On the cyclic decomposition of complete graphs into bipartite graphs

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Abstract

Let G be a graph with n edges. It is known that there exists a cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-labeling. An α-labeling of G easily yields both a cyclic G-decomposition of $K_{n,n}$ and of K_{2nx+1} for all positive integers x. It is well-known that certain classes of bipartite graphs (including certain trees) do not have α-labelings. Moreover, there are bipartite graphs with n edges which do not cyclically divide $K_{n,n}$. In this article, we introduce the concept of an *ordered ρ-labeling* (denoted by ρ^+) of a bipartite graph, and prove that if a graph G with n edges has a ρ^+-labeling, then there is a cyclic G-decomposition of K_{2nx+1} for all positive integers x. We also introduce the concept of a θ-labeling which is a more restrictive ρ^+-labeling. We conjecture that all forests have a ρ^+-labeling and show that the vertex-disjoint union of any finite collection of graphs that admit α-labelings admits a θ-labeling.

1 Introduction

Only graphs without loops, multiple edges, or isolated points will be considered herein. Undefined graph-theoretic terminology can be found in the textbook by Chartrand and Lesniak [5]. If m and n are integers with $m \leq n$ we denote $\{m, m+1, \ldots, n\}$ by $[m,n]$. Let N denote the set of nonnegative integers and Z_n the group of integers modulo n. If we consider K_m to have the vertex set Z_m, by *clicking* we mean applying the isomorphism $i \to i+1$. Likewise if we consider $K_{m,m}$ to have

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the vertex set $Z_m \times Z_2$, with the obvious vertex bipartition, by *clicking* we mean applying the isomorphism $(i, j) \to (i + 1, j)$.

Let K and G be graphs with G a subgraph of K. A *G-decomposition* of K is a set $\Gamma = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that the edge sets of the graphs G_i form a partition of the edge set of K. In this case, we say *G divides K*. If K is K_m or $K_{m,m}$, a G-decomposition Γ is *cyclic* (*purely cyclic*) if clicking is a permutation (t-cycle) of Γ.

A *labeling* or *valuation* of a graph G is a function from $V(G)$ into N. In 1967, Rosa [20] introduced several types of graph labelings as tools for decomposing complete graphs into isomorphic subgraphs. These labelings are particularly useful in attacking the following conjectures.

Conjecture 1 (Ringel [17], 1964) *Every tree with n edges divides the complete graph K_{2n+1}.*

Conjecture 2 *Every tree with n edges divides the complete bipartite graph $K_{n,n}$.*

Conjecture 2, which is part of the folklore of the subject, is a special case of the conjecture by Häggkvist that every tree with n edges divides every n-regular bipartite graph [13]. Since every tree with n edges divides a tree with nx edges for all positive integers x, Conjecture 1 implies the following.

Conjecture 3 *Every tree with n edges divides K_p for all $p \equiv 1 \pmod{2n}$.*

Let G be a graph with n edges. In 1967, Alex Rosa [20] called a function γ a *ρ-labeling* of G if γ is an injection from $V(G)$ into $[0, 2n]$ such that

$$\{\min\{|\gamma(u) - \gamma(v)|, 2n + 1 - |\gamma(u) - \gamma(v)|\} : \{u, v\} \in E(G)\} = [1, n].$$

(Note that this amounts to saying that the values of $\gamma(u) - \gamma(v)$ over all ordered pairs (u, v) with $\{u, v\} \in E(G)$ are distinct modulo $2n + 1$.) Rosa proved the following result.

Theorem 1 (Rosa [20], 1967) *Let G be a graph with n edges. A purely cyclic G-decomposition of K_{2n+1} exists if and only if G has a ρ-labeling.*

The above result does not necessarily extend to G-decompositions of K_{2nx+1}. Also, if G is bipartite, then a ρ-labeling of G does not necessarily yield a G-decomposition of $K_{n,n}$.

Conjecture 4 (Kotzig, see [20]) *Every tree has a ρ-labeling.*

Rosa [20] also introduced β-labelings. A *β-labeling* of a graph G with n edges is an injection γ from $V(G)$ into $\{0, 1, \ldots, n\}$ such that $\{|\gamma(u) - \gamma(v)| : \{u, v\} \in E(G)\} = \{1, 2, \ldots, n\}$. Golomb [12] subsequently called such a labeling a *graceful* labeling and that is now the popular term.

