New lower bounds of ten classical Ramsey numbers

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Abstract

Based on a study of basic properties of cyclic graphs of prime order, we give an algorithm for computing lower bounds of classical Ramsey numbers. Our algorithm reduces certain amount of computation of cyclic graphs of prime order, since only some of them - normalized cyclic graphs require computation in our method. Using the algorithm, we construct ten cyclic graphs of prime order to obtain new lower bounds of ten classical Ramsey numbers:

$R(3,31) \geq 198$, $R(4,17) \geq 182$, $R(5,16) \geq 278$, $R(5,20) \geq 380$, $R(5,25) \geq 458$, $R(7,15) \geq 444$, $R(7,16) \geq 462$, $R(8,13) \geq 422$, $R(8,15) \geq 618$, $R(10,16) \geq 1052$.

1. Introduction

The classical Ramsey number $R(q_1, q_2)$ is the least positive integer r such that every graph with r vertices contains either a clique of order q_1 or an independent set of order q_2. We refer to the book of Graham, Rothschild and Spencer [4] for Ramsey Theory and the classical Ramsey numbers. For concrete values of q_1, q_2, there are only a few nontrivial Ramsey numbers that are known. In 1992, McKay and Zhang [6] showed that $R(3,8) = 28$ by using a substantial amount of computation. In 1995, McKay and Radziszowski [5] showed that $R(4,5) = 25$ with the help of a long computer search. A number of lower bounds for Ramsey numbers are known; see the survey paper paper by Radziszowski [8].

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When q_1 and q_2 increase, finding reasonable lower bounds is a greater and greater challenge. As evidence, we note that it has taken many years to find the following lower bounds for $R(4, q_2)$. In 1988, Bannani [1] obtained the lower bounds $R(4,10) \geq 72$, $R(4,14) \geq 103$, $R(4,16) \geq 120$, $R(4,17) \geq 128$, $R(4,18) \geq 135$. In 1996, Calkin, Erdös,Tovey [2] obtained $R(4,12) \geq 98$, $R(4,15) \geq 128$. Then Piwakowski [7] improved the previous results and obtained $R(4,10) \geq 80$, $R(4,11) \geq 96$, $R(4,12) \geq 106$, $R(4,13) \geq 118$, $R(4,14) \geq 129$, $R(4,15) \geq 134$. In 1997, Su, Luo, Li [9] obtained $R(4,12) \geq 128$. This implies that $R(4,13) \geq 131$, $R(4,14) \geq 136$, $R(4,15) \geq 145$ by using the observation (see [8]):

$$\text{If} \quad R(k,p) \geq s \quad \text{and} \quad R(k,q) \geq t, \quad \text{then} \quad R(k,p+q-1) \geq s+t-1. \qquad (1.1)$$

Using cyclic graphs of prime order to obtain lower bounds for classical Ramsey numbers has been fruitful in the past. We refer to [2] for theoretical motivation and computational evidence. In this paper, we first study some properties of cyclic graphs of prime order and introduce a special class of them, normalized cyclic graphs of prime order, in Sections 2 and 3. We show that every cyclic graph of prime order is isomorphic to some normalized cyclic graph of prime order. We then propose an algorithm to find effective parameter sets for constructing cyclic graphs of prime order in Section 4. The new algorithm reduces certain amount of computation of cyclic graphs of prime order, since only some of them - normalized cyclic graphs require computation. For the two-color Ramsey problems, it is an improvement of the algorithm in [10]. Several examples to illustrate the algorithm are provided. Using the algorithm, we construct ten cyclic graphs of prime order to obtain new lower bounds of ten classical Ramsey numbers. We list the results in Theorem 1.1, which will be proved in Section 5:

Theorem 1.1. $R(3,31) \geq 198^{\dagger}$, $R(4,17) \geq 182^{\dagger}$, $R(5,16) \geq 278$, $R(5,20) \geq 380^{\dagger}$, $R(5,25) \geq 458$, $R(7,15) \geq 444$, $R(7,16) \geq 462$, $R(8,13) \geq 422$, $R(8,15) \geq 618^{\dagger}$, $R(10,16) \geq 1052$.

