

On diagonal cycle systems

Andrew Blinco

Centre for Discrete Mathematics and Computing
Department of Mathematics, The University of Queensland
Queensland 4072, Australia

Abstract

A diagonal cycle is a cycle of length $2n$ together with one edge between two vertices at distance n in the cycle. For example, $K_4 - e$ is a cycle of length four with one extra edge between two vertices at distance two in the cycle. Methods for decomposing the complete graph K_v into diagonal cycles are given when $v \equiv 0 \pmod{2n+1}$ if n is odd, and when $v \equiv 1 \pmod{2n+1}$, if $n \equiv 3 \pmod{4}$; in the case $n \equiv 1 \pmod{4}$ further partial results are obtained.

1 Introduction and necessary conditions

The necessary and sufficient conditions for the existence of a decomposition of the complete graph on v vertices, K_v , into cycles of length n are now known. There have been many papers written on the properties of cycle systems. For a survey of cycle systems see [6]. Until recently, the existence of some cycle systems was not known. These cases are covered in [2] and [7]. A cycle of length $2n$ ($n \geq 2$) is a graph on the vertices a_1, a_2, \dots, a_{2n} with edge set $\{a_i, a_{i+1}\}$, $1 \leq i \leq 2n-1$, $\{a_1, a_{2n}\}$. A *diagonal edge* adjoined to such a cycle is any edge between two vertices at distance n apart around the cycle. In the above cycle, if a diagonal edge such as $\{a_1, a_{n+1}\}$ is adjoined to the cycle, we call the resulting graph a *diagonal cycle* and denote it by $[a_1, a_2, \dots, a_n; a_{n+1}, a_{n+2}, \dots, a_{2n}]$. Furthermore, we denote any such diagonal cycle by D_{2n} . Thus, for instance, $K_4 - e$ is a diagonal cycle of the smallest possible size, denoted by D_4 . There have also been many papers written on the subject of $K_4 - e$ designs. The necessary and sufficient conditions for the existence of $K_4 - e$ designs are given in [3], the intersection problem is solved in [4] and blocking sets are investigated in [5].

A decomposition of a graph into edge-disjoint copies of D_{2n} is called a D_{2n} -*decomposition*. A D_{2n} -decomposition of the complete graph on v vertices is also called a D_{2n} -*design of order v* . A D_{2n} -decomposition of a graph with vertex set V may be written as a pair (V, B) , where B is a collection of copies of D_{2n} that partitions the edge set of the graph.

The graph D_{2n} has $2n + 1$ edges, so the expected spectrum (or the set of possible orders) of D_{2n} -designs will consist of all v for which

$$\frac{v(v - 1)}{2(2n + 1)}$$

is an integer. When $2n + 1 = p^a$, where p is prime and a is an integer, the expected spectrum for D_{2n} -designs is

$$v \equiv 0 \text{ or } 1 \pmod{2n + 1}. \tag{*}$$

When $2n + 1 \neq p^a$, the set of v satisfying (*) will form a subset of the expected spectrum for D_{2n} -designs.

In this paper we demonstrate methods for constructing D_{2n} -designs of order v where n is odd and v satisfies (*), except when $v \equiv 2n + 2 \pmod{4n + 2}$ where $n \equiv 1 \pmod{4}$ and $n \geq 13$. Thus, this paper gives the necessary and some sufficient conditions for the existence of certain G -designs, where G can be thought of as a generalization of $K_4 - e$ or an extension of an even length cycle.

2 D_6 -designs

In this section we show that for all $v \equiv 0 \text{ or } 1 \pmod{7}$, $v > 7$, there is a D_6 -design of order v .

LEMMA 2.1 *There is no D_6 -design of order 7.*

Proof

In any D_6 -design of order 7 we expect 3 copies of D_6 . Let the vertex set of K_7 be $V = \{1, 2, 3, 4, 5, 6, 7\}$. Without loss of generality, we can label one of the expected copies of D_6 as $[1, 2, 3; 4, 5, 6]$. (See Figure 1a.) This forces the vertices shown below in Figures 1b and 1c on the remaining two expected copies of D_6 .

