+1 = [B]$ and this concludes the proof of the theorem. □

5 A method to obtain lower bounds for diagonal Ramsey numbers

Based on the analysis of the previous sections we obtain a new method to compute $c(G_p)$ and thus to obtain lower bounds for diagonal Ramsey numbers.

The algorithm is described as follows:

Step 1:

Choose a prime number $p = 4m+1 \ge 5$. Let $\mathbb{Z}_p = \{-2m, \ldots, -1, 0, 1, \ldots, 2m\}$ and choose a generator g of the multiplicative group $\mathbb{Z}_p^*$. Find $|B| = (p-5)/4$. If $|B| = 0$,(which means $p = 5$) then let $[B] = 0$ and go to Step 7.

Step 2:

Set $A = \{g^{2i} \in \mathbb{Z}_p | 0 \le i \le 2m-1\}$, $B = \{x \in A | x - 1 \in A\}$.

Step 3:

Determine all equivalence classes in B by virtue of Lemma 5 and find the set N of the representatives of all classes.

Step 4:

Find the number of elements $d(a)$ of the set $\{x \in B | x - a \in A\}$ for every $a \in N$. If $\max\{d(a)|a \in N\} = 0$ then $[B] = 1$ and go to Step 7.

Step 5:

Construct the totally ordered set $(B, \prec)$ in terms of Definition 5.

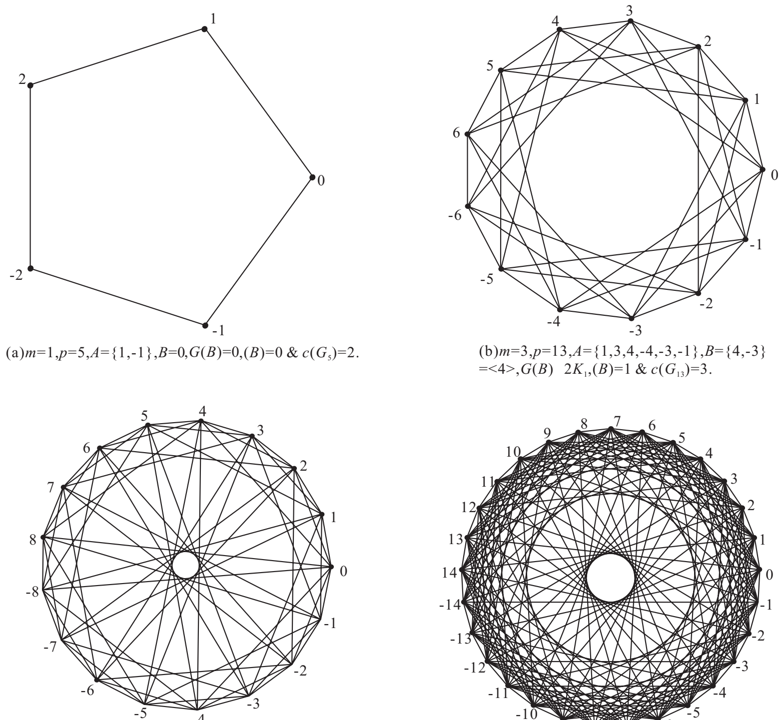
Step 6:

Find $l(a)$ for every $a \in N$ in terms of Definition 6 and determine $[B] = 1 + \max\{l(a)|a \in N\}$.

Step 7:

Set $k = c(G_p) = [B] + 2$. Conclude that $R(k+1, k+1) \geq p+1$, $R(k+2, k+2) \geq 2p+3$ and the algorithm terminates.

To explain the algorithm more explicitly we apply it to obtain some known results. The calculations can be easily carried out manually when $p = 5, 13, 17, 29$. Figure 1 illustrates these examples, among which the ones with $p = 5, 17$ are particularly nice.



(a)m=1,p=5,A={1,-1},B=0,$G(B)$=0,(B)=0 & $c(G_5)$=2.

(b)m=3,p=13,A={1,3,4,-4,-3,-1},B={4,-3} =<4>,$G(B)$ $2K_1$,(B)=1 & $c(G_{13})$=3.

(c)m=4,p=17,A={1,2,4,8,-8,-4,-2,-1}, B={2,-8,-1}=<2>,$G(B)$ $3K_1$,(B)=1 & $c(G_{17})$=3.