The four bounds marked with "$\dagger$" are better than the previously known results (see [8]) and the remaining bounds appear for the first time. Using Theorem 1.1 and (1.1), we obtain new lower bounds for another four classical Ramsey numbers:

Corollary 1.2. $R(3,32) \geq 200$, $R(4,18) \geq 185$, $R(5,21) \geq 384$, $R(8,16) \geq 625$.

2. Isomorphism of Cyclic Graphs of Prime Order

Given a prime $p = 2m+1$, denote $Z_p = \{-m, ..., -1, 0, 1, ..., m\} \equiv [-m, m]$. We write $s = t$ in Z_p iff $s \equiv t \bmod p$. For any nonempty set $S \subset Z_p^+ = [1, m]$, the graph $G_p(S)$ with vertex set $V = Z_p$ and edge set $E = \{\{x,y\} : x, y \in Z_p,\ |x-y| \in S\}$, (where $x - y$ is calculated modulo p), is called *the cyclic graph of order* p *with parameter set* S.

Lemma 2.1. *Let g be a primitive root modulo p. For any $j, b \in Z_p$, the linear transformation $f : x \mapsto g^j x + b$ is an isomorphism $f : G_p(S) \to G_p(S^*)$, where $S^* = \{|g^j x| : x \in S\}$.*

Proof: For any $x, y \in Z_p$, we have

$$|x - y| \in S \Leftrightarrow |f(x) - f(y)| = \left|g^j(x - y)\right| \in S^*.$$

Therefore f is an isomorphism between $G_p(S)$ and $G_p(S^*)$.

3. Normalized Cyclic Graphs of Prime Order

We consider only $t = |S| \geq 2$. Let g be a primitive root modulo p, so $Z_p^+ = [1, m] = \{|g^a| : a \in Z_p^+\}$, and there are $a_j \in Z_p^+$ for $j \in [0, t-1]$ such that

$$S = \{|g^{a_j}| : j \in [0, t-1]\}. \tag{3.1}$$

A cyclic graph of order p is called *normalized* if the following conditions are satisfied in (3.1):

$$(a) \quad 0 = a_0 < a_1 < \ldots < a_{t-1} < m, \tag{3.2}$$

$$(b) \quad a_1 = min\{a_j - a_{j-1} : j \in [1, t-1]\} < m/(t-1). \tag{3.3}$$

Now we have:

Theorem 3.1. *Any cyclic graph $G_p(S)$ with $|S| \geq 2$ is isomorphic to a normalized cyclic graph $G_p(S^*)$ with $|S^*| = |S|$.*

Proof: Without loss of generality, we may assume that in (3.1)

$$0 < a_0 < a_1 < \ldots < a_{t-1} \leq m.$$

We denote $a_{t+j} = a_j + m$, $j \in [0, t-1]$, and note $g^m = g^{(p-1)/2} = -1$ since g is a primitive root modulo p. Then

$$|g^{a_{t+j}}| = |g^{a_j + m}| = |g^{a_j}|, \quad j \in [0, t-1], \tag{3.4}$$

and

$$0 < a_0 < a_1 < \ldots < a_{t-1} \leq m < a_t < a_{t+1} < \ldots < a_{2t-1} \leq 2m. \tag{3.5}$$