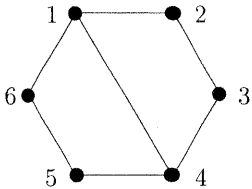


Figure 1a.
Expected Copy 1

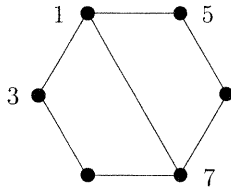


Figure 1b.
Expected Copy 2

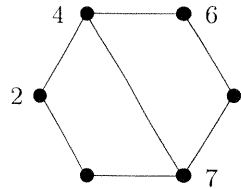


Figure 1c.
Expected Copy 3

This leaves nowhere for the edges $\{3, 5\}$ or $\{2, 6\}$. Hence there is no D_6 -design of order 7. \square

We now present examples of D_6 -decompositions of graphs required for our constructions.

EXAMPLE 1

Let $V = \mathbb{Z}_8$ and let $B = \{[0, 1, 7; 4, 5, 3] + i : i = 0, 1, 2, 3\}$.

Then (V, B) is a D_6 -design of order 8.

EXAMPLE 2

Let $V = \mathbb{Z}_{14}$ and let

$$B = \left\{ \begin{array}{llll} [0, 1, 2; 4, 5, 7], & [0, 2, 9; 5, 8, 3], & [0, 6, 3; 10, 2, 12], & [1, 5, 2; 11, 7, 10], \\ [1, 8, 9; 13, 2, 3], & [1, 12, 6; 9, 11, 4], & [2, 7, 1; 6, 10, 8], & [3, 9, 12; 5, 6, 4], \\ [4, 8, 6; 7, 9, 10], & [4, 9, 0; 13, 3, 12], & [5, 10, 12; 13, 6, 11], & [8, 13, 10; 11, 3, 7], \\ [11, 13, 7; 12, 8, 0] \end{array} \right\}.$$

Then (V, B) is a D_6 -design of order 14.

EXAMPLE 3

Let $V = \mathbb{Z}_3 \times \mathbb{Z}_7$ and let B contain the blocks arising from the following set, with the first components all cycled modulo 3, and the second components fixed.

$$\left\{ \begin{array}{ll} [(0, 0), (0, 2), (0, 4); (0, 5), (2, 5), (1, 2)], & [(0, 0), (2, 0), (2, 1); (1, 3), (1, 2), (2, 6)], \\ [(0, 1), (0, 6), (1, 3); (0, 2), (2, 4), (2, 1)], & [(0, 2), (2, 1), (1, 2); (0, 5), (1, 6), (1, 0)], \\ [(0, 3), (1, 4), (0, 1); (1, 0), (0, 5), (2, 3)], & [(0, 3), (2, 1), (1, 0); (2, 5), (2, 6), (1, 6)], \\ [(0, 4), (1, 4), (2, 1); (2, 5), (1, 6), (2, 2)], & [(1, 1), (1, 3), (1, 0); (2, 6), (1, 4), (2, 5)], \\ [(2, 0), (0, 4), (0, 3); (2, 4), (1, 6), (1, 4)], & [(2, 3), (2, 5), (0, 1); (2, 6), (2, 2), (0, 2)]. \end{array} \right\}.$$

Then (V, B) is a D_6 -design of order 21.

EXAMPLE 4

Let $V = \mathbb{Z}_7 \times \mathbb{Z}_2$ and let $B = \{[(0, 0), (1, 1), (3, 0); (0, 1), (1, 0), (3, 1)] + (i, 0) : i \in \mathbb{Z}_7\}$.

Then (V, B) is a D_6 -decomposition of $K_{7,7}$, where V is partitioned in the obvious way.

The following three lemmas use the D_6 -designs given in Examples 1, 2, 3 and 4 to construct D_6 -designs of order v for all $v \equiv 0$ or $1 \pmod{7}$, $v > 7$.

LEMMA 2.2 *There is a D_6 -design of order $7k + 1$, $k \geq 1$.*

Proof

Let $V = (\mathbb{Z}_k \times \mathbb{Z}_7) \cup \{\infty\}$ and let B contain the copies of D_6 from the following two types of D_6 -decompositions.

Type 1: For each i , $0 \leq i \leq k - 1$, place a D_6 -design of order 8 on $(\{i\} \times \mathbb{Z}_7) \cup \{\infty\}$.