(d)m=7,p=29,A={1,4,5,6,7,9,13,-13,-9,-7, -6,-5,-4,-1},B={5,6,7,-6,-5,-4}=<5>,$G(B)$ C_6,(B)=2 & $c(G_{29})$=4.

Figure 1. Some best simple self-complementary graphs

Example 1 $R(3, 3) \geq 6$ ([2]).

Proof. Set $p = 5$. By Step 1 and Step 7 of the algorithm we obtain $|B| = 0, [B] = 0, c(G_p) = 2$ and $R(3,3) \geq 6$. □

Example 2 $R(4,4) \geq 18$ ([2]).

Proof. Set $p = 17$. From Step 1 we obtain $|B| = 3$. Thus B has only one equivalence class $\{2, -8, -1\}$ by Lemma 5. Since none of $2-(-8), 2-(-1)$ is a quadratic residue modulo 17, we have $d(2) = 0, [B] = 1, c(G_p) = 3$ and $R(4,4) \geq 18$. □

Example 3 $R(6,6) \geq 102$ ([3]), $R(7,7) \geq 205$ ([5], [7]).

Proof. Set $p = 101$ and $g = 2$. Then $|B| = 24$. The set B is divided into 4 equivalence classes, each of which contains 6 elements:

$$\begin{aligned}
\langle 5\rangle &= \{5, -20, 21, -24, 25, -4\},\\
\langle 14\rangle &= \{14, -36, 37, -30, 31, -13\},\\
\langle 22\rangle &= \{22, 23, -22, -23, 24, -21\},\\
\langle 6\rangle &= \{6, 17, -16, -19, 20, -5\}.
\end{aligned}$$

Then $d(5) = 10, d(14) = 10, d(22) = 10, d(6) = 12$. The totally-ordered set $(B, \prec)$ is $\langle 5\rangle, \langle 14\rangle, \langle 22\rangle, \langle 6\rangle$. To find $l(5)$ we first set

$$\begin{aligned}
D(5) &= \{x \in B | x - 5 \in A, 5 \prec x\}\\
&= \{-20, 21, 25, -4, 14, 22, 24, 6, -16, -19\}.
\end{aligned}$$

Then $|D(5)| = 10$. By backtracking we obtain $l(5) = 2$ and the first A-chain of length 2 starting with 5 is $5 \prec -20 \prec 25$. Set

$$\begin{aligned}
D(14) &= \{x \in B | x - 14 \in A, 14 \prec x\}\\
&= \{37, 31, 23, -22, -23, -16, -19, 20, -5\}.
\end{aligned}$$

Then $|D(14)| = 9$. By backtracking we obtain $l(14) = 2$ and the first A-chain of length 2 starting with 14 is $14 \prec 37 \prec 31$. Similarly with

$$\begin{aligned}
D(22) &= \{x \in B | x - 22 \in A, 22 \prec x\}\\
&= \{23, -23, -21, 6, 17\}
\end{aligned}$$

and

$$\begin{aligned}
D(6) &= \{x \in B | x - 6 \in A, 6 \prec x\}\\
&= \{-16, -19, 20\}
\end{aligned}$$

we obtain $l(22) = l(6) = 2$ and the corresponding A-chains $22 \prec 23 \prec 6$ and $6 \prec -16 \prec 20$. Hence $\max\{l(a) | a \in N\} = 2, [B] = 3, c(G_p) = 5$, and we conclude that $R(6,6) \geq 102$ and $R(7,7) \geq 205$. □

Example 4 $R(8,8) \geq 282$ ([1]), $R(9,9) \geq 565$ ([5], [7]).

Proof. Set $p = 281$ and $g = 3$. Then $|B| = 69$. The set B is divided into 12 equivalence classes:

$$\langle 9\rangle = \{9, 125, -124, -34, 35, -8\},$$
$$\langle 59\rangle = \{59, -100, 101, 64, -63, -58\},$$
$$\langle 2\rangle = \{2, -140, -1\},$$
$$\langle 5\rangle = \{5, -56, 57, -69, 70, -4\},$$
$$\langle 10\rangle = \{10, -28, 29, 126, -125, -9\},$$
$$\langle 50\rangle = \{50, -118, 119, -85, 86, -49\},$$
$$\langle 8\rangle = \{8, -35, 36, -39, 40, -7\},$$
$$\langle 17\rangle = \{17, -33, 34, 124, -123, -16\},$$
$$\langle 32\rangle = \{32, -79, 80, 137, -136, -31\},$$
$$\langle 81\rangle = \{81, -111, 112, 138, -137, -80\},$$
$$\langle 18\rangle = \{18, -78, 79, -32, 33, -17\},$$
$$\langle 58\rangle = \{58, 63, -62, -68, 69, -57\}.$$