Take $h \in [1, t]$ such that

$$min\{a_j - a_{j-1} : j \in [1, t]\} = a_h - a_{h-1} \equiv e. \tag{3.6}$$

Since $a_{t+j} - a_{t+j-1} = a_j - a_{j-1}$, $j \in [1, t-1]$,

$$e = min\{a_j - a_{j-1} : j \in [1, 2t-1]\}. \tag{3.7}$$

By (3.1) and (3.4), we have

$$\begin{aligned} S &= \{|g^{a_j}| : j \in [0, h-2]\} \cup \{|g^{a_j}| : j \in [h-1, t-1]\} \\ &= \{|g^{a_{j+h-1}}| : j \in [t-h+1, t-1]\} \cup \{|g^{a_{j+h-1}}| : j \in [0, t-h]\} \\ &= \{|g^{a_{j+h-1}}| : j \in [0, t-1]\}. \end{aligned}$$

Let $f : x \mapsto g^{-a_{h-1}}x$, and denote $a_{j+h-1} - a_{h-1} = b_j$. We have:

$$S^* = \{|g^{-a_{h-1}}x| : x \in S\} = \{|g^{a_{j+h-1}-a_{h-1}}| : j \in [0, t-1]\} = \{|g^{b_j}| : j \in [0, t-1]\}.$$

By Lemma 2.1, $G_p(S)$ is isomorphic to $G_p(S^*)$. Now we verify that $\{b_j : j \in [0, t-1]\}$ satisfies (3.2) and (3.3). From (3.5-7), we have:

$$b_{j+1} - b_j = a_{j+h} - a_{j+h-1} \geq e > 0,\ j \in [0, t-2], \quad 0 = b_0 < b_1 = e < b_2 < \dots < b_{t-1}.$$

We also have $b_{t-1} < m$, since $b_{t-1} = a_{t+h-2} - a_{h-1} = m - (a_{h-1} - a_{h-2}) < m$ if $2 \leq h \leq t$, and $b_{t-1} = a_{t-1} - a_0 < a_{t-1} \leq m$ if $h = 1$. Finally, we have $m > b_{t-1} = (b_{t-1} - b_{t-2}) + (b_{t-2} - b_{t-3}) + \dots + (b_1 - b_0) \geq e(t-1)$ which implies that $e = b_1 < m/(t-1)$, and this finishes the proof.

4. An Algorithm

We define the *clique number* and the *independence number* of a graph to be the largest order of any clique and the largest order of any independent set contained in the graph, respectively. We know that a cyclic graph is vertex-transitive. So the clique number of $G_p(S)$ is equal to the maximal order of any clique in $G_p(S)$ containing 0. Define

$$A = \{x : |x| \in S\}, \quad \bar{S} = \{x \in Z_p^+ : x \notin S\}, \quad \bar{A} = \{x : |x| \in \bar{S}\}. \tag{4.1}$$

Denote the induced sub-graph in $G_p(S)$ whose vertex set is A by $G_p[A]$. Denote the clique number of $G_p(S)$ and $G_p[A]$ by $[G_p(S)]$ and $[A]$, respectively. We have $[G_p(S)] = [A]+1$. The computation is much easier if we use $G_p[A]$ instead of $G_p(S)$. The independence number of $G_p(S)$ is equal to the clique number of $G_p(\bar{S})$. Hence

$$R([A]+2, [\bar{A}]+2) \geq p+1. \qquad (4.2).$$

Now we give an algorithm for finding lower bounds, based on (4.2). We first need to do some preparation for using the algorithm. For given $q_2 \geq q_1 \geq 3$, select a prime $p = 2m+1$ and find a primitive root g modulo p. For the selected m, select a positive integer t such that $2 \leq t \leq m$.

Algorithm 4.1.

(1) For the given p, t, do steps (2)-(6).

(2) Select a subset of $[0, m-1]$ with t elements that satisfies (3.2-3). Denote the set of such subsets by W. We list all the subsets in W according to the lexicographic order of their elements. Assume

$$B_1 = [0, t-1], \quad B_r = \{a_j \in [0, m-1] : j \in [0, t-1]\}.$$

Set $r = 1$.

(3) If $r > |W|$, stop. Otherwise construct $S = \{|g^{a_j}| : a_j \in B_r\}$ and $G_p(S)$.