Type 2: For each i and j satisfying $0 \leq i < j \leq k - 1$, place a D_6 -decomposition of $K_{7,7}$ on $\{i, j\} \times \mathbb{Z}_7$.

Then (V, B) is a D_6 -design of order $7k + 1$. □

LEMMA 2.3 *There is a D_6 -design of order $14k$, $k \geq 1$.*

Proof

Let $V = \mathbb{Z}_{2k} \times \mathbb{Z}_7$ and let B contain the copies of D_6 from the following two types of D_6 -decompositions.

Type 1: For each i , $0 \leq i \leq k-1$, place a D_6 -design of order 14 on $\{2i, 2i+1\} \times \mathbb{Z}_7$.

Type 2: For each i and j satisfying $0 \leq i < j \leq 2k-1$ and $(i, j) \neq (2x, 2x+1)$ for any x , $0 \leq x \leq k-1$, place a D_6 -decomposition of $K_{7,7}$ on $\{i, j\} \times \mathbb{Z}_7$.

Then (V, B) is a D_6 -design of order $14k$. □

LEMMA 2.4 *There is a D_6 -design of order $14k+7$, $k \geq 1$.*

Proof

Let $V = \mathbb{Z}_{2k+1} \times \mathbb{Z}_7$ and let B contain the copies of D_6 from the following three types of D_6 -decompositions.

Type 1: Place a D_6 -design of order 21 on $\mathbb{Z}_3 \times \mathbb{Z}_7$.

Type 2: For each i , $2 \leq i \leq k$, place a D_6 -design of order 14 on $\{2i-1, 2i\} \times \mathbb{Z}_7$.

Type 3: For each i and j satisfying $0 \leq i < j \leq 2k$, $(i, j) \neq (2x-1, 2x)$ for any x , $2 \leq x \leq k$, and $\{i, j\} \not\subset \{0, 1, 2\}$, place a D_6 -decomposition of $K_{7,7}$ on $\{i, j\} \times \mathbb{Z}_7$.

Then (V, B) is a D_6 -design of order $14k+7$. □

3 General Cases

In this section we give methods for constructing the D_{2n} -decompositions which will be used in the constructions in Section 5. In Figures 2 and 3 the heavy edge denotes the diagonal edge of the diagonal cycle.

LEMMA 3.1 *There is a D_{2n} -design of order $2n+1$, where n is odd and $n \geq 5$.*

Proof

If $n = 5$, let $V = (\mathbb{Z}_5 \times \mathbb{Z}_2) \cup \{\infty\}$ and let $B = \{[(1, 0), (0, 0), (0, 1), (4, 0), (1, 1); (3, 0), (2, 1), (4, 1), (3, 1), \infty] + (i, 0) : i \in \mathbb{Z}_5\}$. If $n = 7$, see Example 5 below. If $n = 9$, let $V = (\mathbb{Z}_9 \times \mathbb{Z}_2) \cup \{\infty\}$ and let $B = \{[(1, 0), (0, 0), (0, 1), (8, 0), (1, 1), (7, 0), (2, 1), (6, 0), (3, 1); (5, 0), (4, 1), (8, 1), (5, 1), (7, 1), (6, 1), \infty, (2, 0), (4, 0)] + (i, 0) : i \in \mathbb{Z}_9\}$. If $n \geq 11$, let $V = (\mathbb{Z}_n \times \mathbb{Z}_2) \cup \{\infty\}$ and let

$$B = \left\{ [(1, 0), (0, 0), (0, 1), (n-1, 0), (1, 1), (n-2, 0), (2, 1), \dots, (n-m, 0), (m, 1), \dots, \left(\frac{n+3}{2}, 0\right)\left(\frac{n-3}{2}, 1\right); \left(\frac{n+1}{2}, 0\right), \left(\frac{n-1}{2}, 1\right), (n-1, 1), \left(\frac{n+1}{2}, 1\right), (n-2, 1), \left(\frac{n+3}{2}, 1\right), \dots, (n-m, 1), \left(\frac{n+(2m-1)}{2}, 1\right), \dots, (\lfloor \frac{3n}{4} \rfloor, 1), \infty, (n \pmod{4}) + \lfloor \frac{n-5}{4} \rfloor, 0), \dots, (m, 0), \left(\frac{n-(2m-3)}{2}, 0\right), \dots, (2, 0), \left(\frac{n-1}{2}, 0\right)] + (i, 0) : i \in \mathbb{Z}_n \right\}.$$

