Packings and coverings of $v = 3m + 1$ points with near-triangle factors

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Abstract

We determine, for each $v \equiv 1 \bmod 3$, the maximum (*resp.* minimum) number of near-triangle factors possible on v points so that each pair of distinct points occurs in a triple in at most (*resp.* at least) one of the near-triangle factors. In particular, we show that for each $v \equiv 1 \bmod 3, v \neq 7$, there exists a near-resolvable exact 2-covering of v points by triples whose near-triangle factors admit a partition into a maximum packing and a minimum covering of v points by near-triangle factors.

1 Introduction

We assume a basic familiarity with the terminology and notations of design theory and graph theory; we refer the reader to [3] and [12] for general references.

Let X be a set of v points. A *packing* (*resp. covering*) of X is a collection B of subsets of X (called *blocks*) such that any pair of distinct points in X occur together in at most (*resp.* at least) one block in the collection. The case where each pair of points belongs to *exactly* one block is called an *exact covering.* A packing or covering is called *near-resolvable* if its block set can be partitioned into *near-parallel classes*, each near-parallel class being a partition of $X \backslash \{x\}$ for some $x \in X$ (we will refer to $x = x(P)$ as being the *residue* of the near-parallel class P). In this paper, we are concerned with near-resolvable packings and coverings of a v-set by triples (whence $v \equiv 1 \bmod 3$); we refer to a near-parallel class of triples as being a near-triangle factor. We prove that for each $v \equiv 1 \bmod 3$, $v \neq 7$, there exists a near-resolvable exact 2-covering of v points by triples whose near-triangle factors admit a partition into a maximum packing and a minimum covering of v points by near-triangle factors.

We will determine, for each $v \equiv 1 \bmod 3$, the maximum (*resp.* minimum) number of near-triangle factors possible on v points so that each pair of distinct points occurs in a triple in at most (*resp.* at least) one of the near-triangle factors. Now, Colbourn and Zhao [5] have in fact determined the solution for the packing problem, in the context of unipolar communication systems. We will require that the packings have an additional property, however, that being that there be a minimum covering of the same v points such that the uncovered pairs in the packing coincide precisely with the doubly-covered pairs in the covering: see Theorem 1.8.

Thus we are, in a sense, considering an analogue of Kirkman's famous Schoolgirl Problem in the case where the number of schoolgirls is congruent to 1 mod 3. Such analogues have been considered in the past. In particular, Hanani [7] determined the spectrum for exact 2-coverings of a v-set by near-triangle factors, i.e. each pair of distinct points occur together in exactly two triples:

Theorem 1.1 *There exists a near-resolvable exact 2-covering of v points by triples if and only if $v \equiv 1$ mod 3.*

Carter *et al.* [1] considered what they called Hanani Triple Systems (HTS's), i.e. a Steiner Triple System on v points whose block set can be partitioned into $(v-1)/2$ near-triangle factors, together with one further class of $(v-1)/6$ disjoint triangles (which exactly cover the residues of the $(v-1)/2$ near-triangle factors):

Theorem 1.2 *There exists a Hanani Triple System HTS(v) if and only if $v \equiv 1$ mod 6 and $v \geq 19$.*

More recently, Černý, Horák and Wallis [2] considered the problem of determining the maximum number of factors possible in a packing on $v \equiv 1$ mod 3 points, where each factor is composed of the disjoint union of a block of size 4 with $(v-4)/3$ triples. A maximum packing of this type whose leave is a triangle (K_3) when $v \equiv 1$ mod 6, or the disjoint union of a K_4 and $(v-4)/2$ edges (K_2's) when $v \equiv 4$ mod 6, is called a Canonical Kirkman Packing Design (CKPD(v)). The following was determined in [2], [9], and [4]:

Theorem 1.3 *There exists a Canonical Kirkman Packing Design CKPD(v) if and only if $v \equiv 1$ mod 3 and $v \geq 22$, except possibly for v =55,61,67,73,85 and 109.*

The authors in [9] also defined a Canonical Kirkman Covering Design (CKCD(v)) to be a minimum covering of $v \equiv 1$ mod 3 points by factors of the foregoing type, whose excess is a near-triangle factor when $v \equiv 1$ mod 6, or the disjoint union of $(v-4)/2$ edges (K_2's) when $v \equiv 4$ mod 6. They determined the following result.

Theorem 1.4 *A Canonical Kirkman Covering Design CKCD(v) exists if and only if $v \equiv 1$ mod 3 and $v > 10$, except possibly for v =13,16 and 67.*

The problem that we are considering here may be posed in terms of Kirkman's original problem, as follows: Given a class of $v \equiv 1 \bmod 3$ students, what is the maximum (*resp.* minimum) number of outings possible if, on each outing, some student is designated the *leader* and the remaining students line up in rows of three behind the leader, and if, over the course of the outings, each pair of students walks together in the same row at most (*resp.* at least) once? Now, as with the packing and covering designs introduced in [2], there are many types of solutions in terms of the various leaves and excesses that may occur. For example, if $v \equiv 4 \bmod 6$, then a solution to our packing problem can be obtained by simply designating one student to be the leader on all outings and constructing a Kirkman Triple System on the remaining $v - 1$ students, for a total of $v/2 - 1$ outings. This is not a particularly interesting solution, however. (Indeed, if our packing was designed to be a tournament schedule on v players (where each game involves 3 simultaneous participants) then the foregoing solution would be useless, since it would assign the *bye* to the same player in every round.) Thus, we will say that a solution to our foregoing packing/covering problem is *equitable* if

(i) no point (or student, or player) occurs as the residue (or leader, or bye) in more than one near-triangle factor (or outing, or round), and, for the covering problem,

(ii) no pair of points occur together in more than two triples in the covering.