Since a β-labeling is also a ρ-labeling, Theorem 1 also applies to "graceful" graphs. Unfortunately, from a graph decomposition point of view, a graceful labeling, which is far more restrictive than a ρ-labeling, offers no additional applications.

Theorem 2 *Let G be a graph with n edges that has a β-labeling. Then there exists a purely cyclic G-decomposition of the complete graph K_{2n+1}.*

Again, Theorem 2 does not necessarily extend to G-decompositions of K_{2nx+1} nor does it necessarily yield a G-decomposition of $K_{n,n}$ when G is bipartite.

The following conjecture is attributed to both Ringel and Kotzig.

Conjecture 5 *Every tree has a β-labeling.*

Conjecture 5 is known as *the graceful tree conjecture.* It is one of the best-known problems in the theory of graphs. Since Rosa's 1967 article [20], there have been over 200 research papers related to this conjecture (see Gallian [11]). In spite of many partial results, the conjecture remains open.

A more restrictive labeling than either ρ or β was also introduced by Rosa [20]. An *α-labeling* of G is a β-labeling having the additional property that there exists an integer λ such that if $\{u, v\} \in E(G)$, then $\min\{\gamma(u), \gamma(v)\} \leq \lambda < \max\{\gamma(u), \gamma(v)\}$. The integer λ, which is unique for a given α-labeling, is called its *critical value.* Note that if G admits an α-labeling then G is bipartite with parts A and B, where $A = \{u \in V(G) : \gamma(u) \leq \lambda\}$, and $B = \{u \in V(G) : \gamma(u) > \lambda\}$. Rosa proved the following result.

Theorem 3 (Rosa [20], 1967) *Let G be a graph with n edges that has an α-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

It can also be easily shown (see [7]) that α-labelings are useful in finding purely cyclic G-decompositions of $K_{n,n}$.

Theorem 4 *If a graph G with n edges has an α-labeling, then there exists a purely cyclic decomposition of the complete bipartite graph $K_{n,n}$ into isomorphic copies of G.*

Labelings that are useful in tackling Conjecture 2 have also been introduced. In [18], Ringel, Lladó and Serra introduced the concept of a bigraceful labeling of a tree. Let T be a tree with n edges and let (A, B) be a vertex bipartition of T. A labeling f of T is *bigraceful* if it satisfies the following properties:

(i) $f(a) \in [1, n]$ and $f(b) \in [1, n]$ for all $a \in A$ and all $b \in B$;

(ii) $|\{f(a) : a \in A\}| = |A|$ and $|\{f(b) : b \in B\}| = |B|$;

(iii) $\{f(b) - f(a) : a \in A,\ b \in B,\ \{a, b\} \in E(T)\} = [0, n-1]$.

The comet $S_{3,2}$ on the left in Figure 2 is shown with a bigraceful labeling. It is simple to check that if a tree T with n edges has a bigraceful labeling, then there exists a purely cyclic T-decomposition of $K_{n,n}$. Clearly bipartite graphs other than trees can admit bigraceful labelings. Note that the graph consisting of two disjoint edges is the smallest bipartite graph that does not admit a bigraceful labeling.

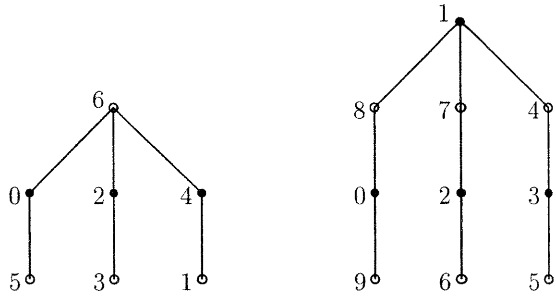

Figure 1: A β-labeling of $S_{3,2}$ and an α-labeling of $S_{3,3}$

The condition of having an α-labeling is the most restrictive applied by Rosa, and there are trees which do not admit α-labelings. In particular, he points out [20] that trees of diameter four that contain the comet $S_{3,2}$ as a subtree (See Figure 1) do not admit α-labelings. The *comet* $S_{k,n}$ is the graph obtained from the k-star $K_{1,k}$ by replacing each edge by a path with n edges. We note that not all comets fail to admit an α-labeling (see Figure 1 for an α-labeling of $S_{3,3}$).