Then a straightforward check shows that (V, B) is a D_{2n} -design of order $2n+1$. □

EXAMPLE 5

Let $V = (\mathbb{Z}_7 \times \mathbb{Z}_2) \cup \{\infty\}$ and let $B = \{[(1, 0), (0, 0), (0, 1), (6, 0), (1, 1), (5, 0), (2, 1); (4, 0), (3, 1), (6, 1), (4, 1), (5, 1), \infty, (3, 0)] + (i, 0) : i \in \mathbb{Z}_7\}$.

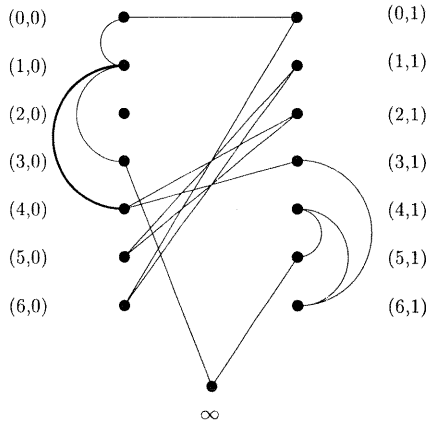


Figure 2. The starter configuration for a D_{14} -design of order 15.

Then (V, B) is a D_{14} -design of order 15.

LEMMA 3.2 *There is a D_{2n} -design of order $2n + 2$, where $n \equiv 3 \pmod{4}$.*

Proof

If $n = 3$ see Example 1, otherwise let $V = \mathbb{Z}_{2n+2}$, where $n \equiv 3 \pmod{4}$. Define a sequence of differences, $\{d_j\}$, in the following way:

$$\{d_j\} = \langle 1, -2, 3, -4, \dots, 2m - 1, -2m, \dots, \frac{n-1}{2}, -(\frac{n+1}{2}), \frac{n+5}{2}, -(\frac{n+7}{2}), \frac{n+9}{2}, -(\frac{n+11}{2}), \dots, \frac{n+(4m+1)}{2}, -(\frac{n+(4m+3)}{2}), \dots, n - 1, -n, \frac{n+3}{2} \rangle.$$

The sequence of differences, $\{d_j\}$, forms a ‘semi-cycle’ in the starter configuration defined below. By ‘semi-cycle’ we mean the path of length n on the vertices a_1, a_2, \dots, a_{n+1} or $a_{n+1}, a_{n+2}, \dots, a_{2n}$. Let $B = \{[0, \sum_{j=1}^1 d_j, \sum_{j=1}^2 d_j, \dots, \sum_{j=1}^{n-1} d_j; n + 1, n + 1 + \sum_{j=1}^1 d_j, n + 1 + \sum_{j=1}^2 d_j, \dots, n + 1 + \sum_{j=1}^{n-1} d_j] + i : i = 0, 1, \dots, n\}$.

Then (V, B) is a D_{2n} -design of order $2n + 2$. □

EXAMPLE 6

Let $V = \mathbb{Z}_{16}$ and let $\{d_j\} = \langle 1, -2, 3, -4, 6, -7, 5 \rangle$.

Let $B = \{[0, 1, 15, 2, 14, 4, 13; 8, 9, 7, 10, 6, 12, 5] + i : i = 0, 1, \dots, 7\}$.

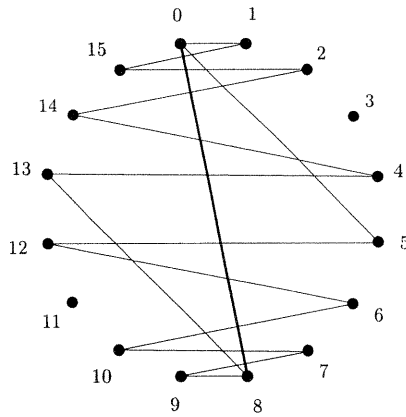


Figure 3. The starter configuration for a D_{14} -design of order 16.

Then (V, B) is a D_{14} -design of order 16.