We will refer to an equitable solution to the packing problem as an *equitable Kirkman Packing Design* (EKPD) if it has $\lfloor (v-1)/2 \rfloor$ near-triangle factors, and we will refer to an equitable solution to the covering problem as an *equitable Kirkman Covering Design* (EKCD) if it has $\lceil (v+1)/2 \rceil$ near-triangle factors. Note that these numbers obviously represent the maximum (*resp.* minimum) possible number of near-triangle factors in *any* solution to the packing (*resp.* covering) problem, as a near-triangle factor covers $v - 1$ pairs. An interesting phenomenon occurs when we examine the leave and excess graphs of these designs. Suppose first that $v \equiv 1 \bmod 6$. Then an EKPD(v) has as its leave a 2-regular graph on the $(v-1)/2$ residues, for a total of $(v-1)/2$ edges, while an EKCD(v) has as its excess a 2-regular graph on the $(v-1)/2$ *non*-residues, again for a total of $(v-1)/2$ edges. On the other hand, suppose that $v \equiv 4 \bmod 6$. Then an EKPD(v) contains $v/2-1$ near-triangle factors, whence its leave is a graph on v vertices in which each of the $v/2 - 1$ residues has degree 3 and each of the $v/2+1$ non-residues has degree 1, for a total of $v-1$ edges; an EKCD(v) contains $v/2+1$ near-triangle factors, whence its excess is a graph on v vertices in which each of the $v/2 - 1$ *non*-residues has degree 3 and each of the $v/2+1$ residues has degree 1, again for a total of $v-1$ edges. It is reasonable to ask, therefore, for which $v \equiv 1 \bmod 3$ can we construct an EKPD(v) and an EKCD(v) in which the leave graph of the former is isomorphic to the excess graph of the latter. In such a case we can, by relabelling the points of one of the designs if necessary, form a near-resolvable exact 2-covering of v points by simply taking the union of the set of near-triangle factors in the EKPD(v) with the set of those in the EKCD(v) (see Lemma 1.9 ahead). A near-resolvable exact 2-covering (of v points by triples) will be called *separable* if it arises in this way, i.e. if its set of near-triangle factors can be partitioned into an EKPD(v) and an EKCD(v). Consider the following small examples:

Proposition 1.5 *There exists a separable exact 2-covering for $v = 4, 10,$ and 13.*

Proof. For each $v = 4, 10,$ and 13 we present the 2-covering by listing the near-triangle factors of the EKPD(v) in the left-hand column and those of the EKCD(v) in the right-hand column.

$v = 4$:

1 2 3 (4)	2 3 4 (1)
	3 4 1 (2)
	4 1 2 (3)

$v = 10$:

2 8 9	1 5 7	4 6 10 (3)	1 2 5	3 4 6	7 8 10 (9)
3 5 9	2 6 7	1 8 10 (4)	1 2 6	3 4 5	7 8 9 (10)
4 7 9	3 6 8	2 5 10 (1)	1 3 7	2 4 8	6 9 10 (5)
1 6 9	4 5 8	3 7 10 (2)	1 3 8	2 4 7	5 9 10 (6)
			1 4 9	2 3 10	5 6 8 (7)
			1 4 10	2 3 9	5 6 7 (8)

$v = 13$:

4 7 8	2 9 10	6 11 12	3 5 13 (1)	1 2 5	4 6 12	8 9 10	3 11 13 (7)
1 7 10	2 4 11	5 8 12	6 9 13 (3)	2 3 6	4 5 10	7 9 11	1 12 13 (8)
3 7 9	4 12 13	1 6 8	5 10 11 (2)	1 3 4	5 6 11	7 8 12	2 10 13 (9)
2 7 12	4 5 9	8 10 13	1 3 11 (6)	3 5 12	1 8 11	2 6 7	4 9 13 (10)
7 11 13	4 6 10	2 3 8	1 9 12 (5)	1 6 10	2 9 12	3 4 8	5 7 13 (11)
5 6 7	8 9 11	1 2 13	3 10 12 (4)	2 4 11	3 7 10	1 5 9	6 8 13 (12)
				1 4 7	2 5 8	3 6 9	10 11 12 (13)

□

It is of interest to note that, up to isomorphism, there is only one partial triple system on 13 points which admits a partition into six edge-disjoint near-triangle factors, since the non-existence of an HTS(13) forces the leave of such a system to be a hexagon. (See [6].)

Proposition 1.6 *There does not exist an EKPD(7) or an EKCD(7).*

Proof. It is easy to see that a maximum packing of near-triangle factors on 7 points contains just one near-triangle factor; i.e. if we start with the factor 1 2 3 4 5 6 (7) then any triangle which is edge disjoint from 1 2 3 and 4 5 6 contains 7. Now suppose that there were a covering of 7 points with 4 near-triangle factors. Then the excess would have to be a triangle (K_3), which we call T. Now since there are only 4 near-triangle factors, then we can assume that T appears as a triangle in one of them. But removing T leaves a Steiner Triple System STS(7), from which no (second) near-triangle factor can be extracted, a contradiction. □

Remark 1.7 *From Proposition 1.6, there is no separable exact 2-covering for $v = 7$. We note that there is a covering of 7 points by 5-near-triangle factors,* viz:

$ab1 \quad 235(4)$
$ab2 \quad 341(5)$
$ab3 \quad 452(1)$
$ab4 \quad 513(2)$
$ab5 \quad 124(3)$

This is not an equitable covering, however, as ab appears together in 5 triples in the covering.

