The existence of bimatching designs

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Abstract

A collection of k-matchings of the bipartite graph $K_{n,n}$ with the property that every pair of independent edges lies in exactly λ of the k-matchings is called a BIMATCH(n, k, λ)-design. In this paper we give a new construction of bimatching designs, and show that the necessary conditions for the existence of a BIMATCH(n, k, λ)-design are also sufficient whenever $k = 3$ and 4.

1 Introduction

A hyperfactorization of index λ of the complete graph K_{2n} consists of a family of perfect matchings of K_{2n}, so that every pair of independent edges of K_{2n} lies in exactly λ of the perfect matchings. This has been studied by Jungnickel and Vanstone [5]. Alspach and Heinrich [1] have given a generalization of the notion of a hyperfactorization, and called it a matching design. A matching design, denoted by MATCH(n, k, λ)-design, is a family of k-matchings (i.e., k independent edges) of K_n so that every pair of independent edges of K_n lies in exactly λ members of the k-matchings.

An analogous definition is given for the bipartite graph $K_{n,n}$, and this is called a bimatching design.

Definition 1.1 [1] A bimatching design, denoted by BIMATCH(n, k, λ)-design, is a collection of k-matchings (i.e., k independent edges) of $K_{n,n}$ so that every pair of independent edges of $K_{n,n}$ lies in exactly λ members of the k-matchings.

We have the following necessary conditions for the existence of BIMATCH(n, k, λ)-designs.

Theorem 1.2 [1] Necessary conditions for the existence of BIMATCH(n, k, λ)-designs are that

(1) $n \geq k$,

(2) $\lambda(n-1)^2 \equiv 0 \pmod{k-1}$, and

(3) $\lambda n^2(n-1)^2 \equiv 0 \pmod{k(k-1)}$.

Now the problem arises whether these necessary conditions are also sufficient, especially for small $k = 3$ and 4. From Alspach and Heinrich [1] and Lin and Heinrich [6] we only have the following known results.

Theorem 1.3 [1,6]

(1) If k is a prime power, $n \geq k$ and $n \equiv 1$ or $k \pmod{k(k-1)}$, there exists a BIMATCH(n, k, λ)-design.

(2) If n is a prime power and $n \geq 3$, there exists a BIMATCH$(n, 3, \lambda)$-design with λ taking on the following values

(i) $\lambda = 1$, when $n \equiv 1$ or $3 \pmod 6$,

(ii) $\lambda = 2$, when $n \equiv 4 \pmod 6$,

(iii) $\lambda = 3$, when $n \equiv 5 \pmod 6$, and

(iv) $\lambda = 6$, when $n \equiv 2 \pmod 6$.

In this paper we give a new construction of bimatching designs, and show that the necessary conditions for the existence of a BIMATCH(n, k, λ)-design are also sufficient whenever $k = 3$ and 4.

2 A Construction

For our construction, we need the concept of a modified group divisible design, which is a generalization of a group divisible design.

Definition 2.1 [2] A modified group divisible design, denoted by MGD(k, λ, m, n), is a pair $(X, \mathcal{B})$, where

$$X = \{(x_i, y_j) : 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\}$$

is a set order mn and $\mathcal{B}$ is a collection of k-subsets (called blocks) of X satisfying the following conditions:

(1) every pair of points (x_i, y_j) and (x_s, y_t) of X is contained in exactly λ blocks when $i \neq s$ and $j \neq t$,

(2) the pair of points (x_i, y_j) and (x_s, y_t) with $i = s$ or $j = t$ is not contained in any block.

The subset $\{(x_i, y_j) : 0 \leq i \leq m-1\}$ where $0 \leq j \leq n-1$ are called groups and the subset $\{(x_i, y_j) : 0 \leq j \leq n-1\}$ where $0 \leq i \leq m-1$ are called rows.

Now we can state our construction.

Theorem 2.2 A BIMATCH(n,k,λ)-design exists if and only if a MGD(k,λ,n,n) exists.

Proof Let V_1 and V_2 be the two partite sets of a $K_{n,n}$, in which

$$V_1 = \{x_i : 0 \le i \le n-1\},$$
$$V_2 = \{y_j : 0 \le j \le n-1\}.$$

Let (x_i, y_j) be the edge of $K_{n,n}$, in which the end vertices are $x_i \in V_1$ and $y_j \in V_2$, and let $E(K_{n,n})$ be the edgeset of $K_{n,n}$. We then have

$$E(K_{n,n}) = \{(x_i, y_j) : 0 \le i \le n-1,\ 0 \le j \le n-1\}.$$

Then B is a k-matching of the BIMATCH(n,k,λ)-design if and only if B is a block of MGD(k,λ,n,n).

This complete the proof.

3 Main Result

From Assaf [2], we have the following necessary conditions for the existence of an MGD(k,λ,m,n).

Lemma 3.1 [2] Necessary conditions for the existence of MGD(k,λ,m,n) are that

(1) $m, n \ge k$,

(2) $\lambda(mn+1-m-n) \equiv 0 \pmod{k-1}$, and

(3) $\lambda mn(mn+1-m-n) \equiv 0 \pmod{k(k-1)}$.

In particular, necessary conditions for the existence of a MGD(k,λ,n,n) are that

(1) $n \ge k$,

(2) $\lambda(n-1)^2 \equiv 0 \pmod{k-1}$, and

(3) $\lambda n^2(n-1)^2 \equiv 0 \pmod{k(k-1)}$.

From Assaf [2] and [3], Wei [7] and Assaf and Wei [4], we have the following known results for the existence of MGD(k,λ,n,n).

Lemma 3.2 [2,3,4,7]

(1) There exists a MGD$(3,\lambda,n,n)$ if and only if $n \ge 3$, $\lambda(n-1)^2 \equiv 0 \pmod 2$ and $\lambda n^2(n-1)^2 \equiv 0 \pmod 6$.

(2) There exists a MGD$(4, \lambda, n, n)$ if and only if $n \geq 4$, $\lambda(n-1)^2 \equiv 0 \pmod 3$ and $\lambda n^2(n-1)^2 \equiv 0 \pmod{12}$.

This establishes our main result.

Theorem 3.3

(1) There exists a BIMATCH$(n, 3, \lambda)$-design if and only if $n \geq 3$, $\lambda(n-1)^2 \equiv 0 \pmod 2$ and $\lambda n^2(n-1)^2 \equiv 0 \pmod 6$.

(2) There exists a BIMATCH$(n, 4, \lambda)$-design if and only if $n \geq 4$, $\lambda(n-1)^2 \equiv 0 \pmod 3$ and $\lambda n^2(n-1)^2 \equiv 0 \pmod{12}$.

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