Bhaskar Rao designs and the alternating group A_4

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Abstract

In this paper we introduce a new construction for generalized Bhaskar Rao designs. Using this construction, we show that a generalized Bhaskar Rao design, $\mathrm{GBRD}(v, 3, \lambda; A_4)$ exists if and only if $\lambda \equiv 0 \pmod{12}$.

1 Introduction

In this paper we give a new construction for generalized Bhaskar Rao designs and then use this construction to establish a set of necessary and sufficient conditions for the existence of a generalized Bhaskar Rao design, $\mathrm{GBRD}(v, 3, \lambda; A_4)$. In particular, we show that a $\mathrm{GBRD}(v, 3, \lambda; A_4)$ exists if and only if $\lambda \equiv 0 \pmod{12}$.

The *alternating group*, A_4, is the group of even permutations on four letters and can be generated by means of the defining relations:

$$a^3 = 1, b^2 = c^2 = d^2 = 1, bc = d, ba = ad, ca = ab, da = ac.$$

2 Bhaskar Rao designs

Let $\mathbb{G}$ be a finite group of order g, written multiplicatively. We write $Z(\mathbb{G})$ for the *group ring* of the group $\mathbb{G}$ over the integers Z. Every element of $Z(\mathbb{G})$ is a formal sum $\sum_{x\in\mathbb{G}} a_x x$, where a_x is an integer and x is an element of the group $\mathbb{G}$. The element $\sum_{x\in\mathbb{G}} a_x x$, where all $a_x = 0$, is called the *zero element* of $Z(\mathbb{G})$ and is denoted by 0. An element $\sum_{x\in\mathbb{G}} a_x x$ of $Z(\mathbb{G})$ which has, for some $x \in \mathbb{G}$, $a_x = 1$ and $a_y = 0$ for

$x \neq y$, is simply written as x and is said to be an element of $Z(\mathbb{G})$ *lying in* $\mathbb{G}$ or a *group element of* $Z(\mathbb{G})$. As we will be concerned exclusively with group rings over the integers Z, we will often, when the meaning is clear, refer to the group element x lying in group ring $Z(\mathbb{G})$ informally as an element of the group $\mathbb{G}$.

We define the element $\left(\sum_{x\in\mathbb{G}} a_x x\right)^{-1}$ of $Z(\mathbb{G})$ by

$$\left(\sum_{x\in\mathbb{G}} a_x x\right)^{-1} = \sum_{x\in\mathbb{G}} a_x x^{-1}.$$

It follows that $0^{-1} = 0$, the zero element of $Z(\mathbb{G})$, and, for any group element x, the group element x^{-1} is $1x^{-1}$, a group element of $Z(\mathbb{G})$. For further information on group rings the reader is referred to, for example, Hall [6, pp. 255–261] or Aschbacher [1, pp. 35–42].

For a matrix A with entries in $Z(\mathbb{G})$, we define the *size* of a column of A to be the number of non-zero entries occurring in that column. If all columns of A have the same size, k (say), we call k the size of A.

Definition 1. Let $\mathbb{G}$ be finite group of order g, written multiplicatively. We define a generalized Bhaskar Rao design GBRD$(v, b, r, k, \lambda; \mathbb{G})$ to be a $v \times b$ matrix with entries from $Z(\mathbb{G})$, all of which are either 0 or group elements such that

1. each row contains exactly r group element entries and exactly $b-r$ zero element entries;

2. each column contains exactly k group element entries and exactly $v-k$ zero element entries;

3. for any pair of distinct rows $(x_1, x_2, \ldots, x_b)$ and $(y_1, y_2, \ldots, y_b)$ the list

$$x_1 y_1^{g-1}, x_2 y_2^{g-1}, \ldots, x_b y_b^{g-1}$$

contains each group element of $Z(\mathbb{G})$ (that is, element of $\mathbb{G}$) exactly λ/g times.

We note that $\lambda \equiv 0 \pmod g$, $k \leq v$ and $\lambda/g \leq r \leq b$.