(4) Construct $A = \{x : |x| \in S\}$ according to (4.1). Use a depth-first search (see [10]) to find $[A]$ for the induced sub-graph $G_p[A]$ of $G_p(S)$. If $[A] \geq q_1 - 1$, set $r = r + 1$ and go to (3).

(5) Construct $\bar{A}$ according to (4.1). Use a depth-first search to find $[\bar{A}]$ for the induced sub-graph $G_p[\bar{A}]$ of $G_p(\bar{S})$. If $[\bar{A}] \geq q_2 - 1$, set $r = r + 1$ and go to (3).

(6) Stop.

Note that if the algorithm stops at step (6), from steps (4) and (5), we have $[A] + 2 \leq q_1$ and $[\bar{A}] + 2 \leq q_2$; therefore $R(q_1, q_2) \geq p + 1$. Since finding a lower bound is a difficult problem for larger q_1, q_2, the algorithm mostly stops at step (3) for the chosen p and t. Then we need to alter p and t in step (1) and repeat the algorithm again.

In step (1), instead of a given value of t, we may run from 2 to m for all the choices of t. In our experience, choosing $m/12 \leq t \leq m/2$ has often worked. All ten values of t used in the proof of Theorem 1.1 are in this range. For a given t, all the subsets in W are determined by step (2). We estimate $|W|$ by the following theorem.

Theorem 4.2. *W satisfies the following estimates:*

(a) $|W| = \sum_{a_1=1}^{\left[\frac{m-1}{t-1}\right]} \binom{m+t-3-a_1(t-1)}{t-2}$;

(b) $|W| / \binom{m}{t} \leq \frac{t}{m}$.

Proof: (a) From (3.2), we have $1 \leq a_1 \leq \left[\frac{m-1}{t-1}\right]$. By (3.3), we have $a_1 - a_0 = a_1$ and $a_j - a_{j-1} = a_1 + c_j, j = 2, 3, ..., t-1$, where $c_j \geq 0$. Defining $c_t = m - 1 - a_{t-1} \geq 0$, we have $\sum_{j=2}^{t} c_j = m - 1 - a_1(t-1)$. Hence $|W|$ is the same as the number of all possibilities of distributing $m - 1 - a_1(t-1)$ ones into $t-1$ places on a line, which can be computed by

$$|W| = \sum_{a_1=1}^{\left[\frac{m-1}{t-1}\right]} \binom{(m-1-a_1(t-1)) + (t-1) - 1}{(t-1) - 1} = \sum_{a_1=1}^{\left[\frac{m-1}{t-1}\right]} \binom{m+t-3-a_1(t-1)}{t-2}.$$

(b) Using (a), we have $|W| / \binom{m}{t} = \sum_{a_1=1}^{\left[\frac{m-1}{t-1}\right]} \frac{\left(\prod_{k=0}^{t-3}(m+k-a_1(t-1))\right)t!}{(t-2)! \prod_{k=0}^{t-1}(m-k)}$

$$\leq \sum_{a_1=1}^{[\frac{m-1}{t-1}]} \frac{\left(\prod_{k=0}^{t-3}(m+k-(t-1))\right)t(t-1)}{m(m-1)\prod_{k=2}^{t-1}(m-k)} = \sum_{a_1=1}^{[\frac{m-1}{t-1}]} \frac{t(t-1)}{m(m-1)} \leq \frac{t}{m}, \tag{4.3}$$

where a product in the summation is defined as one if the subscript in bigger than the superscript.

Note that in our algorithm, we only need to check $|W|$ subsets in W instead of checking $\binom{m}{t}$ t-element subsets of $[1, m]$ for cyclic graphs of order p. By (4.3) we reduce out at least $\frac{m-t}{m}$ of amount of computation. In many cases of m, t, the reduction is even bigger since we use the upper bound of $|W|$ by (4.3). The amount reduced out is considerably large if t is relatively small with respect to m. For the cases which we are interested in mostly, i.e.,$m/12 \leq t \leq m/2$, the part reduced out is at least 50 to 92 percent. The reduction comes from non-normalized cyclic graphs, each of them is isomorphic to one of the normalized cyclic graphs by Theorem 3.1.

