Uniform coverings of 2-paths by 4-paths

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Abstract

We construct a uniform covering of 2-paths by 4-paths in K_n for all $n \geq 5$, i.e., we construct a set S of 4-paths in K_n having the property that each 2-path in K_n lies in exactly one 4-path in S for all $n \geq 5$.

1 Introduction

Let K_n be the complete graph on n vertices. A k-path is a path of length k and a k-cycle is a cycle of length k, where the length of a path [cycle] is the number of edges in the path [cycle]. Note that paths and cycles are undirected. A uniform covering of the 2-paths in K_n by k-paths [k-cycles] is a set S of k-paths [k-cycles] having the property that each 2-path in K_n lies in exactly one k-path [k-cycle] in S. Only the following cases of the problem of constructing a uniform covering of the 2-paths in K_n by k-paths or k-cycles have been solved [2, 8];

1. by 3-cycles,
2. by 3-paths,
3. by 4-cycles,
4. by n-cycles (Hamilton cycles) when n is even.

When n is odd, a uniform covering of the 2-paths in K_n by Hamilton cycles has only been constructed for a few cases: $n = 2^e + 1$, where e is a natural number [7], $n = p + 2$, where p is an odd prime and 2 is a generator of the multiplicative group of $GF(p)$ [1], and some other infinite cases [3, 5]. But in general the problem when n is odd is still open.

In this paper, we solve the problem in the case of 4-paths, that is, we prove,

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Theorem 1.1 *Let $n \geq 5$. Then there exists a set S of 4-paths in K_n having the property that each 2-path in K_n lies in exactly one path in S.*

Finally, we mention the problem in the case of $(n-1)$-paths (Hamilton paths).

Lemma 1.2 *Let $n \geq 3$. If there is a uniform covering of 2-paths by Hamilton cycles in K_{n+1}, there is a uniform covering of 2-paths by Hamilton paths in K_n.*

Proof. Let $V_{n+1} = \{v_0, v_1, \ldots, v_n\}$ be the vertex set of K_{n+1} and let $\mathcal{D}$ be a uniform covering of 2-paths by Hamilton cycles in K_{n+1}. Let K_n be the complete graph with the vertex set $V_n = V_{n+1} \setminus \{v_0\}$. For each Hamilton cycle $H \in \mathcal{D}$, we obtain a Hamilton path in K_n by removing the point v_0 and the two edges incident to v_0 from H. We denote it by H'. Put $\mathcal{D}' = \{H' \mid H \in \mathcal{D}\}$, then $\mathcal{D}'$ is a uniform covering of 2-paths by Hamilton paths in K_n. $\square$

The proof of Theorem 1.3 is immediate from Lemma 1.2 and the existence of a uniform covering of 2-paths by Hamilton cycles in K_n when n is even ≥ 4.

Theorem 1.3 [8] *Let n be an odd integer ≥ 3. Then there exists a set S of Hamilton paths in K_n having the property that each 2-path in K_n lies in exactly one path in S.*

When n is even, the problem of Theorem 1.3 is still open, but Verrall constructed a double covering of 2-paths by Hamilton paths:

Theorem 1.4 [8] *Let n be an even integer ≥ 4. Then there exists a set S of Hamilton paths in K_n having the property that each 2-path in K_n lies in exactly two paths in S.*

2 Proof of Theorem 1.1

There are $n(n-1)(n-2)/2$ 2-paths in K_n and three 2-paths in a 4-path, so $n(n-1)(n-2)/6$ 4-paths are needed to cover the 2-paths in K_n. This is an integer for $n \geq 3$.

When $n = 3$ or 4, K_n has 2-paths but doesn't have 4-paths, so there is no uniform covering of 2-paths by 4-paths in K_n. We consider the case $n \geq 5$.

Lemma 2.1 *There is a uniform covering of 2-paths by 4-paths in K_n when $n = 5$.*

Proof. Let $\{0, 1, 2, 3, 4\}$ be the vertex set of K_5. Let S be a set of 4-paths:

$$\begin{aligned} S = \{&[2,4,0,1,3], \quad [3,0,1,2,4], \quad [4,1,2,3,0], \quad [0,2,3,4,1], \\ &[1,3,4,0,2], \quad [1,2,0,3,4], \quad [2,3,1,4,0], \quad [3,4,2,0,1], \\ &[4,0,3,1,2], \quad [0,1,4,2,3]\}, \end{aligned}$$

then S is a uniform covering of 2-paths by 4-paths in K_5. $\square$

Now we prove Theorem 1.1. We use induction on n. When $n = 5$ there is a uniform covering of 2-paths by 4-paths in K_n from Lemma 2.1. Let $n \geq 6$ and assume that there is a uniform covering of 2-paths by 4-paths in K_{n-1}.