Example 2. A GBRD$(5, 10, 6, 3, 3; Z_3)$ is given below:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & a & a^2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & a^2 & a & 0 & a & a^2 & 0 & 1 \\ 0 & 1 & 0 & a^2 & 0 & a & 1 & 0 & a & a \\ 0 & 0 & 1 & 0 & a^2 & a & 0 & a^2 & 1 & a^2 \end{bmatrix}$$

The generalized Bhaskar Rao design, A, appearing in Example 2 comes from Seberry [11, p. 379]. The column size of A is 3 and each entry of A belongs to $\{1, a, a^2\} \cup \{0\}$. For example, for the rows 2 and 3 of A, the list: 1, 0, 0, 0, 0, $a^{-1} = a^2$, $(a^2)^{-1} = a$, 0, 0, contains each of the group elements 1, a and a^2 exactly once.

Example 3. A GBRD$(3, 12, 12, 3, 12; A_4)$ is given below:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a & a^2 & b & ab & a^2b & c & ac & a^2c & d & ad & a^2d \\ 1 & a^2 & a & c & a^2c & ad & d & a^2d & ab & b & a^2b & ac \end{bmatrix}$$

If $v > k$, replacing the group element entries in a GBRD$(v, b, r, k, \lambda; \mathbb{G})$ produces a $(0, 1)$-matrix which is an incidence matrix for a balanced incomplete block design, BIBD(v, b, r, k, λ). It is well-known, see, for example, Street and Street [12], that the five numbers v, b, r, k and λ for a balanced incomplete block design, are not independent: they satisfy the relations $bk = vr$ and $\lambda(v - 1) = r(k - 1)$. For a generalized Bhaskar Rao design in which $v = k$, there are no zero element entries and $b = r = \lambda$, so the relations: $bk = vr$ and $\lambda(v-1) = r(k-1)$ still hold. Thus it is usual to denote the GBRD$(v, b, r, k, \lambda; \mathbb{G})$ by the shorter notation GBRD$(v, k, \lambda; \mathbb{G})$.

Some generalized Bhaskar Rao designs can be described by a set of *GBRD initial blocks*. See, for example, Seberry [11]. In Example 4, we explain this description by means of an example.

Example 4. A set of initial blocks (mod $3, A_4$) for a GBRD$(4, 3, 12; A_4)$ is given below:

$$\begin{array}{ccc} (\infty_1, 1_{a^2d}, 2_b), & (\infty_1, 1_{ab}, 2_c), & (\infty_1, 1_{a^2b}, 2_{a^2c}), \\ (\infty_1, 1_a, 2_{ac}), & (\infty_1, 1_1, 2_{ad}), & (\infty_1, 1_d, 2_{a^2}), \\ (0_1, 1_a, 2_{a^2}), & (0_1, 1_{ac}, 2_{ab}). & \end{array}$$

A GBRD$(4, 3, 12; A_4)$ has 4 rows and 24 columns. We think of the rows of this 4×24 matrix as labelled ∞, 0, 1 and 2. The GBRD$(4, 3, 12; A_4)$ is built up from eight 4×3 submatrices $B_1, \ldots, B_8$ which are developed from the eight initial blocks.

The collection of blocks developed (mod $3, A_4$) from the initial block $(\infty_1, 1_{a^2d}, 2_b)$ is

$$(\infty_1, 1_{a^2d}, 2_b), (\infty_1, 2_{a^2d}, 0_b), (\infty_1, 0_{a^2d}, 1_b).$$

That is, we develop an initial block, $(a_\alpha, b_\beta, c_\gamma)$, by developing a, b and c (mod 3) and leaving the subscripts α, β and γ unaltered.