In this paper we will prove the following result.

Theorem 1.8 *There exists a separable exact 2-covering of v points by near-triangle factors if and only if $v \equiv 1$ mod 3, except for $v = 7$.*

We will use the following observation, which we noted earlier:

Lemma 1.9 *If there exists an EKPD(v) and an EKCD(v) in which the leave graph of the former is isomorphic to the excess graph of the latter, then there exists a separable exact 2-covering of v points by near-triangle factors.*

Proof. Let G be the leave graph of the EKPD(v) and H be the excess graph of the EKCD(v), where each of G and H have v vertices. Let $\alpha : G \to H$ be an isomorphism, and relabel each point x in the EKPD(v) as $\alpha(x)$. Let $P_1, P_2, \ldots, P_{\lfloor (v-1)/2 \rfloor}$ be the near-triangle factors in the EKPD(v) and $\alpha(P_1), \alpha(P_2), \ldots, \alpha(P_{\lfloor (v-1)/2 \rfloor})$ be these same near-triangle factors with the points relabelled. Let $C_1, C_2, \ldots, C_{\lceil (v+1)/2 \rceil}$ be the near-triangle factors in the EKCD(v). We claim that $C = \{\alpha(P_1), \alpha(P_2), \ldots, \alpha(P_{\lfloor (v-1)/2 \rfloor}), C_1, C_2, \ldots, C_{\lceil (v+1)/2 \rceil}\}$ is an exact 2-covering of v points by near-triangle factors. It is clear from the discussion proceeding Proposition 1.5 that each of the v points appears as a residue in exactly one of the near-triangle factors in C. Moreover, if a and b are any pair of distinct points (in $V(H)$), then either $\{a, b\} \in E(H)$, whereupon a and b appear together in two triples in the EKCD(v) and no triples in the (relabelled) EKPD(v), or $\{a, b\} \notin E(H)$, whereupon a and b appear together in exactly one triple in each of the EKCD(v) and the (relabelled) EKPD(v). In either case, a and b appear together in exactly two triples among the near-triangle factors in C. This establishes our claim. □

2 Preliminaries

In this section we establish the terminology, notation, and some preliminary results that will be used in the sequel.

We begin by establishing the existence of separable exact 2-coverings in the easiest case, that being when $v \equiv 1$ mod 6 :

Theorem 2.1 *There exists a separable exact 2-covering of v points by near-triangle factors for every $v \equiv 1$ mod 6 except $v = 7$.*

Proof. With regards $v = 7$ and 13, see Proposition 1.5 and Remark 1.7. Now let $v \equiv 1 \bmod 6, v \geq 19$. From Theorem 1.2, there exists an HTS(v). Let P be the set of $(v-1)/2$ near-triangle factors in this system. Then P is an EKPD(v) whose leave consists of $(v-1)/6$ vertex disjoint triangles (K_3's) together with $(v+1)/2$ isolated vertices. Let T be the set of $(v-1)/6$ disjoint triangles in the HTS(v) together with a further set of $(v-1)/6$ disjoint triangles, each disjoint from those in the first set. Then $P \cup \{T\}$ is an EKCD(v) whose excess consists of $(v-1)/6$ vertex disjoint triangles (K_3's) together with $(v+1)/2$ isolated vertices. Now apply Lemma 1.9. □

We are thus henceforth concerned with the case $v \equiv 4 \bmod 6$.

A *group-divisible design* (GDD) is a triple (X, G, B) where X is a set of points, G is a partition of X into *groups*, and B is a collection of subsets of X (blocks) so that any pair of distinct points occur together in either one group or exactly one block, but not both. A K-GDD of type $g_1^{t_1} g_2^{t_2} \dots g_m^{t_m}$ has t_i groups of size $g_i, i = 1, 2, \dots, m$ and $|B_j| \in K$ for every block $B_j \in B$. A *transversal design* TD (k, m) is a $\{k\}$-GDD of type n^k; it is well known that a TD (k, n) is equivalent to $k-2$ mutually orthogonal Latin squares of order n. A GDD is called *resolvable* if its block set B admits a partition into *parallel* classes, each parallel class being a partition of the point set X. A GDD is called *frame resolvable* if its block set B admits a partition into *holey* parallel classes, each holey parallel class being a partition of $X - G_j$ for some $G_j \in G$. A *Kirkman frame* is a frame resolvable GDD in which all the blocks have size three. It is a simple consequence of the definition that to each group G_j in a Kirkman frame (X, G, B) there correspond exactly $\frac{1}{2}|G_j|$ holey parallel classes of triples that partition $X - G_j$. The groups in a Kirkman frame are often referred to as *holes*. Kirkman frames were formally introduced by Stinson [11], who established their spectrum in the case where all of the holes have the same size.