Then

$$d(9) = 30, d(59) = 30, d(2) = 32, d(5) = 32, d(10) = 32, d(50) = 32,$$

$$d(8) = 34, d(17) = 34, d(32) = 34, d(81) = 34, d(18) = 36, d(58) = 36.$$

The totally-ordered set $(B, \prec)$ is $\langle 9\rangle, \langle 59\rangle, \ldots, \langle 58\rangle$.

To find $l(9)$ we set

$$\begin{aligned} D(9) &= \{x \in B | x - 9 \in A, 9 \prec x\} \\ &= \{125, -34, -8, 59, -100, -63, 2, -140, -1, 5, -69, 10, 29, -9, -49, 8, \\ &\quad 40, -7, 17, 34, -123, -16, 137, -136, -31, 81, 18, 79, 58, -57\}. \end{aligned}$$

Then $|D(5)| = 30$. By backtracking we obtain $l(9) = 4$ and the first A-chain of length 4 starting with 9 is $9 \prec 125 \prec 59 \prec 2 \prec -7$. Similarly with

$$\begin{aligned} D(59) &= \{x \in B | x - 59 \in A, 59 \prec x\} \\ &= \{64, 2, 57, -69, -4, 10, -9, 50, -85, -39, -7, 34, -79, 137, \\ &\quad -136, -31, -111, 112, 138, -137, -78, 79, 58, 63, -62, 69, -57\}, \end{aligned}$$

$$\begin{aligned} D(2) &= \{x \in B | x - 2 \in A, 2 \prec x\} \\ &= \{-56, 70, 10, 126, 36, -7, -33, 34, -123, -16, -79, 80, \\ &\quad -136, -31, 81, 138, 18, -78, -32, 33, 58, -62, -68, -57\}, \end{aligned}$$

$$
\begin{aligned}
D(81) &= \{x \in B | x - 81 \in A, 81 \prec x\} \\
&= \{112, 138, -137, 18, 79, -17, 63, -62, -68, -57\},
\end{aligned}
$$

$$
\begin{aligned}
D(18) &= \{x \in B | x - 18 \in A, 18 \prec x\} \\
&= \{-32, -17, 58, 63, -62, -68\}
\end{aligned}
$$

and

$$
\begin{aligned}
D(58) &= \{x \in B | x - 58 \in A, 58 \prec x\} \\
&= \{63, -68\}
\end{aligned}
$$

we obtain $l(59) = 4, l(2) = 4, l(81) = 4, l(18) = 3, l(58) = 1$ and the corresponding A-chains

$$
\begin{gathered}
59 \prec 64 \prec 2 \prec -79 \prec -136, \\
2 \prec -56 \prec 70 \prec 34 \prec -16, \\
81 \prec 138 \prec 79 \prec -62 \prec -57, \\
18 \prec -32 \prec 58 \prec -68
\end{gathered}
$$

and $58 \prec 63$. Hence $\max\{l(a) | a \in N\} = 4, [B] = 5, c(G_p) = 7$, and we conclude that $R(8,8) \geq 282, R(9,9) \geq 565$. $\square$

6 Three new lower bounds for diagonal Ramsey numbers

Generally speaking, the amount of computation increases exponentially when one uses backtracking methods to compute the clique numbers of G_p. Many algorithms become impractical when p is relatively large (for example $p = 4457$ or $p = 8941$). Our algorithm improves this situation drastically so that we can handle fairly large prime numbers. The efficiency of our algorithm is based on the following two considerations:

A) To compute $[B]$ we only need to handle the A-chains starting with a representative of the equivalence classes in B.

B) The ordering of the totally-ordered set $(B, \prec)$ enables us to give higher priority to the equivalence classes with minimum value of $|D(a)|$ when we compute $l(a)$, so many unnecessary branches are pruned preliminarily during the process of backtracking. The redundant calculation for isomorphic cyclic graphs are avoided. Moreover, the values $|D\langle a_i \rangle|$ of the equivalence classes $\langle a_i \rangle$ become smaller and smaller, which reduces the amount of computation significantly and increases the speed of the computation of $l(a_i)$.