From the three blocks:

$$(\infty_1, 1_{a^2d}, 2_b), (\infty_1, 2_{a^2d}, 0_b), (\infty_1, 0_{a^2d}, 1_b),$$

we construct B_1. The entries of the first column of B_1, corresponding to the first block $(\infty_1, 1_{a^2d}, 2_b)$ are $(1, 0, a^2d, b)$. That is, the group element 1 is placed in the row labelled ∞, a^2d in the row labelled 1, b is placed in row labelled 2, and the entry in row labelled 0 is 0, the zero element in $Z(A_4)$.

Similarly, the entries of the second column of B_1, corresponding to the second block $(\infty_1, 2_{a^2d}, 0_b)$ are $(1, b, 0, a^2d)$. Finally, the entries of the last column, corresponding to the third block $(\infty_1, 0_{a^2d}, 1_b)$ are $(1, a^2d, b, 0)$.

That is,

$$B_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & b & a^2d \\ a^2d & 0 & b \\ b & a^2d & 0 \end{bmatrix}.$$

Similarly, we construct the other seven matrices $B_2, \dots, B_8$ each constructed from the blocks arising from developing the remaining seven initial blocks. The 4×24 matrix

$$B = [B_1, \dots, B_8]$$

is a GBRD$(4, 3, 12; A_4)$. That is, we develop a collection of 24 blocks from the set of eight initial blocks. The GBRD$(4, 3, 12; A_4)$ so constructed is given explicitly below:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & a^2d & 0 & c & ab & 0 & a^2c & a^2b & 0 & ac & a & 0 & ad & 1 & 0 & a^2 & d & 1 & a^2 & a & 1 & ab & ac \\ a^2d & 0 & b & ab & 0 & c & a^2b & 0 & a^2c & a & 0 & ac & 1 & 0 & ad & d & 0 & a^2 & a & 1 & a^2 & ac & 1 & ab \\ b & a^2d & 0 & c & ab & 0 & a^2c & a^2b & 0 & ac & a & 0 & ad & 1 & 0 & a^2 & d & 0 & a^2 & a & 1 & ab & ac & 1 \end{bmatrix}$$

Example 5. A set of initial blocks (mod 5, A_4) for a GBRD$(6, 3, 12; A_4)$ is given below:

$$\begin{array}{cccc} (\infty_1, 0_1, 3_{a^2}), & (0_1, 2_{a^2b}, 3_{a^2d}), & (\infty_1, 0_d, 3_{a^2c}), & (0_1, 2_c, 3_{ac}), \\ (\infty_1, 0_b, 3_c), & (0_1, 2_{a^2c}, 3_1), & (\infty_1, 0_{ac}, 4_{a^2d}), & (0_1, 1_1, 4_a), \\ (\infty_1, 0_a, 4_{ab}), & (0_1, 1_{ac}, 4_{ab}), & (\infty_1, 0_{ad}, 4_{a^2b}), & (0_1, 1_d, 4_{a^2b}). \end{array}$$

3 Construction Theorems

Theorem 6. *Let $\mathbb{G}$ be a finite group (of order g), $\mathbb{N}$ a normal subgroup (of order n) of $\mathbb{G}$, and v, λ and μ positive integers. We write u for g/n, the order of $\mathbb{G}/\mathbb{N}$ and $0_{\mathbb{G}/\mathbb{N}}$ for the zero element in $Z(\mathbb{G}/\mathbb{N})$. Suppose we are given a $v \times b$ matrix, A, with entries taken from $\mathbb{G}/\mathbb{N} \cup \{0_{\mathbb{G}/\mathbb{N}}\}$such that*

- *for any pair of distinct rows $(x_1, x_2, \dots, x_b)$ and $(y_1, y_2, \dots, y_b)$, the list*

$$x_1 y_1^{u-1}, x_2 y_2^{u-1}, \dots, x_b y_b^{u-1}$$

  contains each element of $\mathbb{G}/\mathbb{N}$ exactly λ/u times.

- *for each column size k of A, a GBRD$(k, c, s, j, \mu; \mathbb{N})$, $C(k)$.*

Then we can construct the matrix, X which is a GBRD$(v, j, \lambda\mu; \mathbb{G})$.