Theorem 2.2 *A Kirkman frame of type g^u exists if and only if $u \geq 4$, g is even and $g(u-1) \equiv 0 \bmod 3$.*

We will also require Kirkman frames in which the holes are not all of the same size. To get these, we use Stinson's 'weighting' construction (see [11]):

Construction 2.3 *Suppose that there is a K-GDD of type $g_1^{t_1} g_2^{t_2} \dots g_m^{t_m}$ and that for each $k \in K$ there is a Kirkman frame of type h^k. Then there exists a Kirkman frame of type $(hg_1)^{t_1}(hg_2)^{t_2} \dots (hg_m)^{t_m}$.*

Before proceeding, we will require the notion of an *incomplete* equitable packing (covering) design. Let $v \equiv w \equiv 4 \bmod 6$. An EKPD($v$)–EKPD($w$) is a triple (X, Y, B) where X is a set of v points, Y is a subset of X of size w (Y is called the *hole*), and B is a collection of 3-subsets of X (triples) such that

(i) $|Y \cap B_i| \leq 1$ for all $B_i \in B$;

(ii) any pair of distinct elements in X occur together in Y or in at most one triple;

(iii) B admits a partition into $\frac{1}{2}(v-w)$ near-parallel classes on X, each of which has a distinct residue in $X\backslash Y$, and a further $\frac{1}{2}(w-2)$ holey parallel classes of triples on $X\backslash Y$.

An EKCD(v)–EKCD(w) is defined similarly, changing 'at most' to 'at least' in Condition (ii), and requiring $\frac{1}{2}(w+2)$ holey parallel classes of triples on $X\backslash Y$ in Condition (iii), with the further requirement that no pair of points occur together in more than two triples of B.

The following is an immediate consequence of the definitions.

Proposition 2.4 *If there exist an EKPD(v)–EKPD(w) (resp. EKCD(v)–EKCD(w)) and an EKPD(w) (resp. EKCD(w)) then there exists an EKPD(v) (resp. EKCD(v)).*

Example 2.5 *An EKPD(16)–EKPD(4) and an EKCD(16)–EKCD(4).*

EKPD(16)–EKPD(4)

$$X = (\mathbb{Z}_6 \times \{1,2\}) \cup \{a_0, a_1, b_0, b_1\}$$
$$Y = \{a_0, a_1, b_0, b_1\}$$

Near-Parallel Classes:

Develop the class $0_1 0_2 1_2$ $a_0 1_1 5_2$ $a_1 2_1 4_2$ $b_0 4_1 3_2$ $b_1 5_1 2_2$ (3_1) *mod* 6, *where* a_0, a_1, b_0, b_1 *are fixed points;*

Holey Parallel Classes:

$0_1 2_1 4_1$ $1_1 3_1 5_1$ $0_2 2_2 4_2$ $1_2 3_2 5_2$.

EKCD(16)–EKCD(4)

X and Y as above.

Near-Parallel Classes:

Develop the class $2_1 5_1 2_2$ $a_0 0_2 3_2$ $a_1 0_1 1_1$ $b_0 4_2 5_2$ $b_1 3_1 4_1$ (1_2) *mod* 6, *where the subscripts on* a *and* b *are evaluated mod* 2;

Holey Parallel Classes:

Develop the triples $0_1 2_1 1_2$ *and* $4_1 0_2 2_2$ *mod* 6, *for 3 holey parallel classes.*

Lemma 2.6 *There is a separable exact 2-covering for* $v = 16$.

Proof. For an EKPD(16), apply Proposition 2.4 to the EKPD(16)–EKPD(4) in Example 2.5 and the EKPD(4)($a_0a_1b_0$ (b_1)) in Proposition 1.5; for an EKCD(16), apply Proposition 2.4 to the EKCD(16)–EKCD(4) in Example 2.5 and the EKCD(4) ($a_1b_0b_1$ (a_0) $b_0b_1a_0$ (a_1) $b_1a_0a_1$ (b_0)) in Proposition 1.5. It is easily checked that the leave graph of the EKPD (16) is the same as the excess graph of the EKCD(16); now apply Lemma 1.9. □

By the *leave* of an EKPD(v)–EKPD(w) (X, Y, B), we will mean the leave with respect to the point set $X\backslash Y$; similarly, by the *excess* of an EKCD(v)–EKCD(w) (X, Y, B), we will mean the excess with respect to the point set $X\backslash Y$.

We now illustrate the main technique that we will be using throughout the remainder of the paper (which is a variant of Stinson's 'Filling in Holes' construction, see [11]) by establishing the following result.

Theorem 2.7 *There exists a separable exact 2-covering of v points by near-triangle factors for every $v \equiv 4 \bmod 12$ with $v \geq 52$.*

Proof. From Theorem 2.2, there exists a Kirkman frame of type $12^{(v-4)/12}$ on the point set $X = \{1,2,3,\ldots,12\} \times \{j : 1 \leq j \leq (v-4)/12\}$, having holes $G_j = \{1,2,3,\ldots,12\} \times \{j\}$ for $j = 1,2,\ldots,(v-4)/12$. To each hole G_j, there correspond 6 holey parallel classes $P_{j1}, P_{j2}, \ldots, P_{j6}$ of triples that partition $X - G_j$. Now adjoin four new points $\{\infty_1, \infty_2, \infty_3, \infty_4\}$ to this frame.

For each hole G_j, construct a copy of the EKPD (16)–EKPD(4) on $G_j \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$, aligning the hole in the incomplete EKPD on $\{\infty_1, \infty_2, \infty_3, \infty_4\}$, having $C_{j1}, C_{j2}, \ldots, C_{j6}$ as its near-parallel classes. Then for each $j = 1,2,\ldots,(v-4)/12, \mathcal{P}_j = \{C_{j1} \cup P_{j1}, C_{j2} \cup P_{j2}, \ldots, C_{j6} \cup P_{j6}\}$ is a set of six near-parallel classes on $X \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$; it is clear that $\cup_j \mathcal{P}_j$, together with the single holey parallel class on X formed by the union of the holey parallel classes on each G_j, forms an EKPD(v)–EKPD(4).