In this article we introduce the concept of an ordered ρ-labeling (denoted by ρ^+) of a bipartite graph, and prove that if a bipartite graph G with n edges has a ρ^+-labeling, then there exists a cyclic G-decomposition K_{2nx+1} for all positive integers x. We also introduce the concept of a θ-labeling which is a more restrictive ρ^+-labeling. We conjecture that all forests have a ρ^+-labeling and show that the vertex-disjoint union of any finite collection of graphs that admit α-labelings admits a θ-labeling.

2 Main Results

Let G be a bipartite graph with n edges and bipartition (A, B). We call h a *θ-labeling* of G if h is a one-to-one function from $V(G)$ to $[0, 2n]$ such that

$$\{h(b) - h(a) : \{a, b\} \in E(G),\ a \in A,\ b \in B\} = [1, n]$$

We call h a *ρ^+-labeling* for G if h is a one-to-one function from $V(G)$ to $[0, 2n]$ such that the integers $h(x) - h(y)$ are distinct modulo $2n + 1$ over all ordered pairs (x, y) with $\{x, y\} \in E(G)$, and if $h(b) > h(a)$ whenever $a \in A$, $b \in B$, and $\{a, b\} \in E(G)$. Any θ-labeling is also a ρ^+-labeling, and any ρ^+-labeling is a ρ-labeling.

Theorem 5 *If a bipartite graph G with n edges has a ρ^+-labeling, and x is any positive integer, then there exists a cyclic G-decomposition of K_{2nx+1}.*

Proof. Let h be the ρ^+-labeling of G. We will start by constructing a graph G^* with nx edges such that G divides G^* and G^* has a ρ^+-labeling.

Let the vertex set of G have the bipartition A, B as in the definition above. Consider x disjoint copies of B called $B_1, B_2, \ldots, B_x$. If $b \in B$, let b_i be the corresponding element of B_i. Let G^* be the graph with vertex set $A \cup B_1 \cup B_2 \cup \ldots \cup B_x$ and edges $\{a, b_i\}, 1 \leq i \leq x$ for each edge $\{a, b\}$ of G with $a \in A$ and $b \in B$. Thus G^* has nx edges.

Now we define a map $h^* : V(G^*) \to [0, 2nx]$ by $h^*(a) = h(a)$ if $a \in A$, and $h^*(b_i) = h(b) + (i-1)2n$ if $b_i \in B_i$. Clearly h^* is one-to-one on $A \cup B_1$, and on each set B_i, $i = 2, 3, \ldots, x$. But $h^*(A \cup B_1) \subseteq [0, 2n]$, and $h^*(B_i) \subseteq [(i-1)2n+1, i2n]$ for $i = 2, 3, \ldots, x$ since $0 \notin h(B)$. Thus h^* is one-to-one.

We claim that h^* is a ρ^+-labeling for G^*, with vertex bipartition A, B^*, where $B^* = B_1 \cup B_2 \cup \ldots \cup B_x$. Clearly if $\{a, b_i\}$ is an edge of G^* with $a \in A$, then $h^*(a) < h^*(b_i)$. Now we will show that if $\{u, v\}$ and $\{x, y\}$ are any two edges of G^* such that $h^*(u) - h^*(v) \equiv h^*(x) - h^*(y) \pmod{2nx+1}$, then $(u, v) = (x, y)$. Let $\{u, v\} = \{a, b_i\}$ and $\{x, y\} = \{a', b'_j\}$, where $a, a' \in A$, $b_i \in B_i$, and $b'_j \in B_j$. All congruences below will be modulo $2nx+1$ unless otherwise specified. Then we have $h^*(b_i) - h^*(a) \equiv \pm(h^*(b'_j) - h^*(a'))$. Thus