Now we illustrate Algorithm 4.1 with the following three examples.

Example 1. $R(3,5) \geq 14$ (see [3]).

Let $p = 13$, $g = 2$. Select $t = 2$. Consider all subsets in $[0,5] \equiv \{0,1,2,3,4,5\}$ satisfying (3.2-3). Note that $a_1 < m/(t-1) = 6$. We have $|W| = 5$ and $B_r = \{0, r\} \in W$, $r \in [1,5]$. Hence $S = \{1, 2^r\}$. There are five parameter sets: $\{1,2\}, \{1,4\}, \{1,5\}, \{1,3\}, \{1,6\}$. When $r = 3$, i.e., $S = \{1,5\}$, we find that the clique number of $G_p(S)$ is 2 and the independence number of $G_p(S)$ is 4. So $R(3,5) \geq 14$. Note that without using normalized cyclic graphs, we would need to search for an effective S from 15 subsets $\{1,2\}, \{1,3\}, ..., \{5,6\}$.

Example 2. $R(4,4) \geq 18$ (see [3]).

Let $p = 17$, $g = 3$. Select $t = 4$. Consider all subsets in $[0,7]$ satisfying (3.2-3). Note that $a_1 < m/(t-1) = 8/3$, i.e., $a_1 \leq 2$. We have $|W| = 18$. There are eighteen parameter sets. For simplicity, we use 0123 to denote $\{0,1,2,3\}$. The eighteen parameter sets are 0123, 0124, 0125, 0126, 0127, 0134, 0135, 0136, 0137, 0145, 0146, 0147, 0156, 0157, 0167, 0246, 0247, 0257. When $r = 16$, $B_r = \{0,2,4,6\}$, $S = \{3^0, |3^2|, |3^4|, |3^6|\} = \{1,2,4,8\}$, we find that the clique number of $G_p(S)$ is 3 and the independence number of $G_p(S)$ is 3. Note that without using normalized cyclic graphs, we would need to search for an effective S from $\binom{8}{4} = 70$ subsets $\{1,2,3,4\}, \{1,2,3,5\}, ..., \{5,6,7,8\}$.

Example 3. $R(4,10) \geq 80$ (see [7]).

Let $p = 79$, $g = 3$. Select $t = 13$. Consider all subsets in $[0,38]$ satisfying (3.2-3). When $B_r = \{0,1,5,7,9,11,14,18,22,26,27,31,35\} \in W$, $S =$

$\{1,3,6,7,10,12,14,20,23,25,28,29,39\}$. We find that the clique number of $G_p(S)$ is 3 and the independence number of $G_p(S)$ is 9. So $R(4,10) \geq 80$.

In this example, we used about 2 hours on a PIII 500 computer to obtain the effective parameter set S. To obtain the clique number and the independence number took less than one second.

From [8], the three lower bounds are the best results so far and the first one is the exact value of the Ramsey number $R(3,5) = 14$.

5. Proof of Theorem 1.1

We use the above algorithm, and search for effective parameter sets.

(1) Let $p_1 = 197, g_1 = 2$, and

$$S_1 = \{1,4,6,19,22,24,33,36,53,62,65,76,83,93\}.$$

Then $[G_{p_1}(S_1)] = 2$ and $[G_{p_1}(\overline{S_1})] = 30$, so $R(3,31) \geq 198$.

(2) Let $p_2 = 181, g_2 = 2$, and $S_2 =$

$$\{1,2,7,14,17,19,26,32,38,39,43,48,49,52,61,62,65,72,73,80,83,85,88,89\}.$$

Then $[G_{p_2}(S_2)] = 3$ and $[G_{p_2}(\overline{S_2})] = 16$, so $R(4,17) \geq 182$.