LEMMA 3.3 *There is a D_{2n} -design of order $4n + 3$, where $n \equiv 1 \pmod{4}$.*

Proof

Let $V = \mathbb{Z}_{4n+3}$, where $n \equiv 1 \pmod{4}$. Define a sequence of differences, $\{d_j\}$, in the following way:

$$\{d_j\} = \langle -2, 4, -6, 8, \dots, -(n-3), (n-1), -(n+3), n+5, -(n+7), n+9, \dots, 2n-4, -(2n-2), 1, 2n+1, 2n-1, -(2n-3), 2n-5, -(2n-7), \dots, 5, -3 \rangle.$$

Once again, the sequence of differences, $\{d_j\}$, forms a ‘semi-cycle’ in the starter configuration defined below. Let $B = \{[\sum_{j=1}^1 d_j, \sum_{j=1}^2 d_j, \dots, \sum_{j=1}^n d_j; \sum_{j=1}^{n+1} d_j, \sum_{j=1}^{n+2} d_j, \dots, \sum_{j=1}^{2n} d_j] + i : i \in \mathbb{Z}_{4n+3}\}$.

Then (V, B) is a D_{2n} -design of order $4n + 3$. □

LEMMA 3.4 *There is a D_{2n} -decomposition of $K_{2n+1, 2n+1}$, where n is odd.*

Proof

Let $V = \mathbb{Z}_{2n+1} \times \mathbb{Z}_2$ where n is odd. We shall consider the case $n \equiv 3 \pmod{4}$ and the case $n \equiv 1 \pmod{4}$ separately.

The case $n \equiv 3 \pmod{4}$. If $n = 3$, see Example 4, otherwise define a sequence of differences, $\{d_j\}$, in the following way:

$$\{d_j\} = \langle 1, 2, 3, 4, \dots, 2m-1, 2m, \dots, \frac{n-5}{2}, \frac{n-3}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \dots, n-1, n, \frac{n-1}{2} \rangle.$$

$$\text{Let } B = \{[(0, 0), (\sum_{j=1}^1 (-1)^{j+1} d_j, 1), (\sum_{j=1}^2 (-1)^{j+1} d_j, 0), \dots, (\sum_{j=1}^{n-2} (-1)^{j+1} d_j, 1), (\sum_{j=1}^{n-1} (-1)^{j+1} d_j, 0); (0, 1), (\sum_{j=1}^1 (-1)^{j+1} d_j, 0), (\sum_{j=1}^2 (-1)^{j+1} d_j, 1), \dots, (\sum_{j=1}^{n-2} (-1)^{j+1} d_j, 0), (\sum_{j=1}^{n-1} (-1)^{j+1} d_j, 1)] + (i, 0) : i \in \mathbb{Z}_{2n+1}\}.$$

Then (V, B) is a D_{2n} -decomposition of $K_{2n+1, 2n+1}$, where $n \equiv 3 \pmod{4}$.

The case $n \equiv 1 \pmod{4}$. Define two sequences of differences, $\{a_j\}$ and $\{b_j\}$, in the following way:

If $n = 5$ let $\{a_j\} = \langle 9, 1, 2, 3, 4 \rangle$ and let $\{b_j\} = \langle 7, 8, 6, 10, 5 \rangle$. If $n > 5$ let $\{a_j\} = \langle \frac{3}{2}(n+1), 1, 2, 3, \dots, n-1 \rangle$ and $\{b_j\} = \langle n+2, n+3, \dots, \frac{1}{2}(3n+1), n+1, 2n, 2n-1, \dots, \frac{1}{2}(3n+5), n \rangle$. Let $B = \{[(0, 0), (\sum_{j=1}^1 (-1)^{j+1} a_j, 1), (\sum_{j=1}^2 (-1)^{j+1} a_j, 0), \dots, (\sum_{j=1}^{n-2} (-1)^{j+1} a_j, 1), (\sum_{j=1}^{n-1} (-1)^{j+1} a_j, 0); (0, 1), (\sum_{j=1}^1 (-1)^{j+1} b_j, 0), (\sum_{j=1}^2 (-1)^{j+1} b_j, 1), \dots, (\sum_{j=1}^{n-2} (-1)^{j+1} b_j, 0), (\sum_{j=1}^{n-1} (-1)^{j+1} b_j, 1)] + (i, 0) : i \in \mathbb{Z}_{2n+1}\}$.