Proof. Let t be the index of $\mathbb{N}$ in $\mathbb{G}$. Fix a set $S = \{g_1 = e, \dots, g_t\}$ of coset representatives of $\mathbb{N}$ in $\mathbb{G}$. We observe that the group element entries of the matrix A are cosets of $\mathbb{N}$ in $\mathbb{G}$ and the group element entries of each matrix $C(k)$ are elements of the normal subgroup $\mathbb{N}$. For each k, we denote the k rows of $C(k)$ by $\mathbf{c}(k)_l$, $l = 1, \dots, k$.

We now form the matrix X from the matrices A and $C(k)$. Select a column of A of size k (say). Replace each entry of this column by row vectors of length c in the following manner: replace the the first non-zero entry, say $g_l\mathbb{N}$ by the row vector $g_l\mathbf{c}(k)_1$, the second non-zero entry, say $g_m\mathbb{N}$ by the row vector $g_m\mathbf{c}(k)_2$, and so on. Finally, we replace the remaining (that is, zero entries) of the selected column of A by the row vector $(0, \dots, 0)$ consisting of c zero entries.

Now select another column of A and repeat the replacement process.

When all columns of A have been replaced by row vectors we have constructed a matrix X which is a GBRD$(v, j, \lambda\mu; \mathbb{G})$. $\square$

The following theorems which can be viewed as immediate consequences of Theorem 6 will be used extensively in the remaining sections of the paper. Theorem 7, based upon pairwise balanced designs (defined below), was first proved in de Launey and Seberry [5, 4], firstly for generalized Bhaskar Rao designs over the group Z_2 and then for generalized Bhaskar Rao designs over any finite group.

For v and λ positive integers and K a set of positive integers. We define a *pairwise balanced design*, denoted by PBD$(v; K; \lambda)$, to be an arrangement of the v elements of a set X into a collection (not necessarily distinct) subsets (called *blocks*) of X, for which:

1. each pair of distinct elements of X appear together in exactly λ blocks.
2. if a block contains exactly k elements of X then k belongs to K.

A pairwise balanced design PBD$(v; \{k\}; \lambda)$, where $K = \{k\}$ consists of exactly one integer, is a BIBD(v, k, λ). It is well-known, see, for example, Street and Wallis [13], that a PBD$(v-1; \{k, k-1\}; \lambda)$ can be obtained from a BIBD(v, b, r, k, λ).

Theorem 8 was proved in Palmer [9].

Theorem 7. *Given a pairwise balanced design PBD$(v; K; \lambda)$, and for each $k \in K$, a GBRD$(k, j, \mu; \mathbb{G})$, we can construct a GBRD$(v, j, \lambda\mu; \mathbb{G})$.*

Proof. In Theorem 6, take $\mathbb{G} = \mathbb{N}$ and A to be an incidence matrix of a PBD$(v; K; \lambda)$. $\square$

Theorem 8. *Suppose that $\mathbb{N}$ is a normal subgroup of a finite group $\mathbb{G}$. Then, given a GBRD$(v, k, \lambda; \mathbb{G}/\mathbb{N})$ and a GBRD$(k, j, \mu; \mathbb{N})$ we can construct a GBRD$(v, j, \lambda\mu; \mathbb{G})$.*

Proof. In Theorem 6, take A to be a GBRD$(v, k, \lambda; \mathbb{G}/\mathbb{N})$. $\square$

In the next section, we apply Theorem 6 to construct a GBRD$(14, 3, 12; A_4)$.