Similarly, for each hole G_j we can construct a copy of the EKCD(16)–EKCD(4) on $G_j \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$, aligning the hole in the incomplete EKCD on $\{\infty_1, \infty_2, \infty_3, \infty_4\}$, having $D_{j1}, D_{j2}, \ldots, D_{j6}$ as its near-parallel classes and H_{j1}, H_{j2}, H_{j3} as its holey parallel classes. Then for each $j = 1,2,\ldots,(v-4)/12, \mathcal{C}_j = \{D_{j1} \cup P_{j1}, D_{j2} \cup P_{j2}, \ldots, D_{j6} \cup P_{j6}\}$ is a set of six near-parallel classes on $X \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. Moreover, each of $\cup_j H_{ji}$ is a holey parallel class on X, where $i = 1,2,3$. Thus, $\cup_j \mathcal{C}_j$ together with these 3 holey parallel classes form an EKCD(v)–EKCD(4).

Now we just fill the hole of size 4 in each of these designs, as we did in Lemma 2.6, to obtain an EKPD(v) and an EKCD(v) in which the leave of the former is the same graph as the excess of the latter. Then apply Lemma 1.9. □

We finish this section by establishing the existence of separable exact 2-coverings in the last two 4(mod 12) cases, namely $v = 28$ and 40. In these and all subsequent direct constructions, points labelled ∞_i are fixed points with respect to the relevant automorphism group.

Lemma 2.8 *There is a separable exact 2-covering for $v = 28$ and 40.*

Proof. For each order, we establish the existence of an EKPD(v) and an EKCD(v) in which the leave of the former is isomorphic to the excess of the latter, and then apply Lemma 1.9.

EKPD(28)

Point Set $(\mathbb{Z}_{12} \times \{1,2\}) \cup \{a_0, a_1, \infty_1, \infty_2\}$.

Near-Parallel Classes:

We get 12 near-parallel classes by developing the triples $2_1 7_2 0_2$ $11_2 10_2 1_2$ $0_1 1_1 9_2$ $6_1 9_1 8_2$ $3_1 5_1 6_2$ $a_0 10_1 5_2$ $a_1 8_1 2_2$ $\infty_1 11_1 3_2$ $\infty_2 4_1 4_2$ (7_1) mod 12, where a_0 and a_1 are fixed points. The last class is given by $0_1 4_1 8_1$ $0_2 4_2 8_2$ mod 12 together with $a_1 \infty_1 \infty_2$ (a_0).

EKCD(28)

Point Set $(\mathbb{Z}_{12} \times \{1,2\}) \cup \{a_0, a_1, \infty_1, \infty_2\}$.

Near-Parallel Classes:

We get 12 near-parallel classes by developing the triples $3_1 5_1 7_2$ $8_1 9_1 9_2$ $1_1 7_1 10_2$ $4_1 0_2 2_2$ $0_1 5_2 11_2$ $a_0 2_1 11_1$ $a_1 3_2 6_2$ $\infty_1 10_1 4_2$ $\infty_2 6_1 1_2$ (8_2) mod 12, where the subscripts on a are evaluated mod 2. The last three classes are obtained by developing the triples $0_1 1_1 5_1$ and $0_2 1_2 5_2$ mod 12 (which generate three holey parallel classes on $\mathbb{Z}_{12} \times \{1,2\}$) together with $\infty_1 \infty_2 a_0$ (a_1), $\infty_2 a_0 a_1$ (∞_1), and $a_0 a_1 \infty_1$ (∞_2).

EKPD(40)

Point Set $(\mathbb{Z}_{18} \times \{1,2\}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$.

Near-Parallel Classes:

We get 18 near-parallel classes by developing the triples $11_1 1_2 9_2$ $6_1 13_1 16_1$ $10_2 14_2 17_2$ $9_1 3_2 8_2$ $10_1 15_2 16_2$ $14_1 5_2 7_2$ $0_1 4_1 0_2$ $5_1 7_1 2_2$ $2_1 15_1 4_2$ $\infty_1 12_1 13_2$ $\infty_2 8_1 12_2$ $\infty_3 3_1 6_2$ $\infty_4 1_1 11_2$ (17_1) mod 18. The last class is given by $0_1 6_1 12_1$ $0_2 6_2 12_2$ mod 18 together with $\infty_1 \infty_2 \infty_3$ (∞_4).

EKCD(40)

Point Set $(\mathbb{Z}_{18} \times \{1,2\}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$.

Near-Parallel Classes:

We get 18 near-parallel classes by developing the triples $10_1 16_1 13_2$ $5_1 12_1 11_2$ $6_1 15_1 17_2$ $9_1 17_1 4_2$ $4_1 14_2 16_2$ $2_1 2_2 10_2$ $14_1 5_2 12_2$ $8_1 11_1 13_1$ $0_2 6_2 9_2$ $\infty_1 3_1 7_2$ $\infty_2 1_1 8_2$ $\infty_3 0_1 1_2$ $\infty_4 7_1 3_2$ (15_2) mod 18. The last three classes are obtained by developing the triples $0_1 1_1 5_1$ and

$0_2 1_2 5_2$ mod 18 (which generate three holey parallel classes on $\mathbb{Z}_{18} \times \{1,2\}$) together with $\infty_2\infty_3\infty_4$ (∞_1), $\infty_3\infty_4\infty_1$ (∞_2), and $\infty_4\infty_1\infty_2$ (∞_3).