Then (V, B) is a D_{2n} -decomposition of $K_{2n+1, 2n+1}$, where $n \equiv 1 \pmod{4}$. □

4 Some useful D_{2n} -designs of order $2n + 2$

In this section we give examples of some D_{2n} -designs of order $2n + 2$, where $n \equiv 1 \pmod{4}$. These designs enable us to complete the spectrum for D_{2n} -designs when $n = 5$ and $n = 9$.

EXAMPLE 7

Let $V = \mathbb{Z}_3 \times \mathbb{Z}_4$ and let $B = \{[(0, 0), (1, 0), (0, 1), (2, 0), (2, 1); (0, 2), (1, 1), (0, 3), (1, 2), (1, 3)] + (i, 0), [(0, 1), (1, 1), (1, 2), (0, 0), (2, 3); (0, 3), (2, 2), (0, 2), (1, 0), (1, 3)] + (i, 0) : i \in \mathbb{Z}_3\}$.

Then (V, B) is a D_{10} -design of order 12.

EXAMPLE 8

Let $V = \mathbb{Z}_5 \times \mathbb{Z}_4$ and let $B = \{[(0, 0), (1, 0), (0, 1), (3, 0), (4, 1), (1, 1), (1, 2), (2, 0), (3, 3); (2, 2), (4, 2), (2, 1), (0, 2), (4, 0), (2, 3), (3, 1), (4, 3), (0, 3)] + (i, 0), [(4, 1), (3, 2), (3, 0), (1, 0), (0, 3), (1, 2), (3, 3), (0, 2), (4, 2); (4, 3), (1, 3), (4, 0), (2, 2), (1, 1), (2, 1), (2, 0), (0, 1), (2, 3)] + (i, 0) : i \in \mathbb{Z}_5\}$.

Then (V, B) is a D_{18} -design of order 20.

5 General Constructions

Since Section 2 completely deals with D_6 -designs, here we consider D_{2n} -designs where n is odd and $n \geq 5$.

LEMMA 5.1 *There is a D_{2n} -design of order $k(2n+1)$, where n is odd, $n \geq 5$, and $k \geq 1$.*

Proof

Let $V = \mathbb{Z}_k \times \mathbb{Z}_{2n+1}$ and let B contain the diagonal cycles from the following two types of D_{2n} -decompositions.

Type 1: For each $i, 0 \leq i \leq k-1$, place a D_{2n} -design of order $2n+1$ on $\{i\} \times \mathbb{Z}_{2n+1}$; such a design exists by Lemma 3.1.

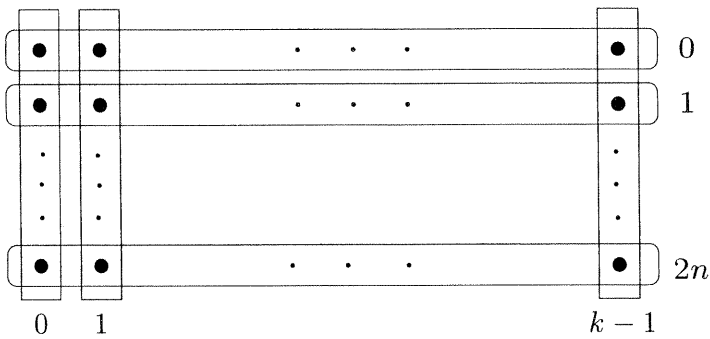


Figure 4. D_{2n} -decompositions of Type 1

Type 2: For each i and j satisfying $0 \leq i < j \leq k - 1$, place a D_{2n} -decomposition of $K_{2n+1, 2n+1}$ on $\{i, j\} \times \mathbb{Z}_{2n+1}$; such a decomposition exists by Lemma 3.4.