Put $m = n - 1$. Let K_n be the complete graph with vertex set $V = \{x\} \cup V'$, where $|V'| = m$. Let K_m be the complete graph with vertex set V'. By the induction hypothesis, there is a uniform covering S' of the 2-paths in K_m by 4-paths. Let T and T' be the sets of all 2-paths in K_n and K_m, respectively.

Put $T_1 = \{(a, b, x) \mid a, b \in V',\ a \neq b\}$, $T_2 = \{(a, x, b) \mid a, b \in V',\ a \neq b\}$, and $T'' = T_1 \cup T_2$, where $(a, b, x), (a, x, b)$ are 2-paths. Then we have $T = T' \cup T''$. We already covered the 2-paths in T' by S', so we will construct a set S'' of 4-paths in K_n that will cover the 2-paths in T''.

We will construct 4-paths of type (a, b, x, c, d) to cover T'', where $a, b, c, d (\in V')$ are all different. Note that $|T_1| = m(m-1)$ and $|T_2| = m(m-1)/2$. We will construct S'' by considering the two cases of m odd and m even separately.

(Case 1) m is odd.

There is a Hamilton cycle decomposition $\mathcal{H}$ in K_m, that is, there is a set $\mathcal{H}$ of Hamilton cycles in K_m such that each edge of K_m lies in exactly one cycle in $\mathcal{H}$. $|\mathcal{H}| = (m-1)/2$. For each Hamilton cycle $H = (v_1, v_2, \ldots, v_m)$ in $\mathcal{H}$, define a set $S(H)$ of 4-paths:

$$\begin{aligned} S(H) = \quad & \{[v_1, v_2, x, v_3, v_4], \qquad [v_2, v_3, x, v_4, v_5], \\ & \cdots \\ & [v_{m-1}, v_m, x, v_1, v_2], \quad [v_m, v_1, x, v_2, v_3]\}. \end{aligned}$$

Define $S'' = \bigcup_{H \in \mathcal{H}} S(H)$. We will show that S'' covers each 2-path in T'' exactly once.

(i) Let (a, b, x) be any 2-path in T_1. There is a Hamilton cycle $H = (v_1, v_2, \ldots, v_m) \in \mathcal{H}$ which contains the edge $\{a, b\}$. So we can write $a = v_i$, $b = v_{i+1}$ or $a = v_{i+1}$, $b = v_i$, for some i, $1 \leq i \leq m$, where subscripts are calculated modulo m. In either case, the 2-path (a, b, x) is in some 4-path in $S(H)$.
(ii) Let (a, x, b) be any 2-path in T_2. There is a Hamilton cycle $H = (v_1, v_2, \ldots, v_m) \in \mathcal{H}$ which contains the edge $\{a, b\}$. So we can write $a = v_i$, $b = v_{i+1}$ or $a = v_{i+1}$, $b = v_i$, for some i, $1 \leq i \leq m$. In either case, the 2-path (a, x, b) is in a 4-path $[v_{i-1}, v_i, x, v_{i+1}, v_{i+2}]$ in $S(H)$.

Since the numbers of 2-paths in T'' and in S'' are equal, S'' covers each 2-path in T'' exactly once.

(Case 2) m is even.

Label the vertices in V' as $\infty, 0, 1, \ldots, m-2$. Put $r = (m-2)/2$. Let σ be the following permutation of the vertices of K_{m+1}: $\sigma = (\infty)(x)(0\ 1\ 2\ \cdots\ m{-}2)$, and put $\Sigma = \langle\sigma\rangle = \{\sigma^j \mid 0 \leq j \leq m-2\}$. Define the set S^0 of 4-paths:

$$\begin{aligned} S^0 = \quad & \{[r+1, \infty, x, 0, 1], \qquad [0, 1, x, m-2, 2], \\ & [m-2, 2, x, m-3, 3], \quad [m-3, 3, x, m-4, 4], \\ & \cdots \\ & [r+3, r-1, x, r+2, r], \quad [r+2, r, x, r+1, \infty]\}. \end{aligned}$$