Note that the leave of the EKPD(28) consists of a 3-edge star, together with edges at pure differences ±5 and ±6 on $\mathbb{Z}_{12} \times \{1\}$, and edges at pure difference ±6 on $\mathbb{Z}_{12} \times \{2\}$; the excess of the EKCD(28) consists of a 3-edge star, together with edges at pure differences ±1 and ±6 on $\mathbb{Z}_{12} \times \{1\}$, and edges at pure difference ±6 on $\mathbb{Z}_{12} \times \{2\}$. The leave of the EKPD(40) consists of a 3-edge star, together with edges at pure differences ±1 and ±9 on $\mathbb{Z}_{18} \times \{1\}$, and edges at pure difference ±9 on $\mathbb{Z}_{18} \times \{2\}$; the excess of EKCD(40) consists of a 3-edge star, together with edges at pure differences ±5 and ±9 on $\mathbb{Z}_{18} \times \{1\}$, and edges at pure difference ±9 on $\mathbb{Z}_{18} \times \{2\}$. For each of the two orders then, the relevant leave and excess graphs are isomorphic. □

Remark 2.9 *Each of the EKPDs and EKCDs constructed in Lemma 2.8 has a subdesign on 4 points; thus (by removing the triples in these subdesigns) we have EKPD*(28)*-EKPD*(4)*, EKCD*(28)*-EKCD*(4)*, EKPD*(40)*-EKPD*(4)*, and EKCD*(40)*-EKCD*(4)*.*

3 The Case $v \equiv 10$ mod 12

In this section, we construct separable exact 2-coverings for the remaining orders $v \equiv 10$ mod 12.

Lemma 3.1 *There exist separable exact 2-coverings for v = 10, 22, 34, 46, 58, 70 and 82.*

Proof. For $v = 10$ see Proposition 1.5. For $v = 22, 34$ and 46, we proceed as in the proof of Lemma 2.8, viz:

EKPD(22)

Point Set $(\mathbb{Z}_9 \times \{1,2\}) \cup \{a_0, a_1, a_2, \infty_1\}$.

Near-Parallel Classes:

We get 9 near-parallel classes by developing the triples $4_1 1_2 5_2$ $6_1 4_2 6_2$ $0_1 5_1 8_2$ $a_0 1_1 8_1$ $a_1 7_1 0_2$ $a_2 2_2 3_2$ $\infty_1 3_1 7_2$ (2_1) mod 9, where the subscripts on a are evaluated mod 3. The last class is given by $0_1 3_1 6_1$ $0_2 3_2 6_2$ mod 9 together with $a_0 a_1 a_2$ (∞_1).

EKCD(22)

Point Set $(\mathbb{Z}_9 \times \{1,2\}) \cup \{a_0, a_1, a_2, \infty_1\}$.

Near-Parallel Classes:

We get 9 near-parallel classes by developing the triples $7_1 0_1 7_2$ $1_1 2_1 5_1$ $0_2 2_2 5_2$ $a_0 6_1 3_2$ $a_1 3_1 6_2$ $a_2 8_1 4_2$ $\infty_1 4_1 8_2 (1_2)$ mod 9, where a_0, a_1, and a_2 are fixed points. The last three classes are obtained by developing the triples $0_1 1_2 2_2$ and $1_1 2_1 0_2$ mod 9 (which generate three holey parallel classes on $\mathbb{Z}_9 \times \{1,2\}$) together with $a_1 a_2 \infty_1$ (a_0), $a_2 \infty_1 a_0$ (a_1), and $\infty_1 a_0 a_1$ (a_2).

EKPD(34)

Point Set $(\mathbb{Z}_{15} \times \{1,2\}) \cup \{a_0, a_1, a_2, \infty_1\}$.

Near-Parallel Classes:

We get 15 near-parallel classes by developing the triples $13_1 6_2 8_2$ $0_1 11_2 14_2$ $2_1 6_1 4_2$ $3_1 3_2 7_2$ $8_1 11_1 5_2$ $a_0 12_1 13_2$ $a_1 14_1 2_2$ $a_2 4_1 10_2$ $\infty_1 7_1 12_2$ $1_1 9_1 10_1$ $0_2 1_2 9_2$ (5_1) mod 15 where a_0, a_1, and a_2 are fixed points. The last class is given by $0_1 5_1 10_1$ $0_2 5_2 10_2$ mod 15 together with $a_0 a_1 a_2$ (∞_1).

EKCD(34)

Point Set $(\mathbb{Z}_{15} \times \{1,2\}) \cup \{a_0, a_1, a_2, \infty_1\}$.

Near-Parallel Classes:

We get 15 near-parallel classes by developing the triples $11_1 13_1 5_2$ $4_1 7_1 6_2$ $5_1 2_2 9_2$ $2_1 10_1 13_2$ $6_1 1_2 7_2$ $0_1 9_1 0_2$ $3_1 8_2 11_2$ $a_0 10_2 12_2$ $a_1 12_1 14_1$ $a_2 8_1 4_2$ $\infty_1 1_1 14_2$ (3_2) mod 15, where the subscripts on a are evaluated mod 3. The last three classes are obtained by developing the triples $0_1 1_1 5_1$ and $0_2 1_2 5_2$ mod 15 (which generate three holey parallel classes on $\mathbb{Z}_{15} \times \{1,2\}$) together with $a_1 a_2 \infty_1$ (a_0), $a_2 \infty_1 a_0$ (a_1), and $\infty_1 a_0 a_1$ (a_2).