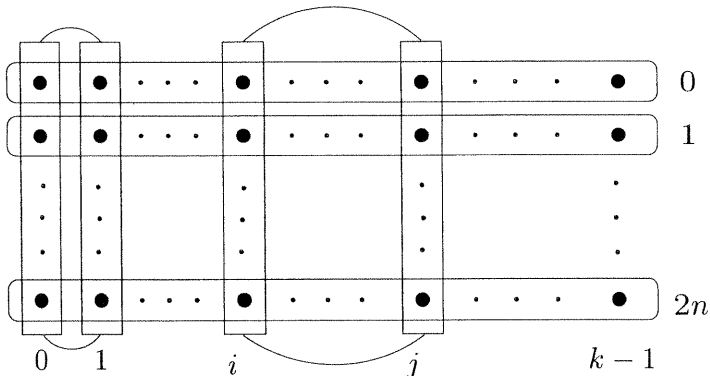


Figure 5. D_{2n} -decompositions of Type 2

Then (V, B) is a D_{2n} -design of order $k(2n + 1)$, where n is odd, $n \geq 5$, and $k \geq 1$. □

LEMMA 5.2 *There is a D_{2n} -design of order $k(2n + 1) + 1$, where $n = 5$, $n = 9$ or $n \equiv 3 \pmod{4}$, $n \geq 7$, and $k \geq 1$.*

Proof

Let $V = (\mathbb{Z}_k \times \mathbb{Z}_{2n+1}) \cup \{\infty\}$ and let B contain the diagonal cycles from the following two types of D_{2n} -decompositions.

Type 1: For each i , $0 \leq i \leq k - 1$, place a D_{2n} -design of order $2n + 2$ on $(\{i\} \times \mathbb{Z}_{2n+1}) \cup \{\infty\}$; such a design exists by Lemma 3.2, Example 7 or Example 8.

Type 2: For each i and j satisfying $0 \leq i < j \leq k - 1$, place a D_{2n} -decomposition of $K_{2n+1, 2n+1}$ on $\{i, j\} \times \mathbb{Z}_{2n+1}$; such a decomposition exists by Lemma 3.4.

Then (V, B) is a D_{2n} -design of order $k(2n + 1) + 1$, where $n = 5$, $n = 9$ or $n \equiv 3 \pmod{4}$, $n \geq 7$ and $k \geq 1$. □

LEMMA 5.3 *There is a D_{2n} -design of order $2k(2n+1)+1$, where $n \equiv 1 \pmod{4}$ and $k \geq 1$.*

Proof

Let $V = (\mathbb{Z}_{2k} \times \mathbb{Z}_{2n+1}) \cup \{\infty\}$ and let B contain the diagonal cycles from the following two types of D_{2n} -decompositions.

Type 1: For each i , $0 \leq i \leq 2k-1$, place a D_{2n} -design of order $4n+3$ on $(\{i, i+1\} \times \mathbb{Z}_{2n+1}) \cup \{\infty\}$; such a design exists by Lemma 3.3.

Type 2: For each i and j satisfying $0 \leq i < j \leq 2k-1$, $(i, j) \neq (2x, 2x+1)$, $0 \leq x \leq k-1$, place a D_{2n} -decomposition of $K_{2n+1, 2n+1}$ on $\{i, j\} \times \mathbb{Z}_{2n+1}$; such a decomposition exists by Lemma 3.4.

Then (V, B) is a D_{2n} -design of order $2k(2n+1)+1$, where $n \equiv 1 \pmod{4}$ and $k \geq 1$. □

6 Summary

When combined, all of the lemmas in this paper give us the following theorem:

THEOREM 6.1

- (i) *When $v \equiv 0 \pmod{2n+1}$ and n is odd, there exists a D_{2n} -design of order v , except when $n = 3$ and $v = 7$.*
- (ii)(a) *When $v \equiv 1 \pmod{2n+1}$ and $n = 5, n = 9$, or $n \equiv 3 \pmod{4}$, there exists a D_{2n} -design of order v .*
- (ii)(b) *When $v \equiv 1 \pmod{4n+2}$ and $n \equiv 1 \pmod{4}$, $n \geq 13$, there exists a D_{2n} - design of order v . (However if $v \equiv 2n+2 \pmod{4n+2}$, $n \geq 13$, the existence of a D_{2n} -design of order v remains open.)*

Proof

Part (i) follows from Lemmas 2.1 and 5.1.

Part (ii)(a) follows from Lemma 5.2.

Part (ii)(b) follows from Lemma 5.3. □

We conjecture that when $v \equiv 1 \pmod{2n+1}$, a D_{2n} -design exists for all odd n and work is proceeding on this.

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