By taking these measures we were able to compute the clique numbers $c(G_p)$ with $p < 15,000$ with the aid of a single computer in a reasonably short period of time. In most cases, the CPU time spent for the computation of $c(G_p)$ is less than 1 second when $p < 1500$ on a Pentium III 800 machine. In our computation of $c(G_p)$ for $p = 4457, 5501, 8941$ (as in Theorem 2) the CPU time is 10 minutes, 30 minutes and 80 hours respectively.

Theorem 2

$$R(17,17) \geq 8917,\ R(18,18) \geq 11005,\ R(19,19) \geq 17885.$$

Proof. We omit details since they are more or less the same as the last two examples in the previous section.

(1) Set $p = 4457$ and $g = 3$. Then $|B| = 1113$. The set B is divided into 186 equivalence classes:

$$\langle 101 \rangle = \{101, 1368, -1367, 313, -312, -100\},$$

$$\langle 443 \rangle = \{443, 825, -824, 2169, -2168, -442\},$$
$$\langle 1145 \rangle = \{1145, 1993, -1992, 1649, -1648, -1144\},$$
$$\langle 1202 \rangle = \{1202, 1346, -1345, -2144, 2145, -1201\},$$
$$\langle 141 \rangle = \{141, -1296, 1297, 2134, -2133, -140\},$$
$$\langle 152 \rangle = \{152, 909, -908, 1772, -1771, -151\},$$
$$\langle 206 \rangle = \{206, 238, -237, -1260, 1261, -205\},$$
$$\langle 431 \rangle = \{431, 1334, -1333, -652, 653, -430\},$$
$$\langle 560 \rangle = \{560, -581, 582, -1187, 1188, -559\},$$
$$\langle 594 \rangle = \{594, -1118, 1119, -2111, 2112, -593\},$$
$$\langle 602 \rangle = \{602, 807, -806, -1764, 1765, -601\},$$
$$\langle 734 \rangle = \{734, -1682, 1683, 1965, -1964, -733\},$$
$$\cdots$$
$$\langle 1067 \rangle = \{1067, -1863, 1864, 1987, -1986, -1066\},$$
$$\langle 1124 \rangle = \{1124, -1257, 1258, -1527, 1528, -1123\}$$

with

$$d(101) = 540, d(443) = 540, d(1145) = 540, d(1202) = 540, d(141) = 542,$$

$$d(152) = 542, d(206) = 542, d(431) = 542, d(560) = 542, d(594) = 542,$$
$$d(602) = 542, d(734) = 542,$$
$$\cdots$$
$$d(1067) = 570, d(1124) = 570.$$

The totally-ordered set $(B, \prec)$ is $\langle 101 \rangle, \langle 443 \rangle, \ldots, \langle 1124 \rangle$. By computation we have

$$l(101) = 12$$

and

$$l(a) \leq 12$$

for all other $a \in N$. The first A-chain of length 12 is

$$101 \prec 1368 \prec -2168 \prec -442 \prec 122 \prec 548 \prec -1592$$

$$\prec 1481 \prec 2173 \prec -1044 \prec -1 \prec 922 \prec 1107.$$

Hence $[B] = 13, c(G_p) = 15$, and we conclude that $R(17,17) \geq 8917$.

(2) Set $p = 5501$ and $g = 2$. Then $|B| = 1374$. The set B is divided into 229 equivalence classes:

$$\langle 601 \rangle = \{601, -897, 898, -925, 926, -600\},$$

$$\langle 677 \rangle = \{677, 1812, -1811, -1537, 1538, -676\},$$

$$\langle 54 \rangle = \{54, -2343, 2344, -2490, 2491, -53\},$$

$$\langle 105 \rangle = \{105, -2148, 2149, 1006, -1005, -104\},$$

$$\langle 196 \rangle = \{196, 421, -420, 537, -536, -195\},$$

$$\langle 213 \rangle = \{213, -594, 595, -1997, 1998, -212\},$$

$$\langle 384 \rangle = \{384, 616, -615, -1780, 1781, -383\},$$

$$\langle 487 \rangle = \{487, 2101, -2100, -2093, 2094, -486\},$$

$$\langle 518 \rangle = \{518, 754, -753, 1746, -1745, -517\},$$

$$\langle 526 \rangle = \{526, -2625, 2626, -2629, 2630, -525\},$$

$$\langle 850 \rangle = \{850, -1948, 1949, -1256, 1257, -849\},$$

$$\langle 860 \rangle = \{860, -2565, 2566, -2266, 2267, -859\},$$

$$\cdots$$

$$\langle 2314 \rangle = \{2314, -2489, 2490, -2344, 2345, -2313\},$$

$$\langle 225 \rangle = \{225, 1198, -1197, 1057, -1056, -224\}$$

with

$$d(601) = 668, d(677) = 668, d(54) = 670, d(105) = 670, d(196) = 670, d(213) = 670,$$

$$d(384) = 670, d(487) = 670, d(518) = 670, d(526) = 670, d(850) = 670, d(860) = 670,$$

$$\cdots$$

$$d(2314) = 702, d(225) = 704.$$

The totally-ordered set $(B, \prec)$ is $\langle 601 \rangle, \langle 677 \rangle, \ldots, \langle 225 \rangle$. By computation we have

$$l(601) = 13$$

and

$$l(a) \leq 13$$

for all other $a \in N$. The first A-chain of length 13 is

$$601 \prec 518 \prec -124 \prec -877 \prec 271 \prec 789 \prec -743 \prec$$

$$-607 \prec 1906 \prec -1163 \prec 1195 \prec 156 \prec -1434 \prec -888.$$

Hence $[B] = 14, c(G_p) = 16$, and we conclude that $R(18, 18) \geq 11005$.

(3) Set $p = 8941$ and $g = 6$. Then $|B| = 2234$. The set B is divided into 373 equivalence classes:

$$\langle 5 \rangle = \{5, -1788, 1789, -2234, 2235, -4\},$$

$$\langle 261 \rangle = \{261, 2535, -2534, -4160, 4161, -260\},$$
$$\langle 627 \rangle = \{627, 713, -712, 2072, -2071, -626\},$$
$$\langle 1415 \rangle = \{1415, -1586, 1587, -1662, 1663, -1414\},$$
$$\langle 1508 \rangle = \{1508, 2674, -2673, 3365, -3364, -1507\},$$
$$\langle 1627 \rangle = \{1627, 2385, -2384, 2239, -2238, -1626\},$$
$$\langle 3258 \rangle = \{3258, 4229, -4228, -3263, 3264, -3257\},$$
$$\langle 3316 \rangle = \{3316, 4031, -4030, -3481, 3482, -3315\},$$
$$\langle 20 \rangle = \{20, -447, 448, -3293, 3294, -19\},$$
$$\langle 132 \rangle = \{132, -3319, 3320, -272, 273, -131\},$$
$$\langle 222 \rangle = \{222, -3665, 3666, 1417, -1416, -221\},$$
$$\langle 397 \rangle = \{397, 3153, -3152, 1875, -1874, -396\},$$
$$\cdots$$
$$\langle 2245 \rangle = \{2245, -3668, 3669, -2298, 2299, -2244\},$$
$$\langle 2393 \rangle = \{2393, -4095, 4096, 2853, -2852, -2392\}$$

with

$$d(5) = 1094, d(261) = 1094, d(627) = 1094, d(1415) = 1094, d(1508) = 1094,$$

$$d(1627) = 1094, d(3258) = 1094, d(3316) = 1094, d(20) = 1096, d(132) = 1096,$$
$$d(222) = 1096, d(397) = 1096,$$
$$\cdots$$
$$d(2245) = 1138, d(2393) = 1138.$$

The totally-ordered set $(B, \prec)$ is $\langle 5 \rangle, \langle 261 \rangle, \ldots, \langle 2393 \rangle$. By computation we have

$$l(5) = 14$$

and

$$l(a) \leq 14$$

for all other $a \in N$. The first A-chain of length 14 is

$$5 \prec 1789 \prec -2234 \prec 2535 \prec -3714 \prec -2372 \prec 320 \prec$$

$$-1516 \prec 1534 \prec 3505 \prec -571 \prec 2554 \prec -3836 \prec -689 \prec -4435.$$

Hence $[B] = 15, c(G_p) = 17$, and we conclude that $R(19, 19) \geq 17885$. □

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