4 GBRD$(14, 3, 12; A_4)$

The subgroup, $\mathbb{N} = \langle b, c\rangle$ is normal in A_4 and is isomorphic to the group $Z_2 \times Z_2$. The factor group, $A_4/\mathbb{N}$ is isomorphic to Z_3. The matrix, Y which is a GBRD$(15, 7, 3; A_4/\mathbb{N})$, found as a result of a Magma [3] search. We exhibit the matrix Y below:

$$\begin{bmatrix}
1\mathbb{N} & 0 & 0 & 0 & 0 & 1\mathbb{N} & 0 & 1\mathbb{N} & 0 & 0 & 1\mathbb{N} & 1\mathbb{N} & 0 & 1\mathbb{N} & 1\mathbb{N} \\
1\mathbb{N} & 1\mathbb{N} & 0 & 0 & 0 & 0 & 1\mathbb{N} & 0 & 1\mathbb{N} & 0 & 0 & a\mathbb{N} & 1\mathbb{N} & 0 & a^2\mathbb{N} \\
1\mathbb{N} & a\mathbb{N} & 1\mathbb{N} & 0 & 0 & 0 & 0 & a\mathbb{N} & 0 & 1\mathbb{N} & 0 & 0 & a^2\mathbb{N} & a^2\mathbb{N} & 0 \\
0 & 1\mathbb{N} & a\mathbb{N} & 1\mathbb{N} & 0 & 0 & 0 & 0 & a\mathbb{N} & 0 & 1\mathbb{N} & 0 & 0 & a^2\mathbb{N} & a\mathbb{N} \\
1\mathbb{N} & 0 & a^2\mathbb{N} & a^2\mathbb{N} & 1\mathbb{N} & 0 & 0 & 0 & 0 & a\mathbb{N} & 0 & a^2\mathbb{N} & 0 & 0 & a\mathbb{N} \\
1\mathbb{N} & a^2\mathbb{N} & 0 & 1\mathbb{N} & a^2\mathbb{N} & a^2\mathbb{N} & 0 & 0 & 0 & 0 & a\mathbb{N} & 0 & a\mathbb{N} & 0 & 0 \\
0 & 1\mathbb{N} & 1\mathbb{N} & 0 & a^2\mathbb{N} & a\mathbb{N} & a^2\mathbb{N} & 0 & 0 & 0 & 0 & a^2\mathbb{N} & 0 & 1\mathbb{N} & 0 \\
0 & 0 & 1\mathbb{N} & a\mathbb{N} & 0 & a^2\mathbb{N} & a\mathbb{N} & 1\mathbb{N} & 0 & 0 & 0 & 0 & 1\mathbb{N} & 0 & a\mathbb{N} \\
1\mathbb{N} & 0 & 0 & a\mathbb{N} & a\mathbb{N} & 0 & a^2\mathbb{N} & a^2\mathbb{N} & a\mathbb{N} & 0 & 0 & 0 & 0 & a\mathbb{N} & 0 \\
0 & 1\mathbb{N} & 0 & 0 & a\mathbb{N} & a^2\mathbb{N} & 0 & a\mathbb{N} & a^2\mathbb{N} & a\mathbb{N} & 0 & 0 & 0 & 0 & 1\mathbb{N} \\
1\mathbb{N} & 0 & a\mathbb{N} & 0 & 0 & a\mathbb{N} & a\mathbb{N} & 0 & a^2\mathbb{N} & a^2\mathbb{N} & a^2\mathbb{N} & 0 & 0 & 0 & 0 \\
0 & 1\mathbb{N} & 0 & a^2\mathbb{N} & 0 & 0 & a\mathbb{N} & a^2\mathbb{N} & 0 & 1\mathbb{N} & a\mathbb{N} & 1\mathbb{N} & 0 & 0 & 0 \\
0 & 0 & 1\mathbb{N} & 0 & 1\mathbb{N} & 0 & 0 & a^2\mathbb{N} & a^2\mathbb{N} & 0 & 1\mathbb{N} & a\mathbb{N} & a\mathbb{N} & 0 & 0 \\
0 & 0 & 0 & 1\mathbb{N} & 0 & 1\mathbb{N} & 0 & 0 & a^2\mathbb{N} & 1\mathbb{N} & 0 & a^2\mathbb{N} & 1\mathbb{N} & a\mathbb{N} & 0 \\
0 & 0 & 0 & 0 & 1\mathbb{N} & 0 & a^2\mathbb{N} & 0 & 0 & a^2\mathbb{N} & a\mathbb{N} & 0 & 1\mathbb{N} & a^2\mathbb{N} & 1\mathbb{N}
\end{bmatrix}$$