(3) Let $p_3 = 277, g_3 = 5$, and

$$S_3 = \{1,3,4,12,13,16,19,21,25,27,29,30,39,41,47,48,49,52,55,57,59,62, 63,64,66,69,70,74,76,79,81,83,84,85,89,90,100,102,108,113,116, 120,121,122,123,131\}.$$

Then $[G_{p_3}(S_3)] = 4$ and $[G_{p_3}(\overline{S_3})] = 15$, so $R(5,16) \geq 278$.

(4) Let $p_4 = 379, g_4 = 2$, and

$$S_4 = \{1,4,11,14,18,24,30,31,38,39,40,43,44,49,50,51,52,53,61,63,65,69,79, 81,84,85,87,94,96,103,105,107,108,111,115,121,128,133,135,140,145, 151,154,160,166,171,175,177,178,180,181,182,185,186\}.$$

Then $[G_{p_4}(S_4)] = 4$ and $[G_{p_4}(\overline{S_4})] = 19$, so $R(5,20) \geq 380$.

(5) Let $p_5 = 457, g_5 = 13$, and

$$S_5 = \{1,3,11,12,13,18,20,22,26,32,35,44,45,46,49,52,54,55,56,58,61,62,63, 70,76,82,85,86,89,92,94,97,99,103,104,105,109,111,113,119,120,122, 125,127,130,133,134,136,139,143,153,159,163,168,170,172,173,175, 177,184,192,198,200,202,206,207,216,217,220,223,225,226\}.$$

Then $[G_{p_5}(S_5)] = 4$ and $[G_{p_5}(\overline{S_5})] = 24$, so $R(5, 25) \geq 458$.

(6) Let $p_6 = 443, g_6 = 2$, and

$$S_6 = \{1, 4, 5, 6, 13, 14, 15, 18, 21, 22, 27, 31, 32, 33, 34, 37, 38, 40, 41, 43, 47, 48, 49, 50, 51, 52, 57, 59, 60, 63, 65, 67, 71, 72, 73, 75, 77, 78, 89, 90, 92, 106, 108, 112, 113, 115, 116, 119, 121, 123, 127, 128, 133, 136, 138, 140, 145, 148, 151, 152, 157, 159, 166, 168, 169, 170, 172, 173, 174, 175, 176, 179, 181, 182, 191, 194, 195, 199, 202, 204, 207, 209, 210, 218, 220\}.$$

Then $[G_{p_6}(S_6)] = 6$ and $[G_{p_6}(\overline{S_6})] = 14$, so $R(7, 15) \geq 444$.

(7) Let $p_7 = 461, g_7 = 2$, and

$$S_7 = \{1, 2, 3, 13, 14, 20, 21, 22, 23, 28, 29, 30, 32, 33, 35, 37, 38, 40, 41, 42, 44, 45, 46, 48, 50, 52, 55, 57, 60, 61, 63, 66, 68, 69, 71, 72, 75, 78, 82, 86, 90, 99, 102, 108, 113, 117, 118, 119, 122, 123, 124, 127, 128, 129, 134, 135, 136, 139, 145, 148, 152, 153, 155, 162, 163, 167, 169, 170, 173, 175, 177, 178, 179, 181, 182, 183, 186, 187, 188, 192, 194, 196, 199, 201, 204, 206, 211, 218, 222, 226, 228, 229\}.$$

Then $[G_{p_7}(S_7)] = 6$ and $[G_{p_7}(\overline{S_7})] = 15$, so $R(7, 16) \geq 462$.