$$h(b) + (i-1)2n - h(a) \equiv \pm(h(b') + (j-1)2n - h(a')). \tag{1}$$

First we will show that the minus sign cannot hold in (1). For if it does then

$$h(b) - h(a) + h(b') - h(a') + (i+j-2)2n \equiv 0. \tag{2}$$

Now $0 < h(b) - h(a) + h(b') - h(a') + (i+j-2)2n \leq 4n + (2x-2)2n = 4nx < 2(2nx+1)$. Thus (2) implies that $h(b) - h(a) + h(b') - h(a') + (i+j-2)2n = 2nx+1$. We see that $2n | h(b) - h(a) + h(b') - h(a') - 1$. This is only possible if $h(b) - h(a) + h(b') - h(a') - 1 = 2n$. But then $h(b) - h(a) \equiv h(a') - h(b') \pmod{2n+1}$. This contradicts the assumption that h is a ρ^+-labeling.

Now take the plus sign in (1). This gives

$$h(b) - h(a) - (h(b') - h(a')) + (i-j)2n \equiv 0. \tag{3}$$

The absolute value of the left side of (3) does not exceed $2n + (x-1)2n = 2nx$, so the congruence implies $h(b) - h(a) - (h(b') - h(a')) + (i-j)2n = 0$. Then $2n | h(b) - h(a) - (h(b') - h(a'))$, so $h(b) - h(a) = h(b') - h(a')$. We must have $a = a'$ and $b = b'$. Then (3) implies $i = j$.

Now we are done unless, say, $(u, v) = (a, b_i)$ and $(x, y) = (b_i, a)$. But in this case the congruence $h^*(u) - h^*(v) \equiv h^*(x) - h^*(y)$ becomes $2(h(b) - h(a)) + (i-1)4n \equiv 0$, which implies $h(b) - h(a) + (i-1)2n \equiv 0$. This is impossible because the left side is positive and less than $2n + (x-1)2n = 2nx$.

Now since h^* is a ρ-labeling of G^*, the complete graph K_{2nx+1} has a purely cyclic G^*-decomposition by Theorem 1. This gives a cyclic G-decomposition of K_{2nx+1}. ∎

The *base* of a tree T is the tree T_B obtained from T by removing all its degree 1 vertices. A tree is called a *caterpillar* if its base tree is a path and it is called a *lobster* if its base tree is a caterpillar. It is known that caterpillars have α-labelings [20] and that lobsters have ρ-labelings [14]. It is not yet proved that all lobsters have β-labelings (see [11]).

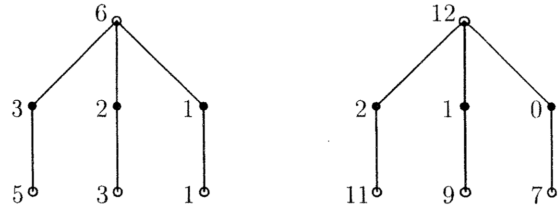

Figure 2: A bigraceful labeling of $S_{3,2}$ and the corresponding ρ^+-labeling

Ringel et al. showed in [18] that lobsters, trees of diameter at most 5, and comets have bigraceful labelings, and they conjectured that all trees have bigraceful labelings. They also note that if f is a bigraceful labeling of a tree T with bipartition (A, B), then

$$\bar{f}(x) = \begin{cases} f(x) - 1 & \text{if } x \in A \\ n + f(x) & \text{if } x \in B \end{cases}$$

is a ρ-labeling of T. It is easy to check that the labeling $\bar{f}$ is in fact a ρ^+-labeling (see Figure 2). Thus the following holds.

Theorem 6 *Lobsters, trees of diameter at most 5, and comets have ρ^+-labelings.*

In [20] Rosa proved that if an Eulerian graph G with n edges has a β-labeling, then $|E(G)| \equiv 0 \text{ or } 3 \pmod 4$. The proof of this fact easily extends to disconnected graphs with Eulerian components. It is also easy to see that the same holds if G has a θ-labeling or a bigraceful labeling. However an Eulerian graph with $|E(G)| \equiv 2 \pmod 4$ can have a ρ^+-labeling as seen by Theorem 7.