EKPD(46)

Point Set $(\mathbb{Z}_{21} \times \{1,2\}) \cup \{a_0, a_1, a_2, \infty_1\}$.

Near-Parallel Classes:

We get 21 near-parallel classes by developing the triples $13_1 10_2 19_2$ $17_1 19_1 20_1$ $1_2 7_2 11_2$ $9_1 15_1 9_2$ $1_1 12_1 8_2$ $5_1 14_1 6_2$ $3_1 7_1 5_2$ $11_1 2_2 4_2$ $2_1 13_2 18_2$ $16_1 0_2 3_2$ $6_1 15_2 16_2$ $a_0 0_1 8_1$ $a_1 12_2 20_2$ $a_2 18_1 17_2$ $\infty_1 10_1 14_2$ (4_1) mod 21, where the subscripts on a are evaluated mod 3. The last class is given by $0_1 7_1 14_1$ $0_2 7_2 14_2$ mod 21 together with $a_0 a_1 a_2$ (∞_1).

EKCD(46)

Point Set $(\mathbb{Z}_{21} \times \{1,2\}) \cup \{a_0, a_1, a_2, \infty_1\}$.

Near-Parallel Classes:

We get 21 near-parallel classes by developing the triples $8_1 10_1 5_2$ $0_1 20_1 20_2$ $4_1 19_1 17_2$ $3_1 12_1 14_2$ $11_1 18_1 4_2$ $13_1 12_2 18_2$ $5_1 6_2 9_2$ $7_1 1_2 15_2$ $16_1 7_2 19_2$ $0_2 2_2 13_2$ $6_1 9_1 17_1$ $a_0 15_1 11_2$ $a_1 1_1 10_2$ $a_2 14_1 3_2$ $\infty_1 2_1 8_2$ (16_2) mod 21, where a_0, a_1, and a_2 are fixed points. The last three classes are obtained by developing the triples $0_1 1_1 5_1$ and $0_2 1_2 5_2$ mod 21 (which generate three holey parallel classes on $\mathbb{Z}_{21} \times \{1, 2\}$) together with $a_1 a_2 \infty_1$ (a_0), $a_2 \infty_1 a_0$ (a_1), and $\infty_1 a_0 a_1$ (a_2).

Note that in each of the foregoing packings (*resp.* coverings) the leave (*resp.* excess) consists of a 3-edge star on $\{a_0, a_1, a_2, \infty_1\}$ together with a 'sun' on $\mathbb{Z}_{(v-4)/2} \times \{1, 2\}$, i.e. a $(v-4)/2$-cycle on $\mathbb{Z}_{(v-4)/2} \times \{1\}$ and a matching M on $\mathbb{Z}_{(v-4)/2} \times \{1, 2\}$ where each edge in M has one end-vertex in orbit 1 and one end-vertex in orbit 2. For each of the three orders then, the relevant leave and excess graphs are isomorphic.

There remain the orders $v = 58, 70$, and 82. For $v = 58$, we construct EKPD(58)–EKPD(16) and EKCD(58)–EKCD(16), in which the leave of the former is isomorphic to the excess of the latter, in the Appendix. Now apply Proposition 2.4, filling in the EKPD(16) (*resp.* EKCD(16)) from Lemma 2.6. Similarly, for $v = 70$ and 82, we construct EKPD(v)–EKPD(22) and EKCD(v)–EKCD(22) in the Appendix, and then apply Proposition 2.4, filling in EKPD(22) or EKCD(22) from above. □

Remark 3.2 *Each of the EKPDs and EKCDs of orders* 22, 34, *and* 46 *constructed in Lemma 3.1 has a subdesign on 4 points; thus as with Remark 2.9 we have EKPD*(v)*–EKPD*(4) *and EKCD*(v)*–EKCD*(4) *for* $v = 22, 34$ *and* 46.

Lemma 3.3 *If there is a GDD on s points with block sizes from the set* $\{k \in \mathbb{Z} : k \geq 4\}$ *and group sizes from the set* $\{1, 2, \ldots, 8\}$, *in which there is at most one group of size 1, then there is a separable exact 2-covering of* $v = 6s + 4$ *points by near-triangle factors.*

Proof. Let the given GDD have type $g_1^{t_1} g_2^{t_2} \ldots g_m^{t_m}$. Apply Construction 2.3 to this GDD, using 'weight' $h = 6$, to yield a Kirkman frame of type $(6g_1)^{t_1}(6g_2)^{t_2} \ldots (6g_m)^{t_m}$. If there is a group of size 1 in the GDD, then we assume that $g_m = t_m = 1$. Adjoin four new points $\{\infty_1, \infty_2, \infty_3, \infty_4\}$ to the frame and apply the 'Filling in Holes' construction (see, e.g. Theorem 2.7), constructing on each hole of size $6g_i$ (except the last hole of size $6g_m$) together with the four new points an EKPD($6g_i + 4$)–EKPD(4) (*resp.* EKCD($6g_i + 4$)–EKCD(4)) aligning the hole in the incomplete packing (*resp.* covering) on the four new points; then on the last hole of size $6g_m$ together with the four infinite points construct an EKPD($6m + 4$) (*resp.* EKCD($6m + 4$)). All the required input designs exist by Proposition 1.5, Lemma 2.6, Theorem 2.7, Lemma 2.8 and Remark 2.9, and Lemma 3.1 and Remark 3.2. The result is an EKPD($6s + 4$) (*resp.* EKCD($6s + 4$)) in which the leave graph of the former is isomorphic to the excess graph of the latter. (Note that with regards the last hole H of size $6g_m, 3g_m$ of the near-parallel classes in the EKPD($6g_m + 4$) (*resp.* EKCD($6g_m + 4$)) are paired with the $3g_m$ holey parallel classes in the Kirkman frame corresponding to H. The remaining near-parallel class or classes is/are paired with the holey parallel class or classes in each of the incomplete packings (*resp.* coverings).) Now apply Lemma 1.9. □