Now we delete the last row of Y to form A a 14×15 matrix. We observe that:

- the size of each column of A lies in $\{6, 7\}$;
- for any pair of distinct rows of A, $(x_1, x_2, \ldots, x_{15})$ and $(y_1, y_2, \ldots, y_{15})$, the list

$$x_1 y_1^{-1}, x_2 y_2^{-1}, \ldots, x_{15} y_{15}^{-1}$$

  contains each element of the factor group $A_4/\mathbb{N}$ exactly once.

The matrix $C(7)$ which is a GBRD$(7, 3, 4; \mathbb{N})$, can be constructed by replacing the 1s in an incidence matrix for a BIBD$(7, 3, 1)$ by the rows of D which is a GBRD$(3, 3, 4; \mathbb{N})$ where

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & b & bc & c \\ 1 & c & b & bc \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix}.$$

The 0s are replaced by the zero row vector $\mathbf{0} = (0, 0, 0, 0)$. The matrix

$$\begin{bmatrix}
\mathbf{c}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_1 & \mathbf{0} & \mathbf{c}_1 \\
\mathbf{c}_1 & \mathbf{c}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_1 & \mathbf{0} \\
\mathbf{0} & \mathbf{c}_2 & \mathbf{c}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_2 \\
\mathbf{c}_3 & \mathbf{0} & \mathbf{c}_2 & \mathbf{c}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{c}_3 & \mathbf{0} & \mathbf{c}_2 & \mathbf{c}_2 & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{c}_3 & \mathbf{0} & \mathbf{c}_3 & \mathbf{c}_2 & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_3 & \mathbf{0} & \mathbf{c}_3 & \mathbf{c}_3
\end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \\ \mathbf{e}_5 \\ \mathbf{e}_6 \\ \mathbf{e}_7 \end{bmatrix}.$$

is then the required matrix $C(7)$ which is a GBRD$(7, 3, 4; \mathbb{N})$.

From Lam and Seberry [8, p. 90], we see that the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & b & c & bc & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & b & c & bc & 0 & 0 & 0 & b & c & bc & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & b & 0 & 0 & c & bc & 0 & 1 & 0 & 0 & c & bc & 0 & c & bc & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & c & 0 & b & 0 & bc & 0 & 1 & 0 & bc & 0 & b & 0 & bc & b & c \\ 0 & 0 & 0 & 1 & 0 & 0 & bc & 0 & b & c & 0 & 0 & 1 & 0 & c & b & b & 0 & c & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \end{bmatrix}$$

is a GBRD$(6, 3, 4; \mathbb{N})$. Set this general Bhaskar Rao design to be the matrix $C(6)$.

We now construct the matrix X from the entries of the matrix A and the rows of the matrices $C(6)$ and $C(7)$ using the construction contained in Theorem 6

Each column of A is of size 6 or 7.

Consider in turn the columns of A of size 6.

- Replace the first non-zero entry, say $1\mathbb{N}$, by the row vector $1\mathbf{f}_1$,
- Replace the second non-zero entry, say $h\mathbb{N}$, where $h \in \{1, a, a^2\}$, by the row vector $h\mathbf{f}_2$, where $\mathbf{f}_2$ is the second row of the matrix $C(6)$; and so on.
- Replace the zero entries by a zero row vector of length 20.

Next consider in turn the columns of A of size 7.

- Replace the first non-zero entry, $1\mathbb{N}$, by the row vector $1\mathbf{e}_1$, where $\mathbf{e}_1$ is the first row of the matrix $C(7)$.
- Replace the second non-zero entry, say $h\mathbb{N}$, where $h \in \{1, a, a^2\}$, by the row vector $h\mathbf{e}_2$, where $\mathbf{e}_2$ is the second row of the matrix $C(7)$; and so on.
- Replace the zero entries by a zero row vector of length 28.