(8) Let $p_8 = 421, g_8 = 2$, and

$$S_8 = \{1, 4, 7, 13, 21, 22, 24, 29, 34, 35, 36, 37, 40, 44, 46, 47, 48, 52, 53, 54, 56, 60, 62, 67, 71, 76, 79, 80, 81, 82, 84, 90, 91, 92, 93, 99, 100, 103, 108, 109, 111, 113, 114, 116, 118, 121, 126, 129, 135, 136, 140, 141, 142, 144, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 162, 165, 166, 168, 169, 171, 173, 176, 177, 180, 181, 183, 185, 186, 188, 190, 194, 196, 197, 198, 202, 203, 204, 206, 207, 210\}.$$

Then $[G_{p_8}(S_8)] = 7$ and $[G_{p_8}(\overline{S_8})] = 12$, so $R(8, 13) \geq 422$.

(9) Let $p_9 = 617, g_9 = 3$, and

$$S_9 = \{1, 3, 4, 8, 10, 11, 14, 23, 25, 26, 27, 28, 32, 33, 34, 35, 36, 37, 38, 46, 47, 49, 55, 62, 65, 75, 79, 81, 82, 85, 86, 87, 89, 90, 91, 94, 98, 99, 108, 110, 112, 113, 115, 116, 119, 120, 121, 125, 126, 128, 129, 132, 133, 134, 137, 138, 142, 143, 144, 148, 152, 155, 157, 159, 161, 163, 166, 169, 171, 174, 176, 177, 179, 181, 184, 186, 187, 191, 192, 193, 194, 195, 197, 202, 209, 211, 212, 214, 217, 219, 221, 225, 226, 228, 232, 233, 234, 236, 239, 241, 243, 248, 250, 251, 255, 257, 258, 262, 270, 271, 274, 275, 283, 286, 287, 288, 289, 290, 292, 298, 299, 300, 301, 302, 305, 306\}.$$

Then $[G_{p_9}(S_9)] = 7$ and $[G_{p_9}(\overline{S_9})] = 14$, so $R(8, 15) \geq 618$.

(10) Let $p_{10} = 1051, g_{10} = 7$, and

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$$S_{10} = \{1, 5, 7, 9, 11, 12, 15, 16, 18, 21, 24, 25, 29, 31, 37, 38, 41, 44, 46, 47, 53, 54, 55, 57, 58, 60, 61, 64, 67, 69, 71, 75, 76, 77, 78, 79, 80, 81, 84, 86, 90, 93, 96, 97, 99, 101, 102, 104, 105, 111, 112, 117, 118, 120, 123, 127, 128, 129, 132, 136, 139, 140, 141, 142, 146, 147, 153, 155, 156, 157, 163, 167, 168, 171, 172, 174, 179, 181, 182, 183, 184, 185, 190, 191, 193, 196, 198, 201, 202, 204, 206, 210, 211, 212, 216, 219, 223, 224, 226, 227, 230, 231, 236, 238, 242, 247, 251, 254, 261, 262, 265, 266, 267, 269, 270, 271, 273, 274, 275, 285, 288, 290, 292, 294, 295, 299, 300, 301, 302, 304, 305, 307, 309, 314, 316, 317, 318, 319, 320, 321, 322, 323, 324, 327, 334, 335, 339, 341, 343, 345, 346, 347, 351, 353, 355, 356, 357, 359, 367, 368, 371, 372, 375, 377, 378, 380, 385, 386, 387, 388, 389, 390, 391, 394, 395, 397, 398, 400, 401, 405, 407, 409, 411, 413, 416, 418, 419, 424, 428, 430, 432, 436, 444, 445, 446, 451, 453, 454, 459, 460, 465, 478, 483, 484, 485, 487, 491, 492, 493, 494, 495, 496, 497, 498, 500, 503, 504, 506, 507, 510, 517, 519, 520, 523, 525\}.$$

Then $[G_{p_{10}}(S_{10})] = 9$ and $[G_{p_{10}}(\overline{S_{10}})] = 15$, therefore $R(10, 16) \geq 1052$.

This completes the proof of Theorem 1.1.

We used about twelve hours on a PIII 500 computer to verify $R(3, 31) \geq 198$ and used about one hour for the remaining nine bounds.

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