From a graph-decomposition perspective (Rosa's motivation for introducing labelings in [20]), α-labelings of bipartite graphs are the most attractive labelings. As stated earlier, an α-labeling of a graph G with n edges yields both a cyclic G-decomposition of K_{2nx+1} and of $K_{n,n}$. Bigraceful labelings [18] are also attractive for these same reasons. However, for bipartite graphs in general, both of these labelings are quite restrictive and it is easy to construct graphs which do not admit either labeling. For example, if G is a forest with more than one component, then G cannot admit an α-labeling. Similarly, if G consists of Eulerian components and if $|E(G)|$ is congruent to 2 modulo 4, then G cannot admit either labeling. Neither of the restrictions in the previous examples would automatically prevent G from admitting a ρ^+-labeling. Although a ρ^+-labeling of a graph G with n edges does not necessarily yield a cyclic G-decomposition of $K_{n,n}$, Theorem 5 makes ρ^+-labelings quite useful. In addition to the graphs in Theorem 6, we will show that all even cycles have ρ^+-labelings and that the disjoint union of graphs with α-labelings has the stronger θ-labeling.

Theorem 7 *Let n be an even integer greater than 2. Then C_n has a ρ^+-labeling.*

Proof. Note that if 4 divides n, then C_n has an α-labeling [20], and so a ρ^+-labeling. Thus we can assume $n = 2m$, where m is odd. Let the vertices of C_{2m} be 1, 2, ..., $2m$, where i and j are adjacent whenever $j \equiv i+1 \pmod{2m}$. We define h on the vertices of C_{2m} by

$$h(i) = \begin{cases} (i-1)/2, & i \text{ odd}, \\ 2m - i/2, & i \text{ even}, 0 < i < m-1, \\ 2m-1-i/2, & i \text{ even}, m-1 \le i < 2m, \\ 2m+1, & i = 2m. \end{cases}$$

We will show that h is a ρ^+-labeling (see Figure 3). This is easily checked for $m = 3$ so we assume $m > 3$.

Let $A = \{i : 0 < i < 2m, i \text{ odd}\}$, $B_1 = \{i : 0 < i < m-1, i \text{ even}\}$, and $B_2 = \{i : m-1 \le i < 2m, i \text{ even}\}$. Clearly h is increasing on A and decreasing on $B_1 \cup B_2$. Also, if $a \in A$ and $b \in B_1 \cup B_2$, then $0 \le h(a) \le h(2m-1) = m-1 < m = h(2m-2) \le h(b) \le h(2) = 2m-1 < 2m+1 = h(2m)$. Thus h is a one-to-one function from the vertices of C_{2m} into $[0, 2(2m)]$. Furthermore, C_{2m} is bipartite, with bipartition (A, B), where $B = B_1 \cup B_2 \cup \{2m\}$, and if $\{a, b\}$ is an edge with $a \in A$ and $b \in B$, then $h(a) < h(b)$.

It remains to show that as $\{x, y\}$ runs through the edges of C_{2m}, the values of $h(x) - h(y)$ are distinct modulo $4m+1$. Note that if $i \in B_1 \cup B_2$, then i is adjacent to $i-1$ and $i+1$ in A. For $i \in B_1 = \{2, 4, \ldots, m-3\}$ the values of $h(i) - h(i-1) = 2m - i + 1$ are $m+4, m+6, \ldots, 2m-1$, while the values of $h(i) - h(i+1) = 2m - i$ are $m+3, m+5, \ldots, 2m-2$. Together these comprise exactly the set $[m+3, 2m-1]$.

Similarly for $i \in B_2 = \{m-1, m+1, \ldots, 2m-2\}$ the values of $h(i) - h(i-1) = 2m - i$ are $2, 4, \ldots, m+1$, while the values of $h(i) - h(i+1) = 2m - i - 1$ are $1, 3, \ldots, m$. Together these comprise the set $[1, m+1]$.