The matrix X is a GBRD$(14, 3, 12; A_4)$.

5 Existence result

Fix $\mathbb{N} = \langle b, c \rangle$, a normal subgroup of A_4.

Lemma 9. *Necessary conditions for the existence of a GBRD$(v, 3, \lambda; A_4)$ are $v \geq 3$ and $\lambda \equiv 0 \pmod{12}$.*

Proof. For a GBRD$(v, 3, \lambda; A_4)$ to exist

$$\lambda \equiv 0 \pmod{12}$$

and there must exist a BIBD$(v, 3, \lambda)$ which exists only if:

$$\begin{aligned} v &\geq 3 \\ \lambda(v-1) &\equiv 0 \pmod{2} \\ \lambda v(v-1) &\equiv 0 \pmod{6}. \end{aligned}$$

These conditions are equivalent to the necessary conditions:

$$\begin{aligned} v &\geq 3 \\ \lambda &\equiv 0 \pmod{12} \end{aligned}$$

□

From Hall [7, Lemma 15.4.2] we have the useful

Lemma 10. *If $v \geq 3$ then a PBD$(v; K_3^2; 1)$ exists, where $K_3^2 = \{3, 4, 5, 6, 8, 11, 14\}$.*

We now construct a GBRD$(u, 3, 12; A_4)$ where $u \in \{3, 4, 5, 6, 8, 11, 14\}$. Whence we apply Lemma 10 and Theorem 7 to construct a GBRD$(v, 3, 12; A_4)$ for all $v \geq 3$.

Theorem 11. *If $v \geq 3$ and odd. Then we can construct a GBRD$(v, 3, 12; A_4)$.*

Proof. In Seberry [11] it was shown that a GBRD$(v, 3, 3; Z_3)$ exists when $v \geq 3$ and odd. Hence there exists a GBRD$(v, 3, 3; A_4/\mathbb{N})$ Also a GBRD$(3, 3, 4; Z_2 \times Z_2)$ exists (Lam and Seberry [8, Corollary 3.5.]) so a GBRD$(3, 3, 4; \mathbb{N})$. Hence, by Theorem 8, we can construct a GBRD$(v, 3, 12; A_4)$ when $v \geq 3$ and odd. □

Corollary 12. *A GBRD$(v, 3, 12; A_4)$ exists for $v \in \{3, 5, 7, 11\}$.*

Lemma 13. *We can construct a GBRD$(8, 3, 12; A_4)$.*

Proof. GBRD$(8, 4, 3; A_4/\mathbb{N})$ exists as a GBRD$(8, 4, 3; Z_3)$ exists (de Launey and Seberry [4]) A GBRD$(4, 3, 4; \mathbb{N})$ exists as a GBRD$(4, 3, 4; Z_2 \times Z_2)$ exists (Lam and Seberry [8]). Hence, using Theorem 8, we can combine these designs to construct a GBRD$(8, 3, 12; A_4)$. □

Theorem 14. *A generalized Bhaskar Rao design, GBRD$(v, 3, \lambda; A_4)$ exists if and only if $\lambda \equiv 0 \pmod{12}$.*

Proof. We have constructed a generalized Bhaskar Rao design, GBRD$(u, 3, \lambda; A_4)$ for each $u \in \{3, 4, 5, 6, 8, 11, 14\}$. Hence, by Lemma 10 and Theorem 7, we can construct a GBRD$(v, 3, 12; A_4)$ for all $v \geq 3$. Finally, for all $v \geq 3$ and for all $\lambda = 12s$, we can construct a GBRD$(v, 3, \lambda; A_4)$ by taking s copies of a GBRD$(v, 3, 12; A_4)$.

In Lemma 9 we proved that a GBRD$(v, 3, \lambda; A_4)$ exists only if $v \geq 3$ and $\lambda \equiv 0 \pmod{12}$. □

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