Note that the set of edges $\{\{u_2, u_3\} \mid [u_1, u_2, x, u_3, u_4] \in S^0\}$ is F_0 and the set of arcs $\{(u_1, u_2) \mid [u_1, u_2, x,\ u_3, u_4] \in S^0\}$ which equals the set $\{(u_4, u_3) \mid [u_1, u_2, x, u_3, u_4] \in S^0\}$ is F^*_{r+1}, where

$$\begin{aligned} F_0 &= \{\{\infty, 0\}\} \cup \{\{u, v\} \mid u + v \equiv 0 \pmod{m-1},\ u, v \in V',\ u, v \neq \infty,\ u \neq v\} \\ F^*_{r+1} &= \{(\infty, r+1), (r+1, \infty)\} \cup \{(u, v) \mid u + v \equiv 1 \pmod{m-1}, \\ & \qquad\qquad u, v \in V',\ u, v \neq \infty,\ u \neq v\}. \end{aligned}$$

Put $S'' = \Sigma S^0 = \{P^{\sigma^j} \mid P \in S^0,\ 0 \leq j \leq m-2\}$. We will show that S'' is a set of 4-paths in K_n that covers each 2-path in T'' exactly once.

(i) Let (a, b, x) be any 2-path in T_1. Then there is an arc $(u, v) \in F^*_{r+1}$ such that $(a, b) = (u, v)^{\sigma^j}$ for some j. Since $\{(u_1, u_2) \mid [u_1, u_2, x, u_3, u_4] \in S^0\} = F^*_{r+1}$, $[u, v, x, u_3, u_4] \in S^0$ for some $u_3, u_4 \in V'$. Therefore $[u, v, x, u_3, u_4]^{\sigma^{-j}} = [a, b, x, u_3^{\sigma^{-j}}, u_4^{\sigma^{-j}}] \in S''$. Thus S'' covers the 2-path (a, b, x).

(ii) Let (a, x, b) be any 2-path in T_2. There is an edge $\{u, v\} \in F_0$ such that $\{a, b\} = \{u, v\}^{\sigma^j}$ for some j. Since $\{\{u_2, u_3\} \mid [u_1, u_2, x, u_3, u_4] \in S^0\} = F_0$, $[u_1, u, x, v, u_4] \in S^0$ for some $u_1, u_4 \in V'$. Therefore $[u_1, u, x, v, u_4]^{\sigma^{-j}} = [u_1^{\sigma^{-j}}, a, x, b, u_4^{\sigma^{-j}}] \in S''$. Thus S'' covers the 2-path (a, x, b).

Hence S'' covers each 2-path in T'' exactly once.

Put $S = S' \cup S''$, then S is a set of 4-paths with the property that each 2-path in T lies in exactly one path in S. This completes the proof of Theorem 1.1. □.

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References

[1] K. Heinrich, M. Kobayashi and G. Nakamura, Dudeney's Round Table Problem, *Annals of Discrete Math.* **92** (1991), 107–125.

[2] K. Heinrich, D. Langdeau and H. Verrall, Covering 2-paths uniformly, *J. Combin. Des.* **8** (2000), 100–121.

[3] M. Kobayashi, J. Akiyama and G. Nakamura, On Dudeney's round table problem for $p+2$, *Ars Combinatoria*, to appear.

[4] M. Kobayashi, Kiyasu-Z. and G. Nakamura, A solution of Dudeney's round table problem for an even number of people, *J. Combinatorial Theory* (A) **62** (1993), 26–42.

[5] M. Kobayashi, N. Mutoh, Kiyasu-Z. and G. Nakamura, New Series of Dudeney Sets for $p+2$ Vertices, *Ars Combinatoria*, to appear.

[6] J. McGee and C.A. Rodger, Path coverings with paths, *J. Graph Theory*, to appear.

[7] G. Nakamura, Kiyasu-Z. and N. Ikeno, Solution of the round table problem for the case of p^k+1 persons, *Commentarii Mathematici Universitatis Sancti Pauli* **29** (1980), 7–20.

[8] H. Verrall, Pairwise Compatible Hamilton Decompositions of K_n, *J. Combinatorial Theory* (A) **79** (1997), 209–222.

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