A trivalent graph of girth 17

Geoffrey Exoo

Department of Mathematics and Computer Science
Indiana State University
Terre Haute, IN 47809
ge@ginger.indstate.edu

Abstract

A family of trivalent graphs is described that includes most of the known trivalent cages. A new graph in this family is the smallest trivalent graph of girth 17 yet discovered.

Introduction

We consider the trivalent cage problem. Recall that a **cage** is a minimal regular graph of given degree and girth. The case where the degree is three is perhaps the most studied variant of this problem. See [1] for a recent comprehensive survey. Our focus here is on a family of trivalent graphs whose definition is motivated by drawings of trivalent cages found in popular Graph Theory books (e.g., [2], p43). Three such drawings are reproduced in Figures 1, 2 and 3.

Observe that each of the three graphs depicted consists of a hamiltonian cycle plus some chord edges. Each vertex is incident to exactly one of the chord edges, and the chord edges of a given length are regularly spaced around the outer cycle. This observation leads us to a definition. Given positive integers n and s, where s divides n, define the graph

$$n_s(c_0, \ldots, c_{s-1})$$

as follows. The graph has n vertices, which we label v_0 to v_{n-1}. This ordering of the vertices determines a Hamiltonian cycle, i.e., v_i is adjacent to v_{i+1}, for $0 \leq i < n$ and where addition of subscripts is modulo n. The remaining edge incident to vertex v_i is determined taking $j = i \ mod \ s$ and joining v_i to v_{i+c_j}. In order for the construction to give us a trivalent graph, we need $c_i = -c_k$, where $k = c_i \ mod \ s$.

Beginning with the Heawood graph, the (3,6)-cage, several of the known trivalent cages can be represented in the form described above. For example, the Heawood graph is $14_2(5, -5)$, as shown in the first figure. In each of the figures, imagine that the vertices are labeled in clockwise order, beginning at the twelve o'clock position.

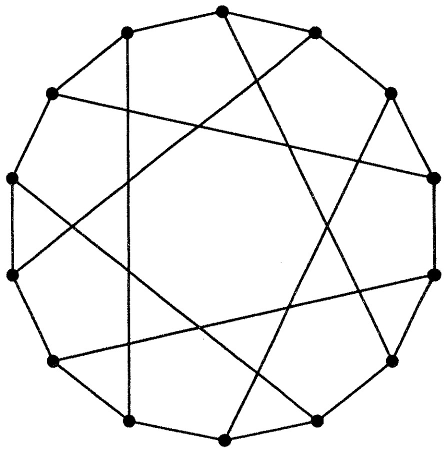

Figure 1. The Heawood Graph.

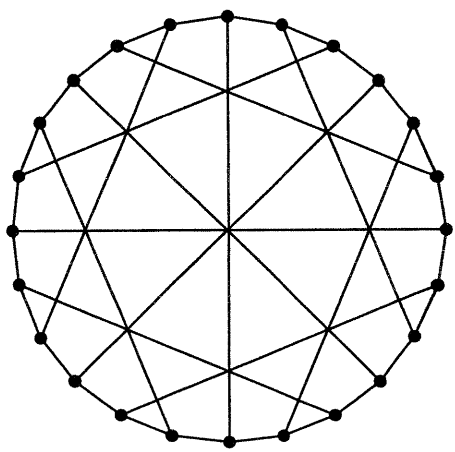

Figure 2. The McGee Graph is the (3,7)-cage.

The McGee graph is the (3,7)-cage. In our notation it can be represented as $24_3(12, 7, -7)$. It is show in Figure 2. Tutte's (3,8)-cage is also a member of our family of graphs. It can be denoted $30_6(7, -7, 9, 13, -13, -9)$ and is shown in Figure 3.

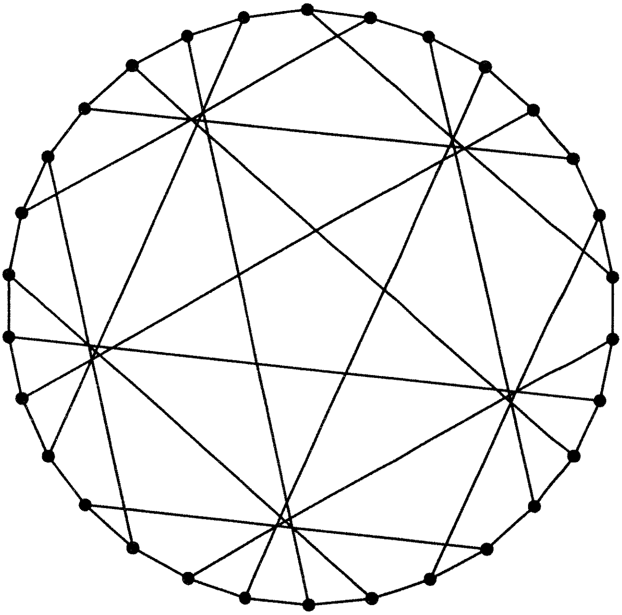

Figure 3. Tutte's (3,8)-cage.

Cages of girths 10 and 12 can also be represented in this format. The graph

$$70_{14}(9,15,-15,23,-27,27,35,15,-15,-9,-27,27,-23,-35),$$

is a 10-cage, and the graph

$$126_{18}(13,45,19,-19,49,75,111,25,-75,-111,-45,55,-55,-13,-25,91,-91,-49).$$

is the 12-cage. For girths 13 and 14, the smallest known trivalent graphs are Cayley graphs on 272 and 406 vertices, respectively [1]. These graphs are also members of our family. They can be described as

$$272_{16}(45,59,-119,-89,101,-109,-72,72,109,-101,89,119,-59,-45,72,-72)$$

and

$$406_{14}(23,63,173,199,77,-77,-199,-173,-63,-23,143,-77,77,-143).$$

These findings suggest that our family of graphs might be worth considering when looking for cages of larger girths. Looking at the known bounds in [1], it appeared to us that the bounds on the order of the trivalent cage of girth 17 were the weakest. So using a variety of computer searches, we investigated this case and found several members of our family that have girth 17. The smallest found so far is the graph

$$2520_{14}(61,76,1283,495,2206,-61,1852,-76,-495,382,-1852,-1283.-2206,-382)$$

The automorphism group of this graph is cyclic of order 180 [3], the minimum size guaranteed by the construction method. So we have the smallest known trivalent graph of girth 17, improving the value of 2978 cited in [1] .

A Problem

It is evident from the definition that the girth of a graph $n_s(c_0, \ldots, c_{s-1})$ is at most $2s+2$, since the sequence of vertices

$$v_0, v_{c_0}, v_{c_0+1}, \ldots, v_{c_0+s}, v_s, v_{s-1}, \ldots, v_1$$

forms a $2s+2$ cycle. The graph $80_4(9, -9, -31, 31)$ which has girth 10 is an example of a graph that achieves this bound. It can be shown that 10 is the largest girth for which this can happen. It would greatly facilitate computer searches if we had tighter bounds for the girth in terms of s. Define $f(g)$ to be the minimum value of s such that one of the graphs $n_s(c_0, \ldots, c_{s-1})$ has girth g. We ask for the determination of $f(g)$.

References

[1] N. Biggs, Constructions for Cubic Graphs of Large Girth. Electronic J. Comb. **5** (1998) A1.

[2] G. Chartrand and L. Lesniak, Graphs and Digraphs, 2nd ed. Wadsworth. Monterey, CA, 1986.

[3] B.D. McKay, *nauty User's Guide*, preprint, 1992.

(Received 4/9/2000)