Note that $h(2m) - h(2m-1) = m+2$ and $h(2m) - h(1) = 2m+1$. Thus for edges $\{a, b\}$ with $a \in A$ and $b \in B$ the values of $h(b) - h(a)$ form the set $S = [1, 2m-1] \cup \{2m+1\}$. Then the differences $h(a) - h(b)$ comprise the set $\{-2m-1\} \cup [-2m+1, -1]$, which is elementwise congruent to $S' = \{2m\} \cup [2m+2, 4m]$ modulo $4m+1$. Since S and S' are disjoint subsets of $[1, 4m]$, h is a ρ^+-labeling. ∎

Corollary 8 *If $n \ge 4$ is even, then there exists a cyclic C_n-decomposition of K_{2nx+1} for all positive integers x.*

Kotzig [15] was the first to construct cyclic C_n-decompositions of K_{2nx+1} for $n \equiv 0 \pmod 4$ and Rosa [19] constructed cyclic C_n-decompositions of K_{2nx+1} for $n \equiv 2 \pmod 4$. Thus Corollary 8 is not new. However, Theorem 7 is a new result.

Next we turn our attention to labelings of disconnected graphs. One section of Gallian's survey [11] is dedicated to such labelings. Some of the investigated disconnected graphs are $K_{m,n} \cup G$ (see [4]), $C_{4t} \cup K_{1,4t-1}$ and $C_{4t+3} \cup K_{1,4t+2}$ (in [3]), $C_s \cup P_n$ (in [10] and more recently in [6]), and the graph consisting of unions of cycles

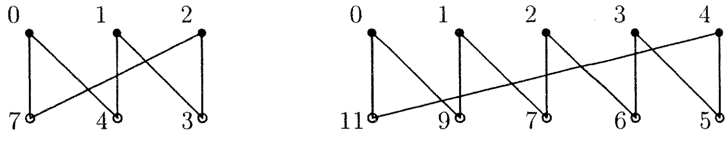

Figure 3: ρ^+-labelings of C_6 and of C_{10}

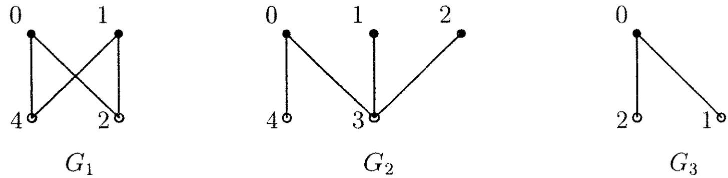

Figure 4: Three graphs each with an α-labeling

(see [16], [1], [2], and [9]). Unfortunately, some authors do not distinguish between β- and α-labelings. In a forthcoming article [8], we investigate sufficent conditions for the vertex-disjoint union G of graphs with α-labelings to have an α-labeling. Such a graph G need not admit an α-labeling. For example if $|E(G)| < |V(G)| - 1$ then G cannot admit an α-labeling. Another example is the graph consisting of 3 disjoint copies of C_4; this graph does not have an α-labeling (see [1] or [9]) although both C_4 and the graph consisting of two disjoint copies of C_4 admit α-labelings.

Theorem 9 *The disjoint union of graphs with α-labelings has a θ-labeling.*

Proof. For $i \in [1, k]$ let G_i be a graph with n_i edges having an α-labeling h_i with critical value λ_i and vertex bipartition (A_i, B_i), where $A_i = \{v \in V(G_i) : h_i(v) \leq \lambda_i\}$. Let $N = \sum_{j=1}^{k} n_j$. We define integers α_i and β_i for $i \in [1, k]$ by

$$\alpha_i = \begin{cases} \dfrac{i-1}{2} + \displaystyle\sum_{j \text{ odd},\ j < i} \lambda_j & i \text{ odd} \\ N + \displaystyle\sum_{j \text{ odd},\ j \leq i} n_j + \displaystyle\sum_{j \text{ even},\ j < i} \lambda_j & i \text{ even,} \end{cases}$$

and

$$\beta_i = \alpha_i + \sum_{j > i} n_j.$$

Assuming the graphs G_i are vertex-disjoint, we define h on their union G by

$$h(v) = \begin{cases} h_i(v) + \alpha_i & v \in A_i, \\ h_i(v) + \beta_i & v \in B_i. \end{cases}$$

We will show that h is one-to-one on $V(G)$. Clearly it is one-to-one on each set A_i and B_i. The idea will be to show that the sequence of sets

$$h(A_1), h(A_3), \ldots, h(B_3), h(B_1), h(A_2), h(A_4), \ldots, h(B_4), h(B_2)$$

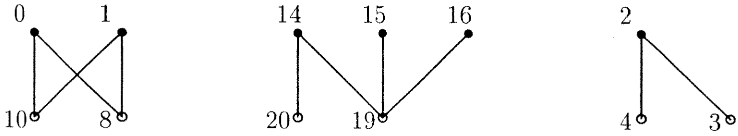

Figure 5: A θ-labeling of $G_1 \cup G_2 \cup G_3$ from Figure 4

has the property that the maximum of each set in the sequence is less than the minimum of the next set in the sequence. Note that since the differences 1 and n_i must be achieved by the α-labeling h_i on G_i, there exist vertices w and x in A_i such that $h_i(w) = 0$ and $h_i(x) = \lambda_i$, and vertices y and z in B_i such that $h_i(y) = \lambda_i + 1$ and $h_i(z) = n_i$.

First assume that i is odd. Note that then

$$\alpha_{i+2} - \alpha_i = 1 + \lambda_i \quad \text{and} \quad \beta_{i+2} - \beta_i = 1 + \lambda_i - n_{i+1} - n_{i+2}.$$

Then $\max h(A_i) = \lambda_i + \alpha_i = \alpha_{i+2} - 1 < 0 + \alpha_{i+2} = \min h(A_{i+2})$. Likewise from the definition of β_i we have $\max h(A_i) = \lambda_i + \alpha_i < \lambda_i + 1 + \beta_i = \min h(B_i)$. Also $\max h(B_{i+2}) = n_{i+2} + \beta_{i+2} = \beta_i + 1 + \lambda_i - n_{i+1} < \lambda_i + 1 + \beta_i = \min h(B_i)$.

Note that $\max h(B_1) = n_1 + \beta_1 = n_1 + \alpha_1 + \sum_{j>1} n_j = 0 + N < 0 + N + n_1 = 0 + \alpha_2 = \min h(A_2)$.

Now assume that i is even. Then

$$\alpha_{i+2} - \alpha_i = \lambda_i + n_{i+1} \quad \text{and} \quad \beta_{i+2} - \beta_i = \lambda_i - n_{i+2}.$$

Thus $\max h(A_i) = \lambda_i + \alpha_i = \alpha_{i+2} - n_{i+1} < 0 + \alpha_{i+2} = \min h(A_{i+2})$. Likewise from the definition of β_i we have $\max h(A_i) = \lambda_i + \alpha_i < \lambda_i + 1 + \beta_i = \min h(B_i)$. Finally $\max h(B_{i+2}) = n_{i+2} + \beta_{i+2} = \lambda_i + \beta_i < \lambda_i + 1 + \beta_i = \min h(B_i)$. This concludes the proof that h is one-to-one.

Notice that the maximum possible value of h is $\max h(B_2) = n_2 + \beta_2 = 2N$, where N is the number of edges of G. Thus h is an injection from $V(G)$ into $[0, 2N]$.

Now the edge labels corresponding to the edges between A_i and B_i are $\{h(b) - h(a) : a \in A_i, b \in B_i, \ \{a, b\} \in E(G_i)\} = \{h_i(b) - h_i(a) + \beta_i - \alpha_i : a \in A_i, b \in B_i, \{a, b\} \in E(G_i)\} = [1 + \beta_i - \alpha_i, n_i + \beta_i - \alpha_i] = [\sum_{j>i} n_j + 1, \sum_{j \geq i} n_j]$, and the union of these sets is $[1, N]$. Thus h is a θ-labeling for G. ∎

Corollary 10 *If G is any graph with n edges that is the disjoint union of graphs with α-labelings, then K_{2nx+1} has a cyclic G-decomposition for every positive integer x.*

Proof. By Theorem 9 G has a θ-labeling, which is a ρ^+-labeling. Thus Theorem 5 applies. ∎

It has become a tradition for researchers in this area to introduce variations on Rosa's original labelings and to conjecture that all trees have these labelings. We join the ranks of the guilty.

Conjecture 6 *All forests have ρ^+-labelings.*

While Conjecture 6 may be a needless addition to an area littered with difficult conjectures, we have exercised some self-restraint. Interested readers are challenged to find a bipartite graph without isolated vertices that does not admit a ρ^+-labeling. We do not believe that such a graph exists.

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