Theorem 3.4 *There exists a separable exact 2-covering of v points by near-triangle factors for every $v \equiv 10 \bmod 12$ with $v \geq 94$.*

Proof. For each odd $s \geq 15$ we construct a GDD on s points with blocks sizes from the set $\{k \in \mathbb{Z} : k \geq 4\}$ and groups sizes from the set $\{1, 2, \ldots, 8\}$ and apply Lemma 3.3.

If $s \geq 49$, we can write $s = 4n + m$ where $n \geq 11$ is odd and $4 \leq m \leq n$ (e.g. let $m = s \bmod 8 + 4$ and $n = (s - m)/4$). Take a TD $(5, n)$ with a parallel class of blocks and truncate a group to m points. By viewing the resulting parallel class of blocks on the truncated TD as groups, we have produced a $\{4, 5, m, n\}$-GDD of type $4^{n-m}5^m$ on $4(n - m) + 5m = s$ points, as desired.

If $15 \leq s \leq 47$, we construct the appropriate GDD according to the following table.

s	GDD	Source
15	$4-$ GDD of type 3^5	TD $(4, 4)$
17, 19	$\{4, 5\}-$ GDD of type $4^41^1, 4^43^1$	TD $(5, 4)$
21, 23, 25	$\{4, 5\}-$ GDD of type $5^41^1, 5^43^1, 5^45^1$	TD $(5, 5)$
27, 29, 31	$\{4, 5\}-$ GDD of type $3^83^1, 3^85^1, 3^87^1$	resolvable 4-GDD of type 3^8 [8]
33, 35, 37, 39	$\{4, 5\}-$ GDD of type $8^41^1, 8^43^1, 8^45^1, 8^47^1$	TD $(5, 8)$
41, 43, 45, 47	$\{5, 6\}-$ GDD of type $8^51^1, 8^53^1, 8^55^1, 8^57^1$	TD $(6, 8)$

□

4 Conclusion

Theorem 1.8 now follows from the results in Sections 2 and 3, together with Propositions 1.5 and 1.6.

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Appendix

EKPD(58)–EKPD(16)

Point Set $(\mathbb{Z}_{21} \times \{1,2\}) \cup \{a_0, a_1, \ldots, a_6\} \cup \{\infty_1, \infty_2, \ldots, \infty_9\}$.

Near-Parallel Classes:

We get 21 near-parallel classes by developing the triples

$$\begin{array}{rrrr}
12_1 0_2 9_2 & a_3 9_1 10_2 & \infty_3 20_1 16_2 & \infty_9 16_1 15_2 \\
10_1 1_2 4_2 & a_4 15_1 19_2 & \infty_4 5_1 3_2 & (2_1) \\
0_1 3_1 14_2 & a_5 17_1 17_2 & \infty_5 14_1 6_2 & \\
a_0 1_1 7_1 & a_6 2_2 8_2 & \infty_6 13_1 18_2 & \\
a_1 18_1 7_2 & \infty_1 11_1 13_2 & \infty_7 19_1 5_2 & \\
a_2 4_1 20_2 & \infty_2 6_1 12_2 & \infty_8 8_1 11_2 &
\end{array}$$

mod 21, where the subscripts on a are evaluated mod 7. We then obtain 7 holey parallel classes on $\mathbb{Z}_{21} \times \{1,2\}$ by developing the triples $0_1 1_1 5_1$ $0_2 1_2 5_2$, $0_1 2_1 10_1$ $0_2 2_2 10_2$, and $0_1 7_1 14_1$ $0_2 7_2 14_2$ mod 21.

EKCD(58)–EKCD(16)

Point Set $(\mathbb{Z}_{21} \times \{1,2\}) \cup \{a_0, a_1, \ldots, a_6\} \cup \{\infty_1, \infty_2, \ldots, \infty_9\}$.

Near-Parallel Classes:

We get 21 near-parallel classes by developing the triples

$$\begin{array}{rrrr}
1_1 8_1 6_2 & a_3 18_1 16_2 & \infty_3 17_1 18_2 & \infty_9 20_1 8_2 \\
7_1 13_1 13_2 & a_4 5_1 20_2 & \infty_4 2_1 15_2 & (10_2) \\
11_1 0_2 7_2 & a_5 3_1 19_2 & \infty_5 4_1 3_2 & \\
a_0 14_1 11_2 & a_6 15_1 5_2 & \infty_6 19_1 12_2 & \\
a_1 12_1 14_2 & \infty_1 9_1 17_2 & \infty_7 16_1 2_2 & \\
a_2 10_1 1_2 & \infty_2 0_1 4_2 & \infty_8 6_1 9_2 &
\end{array}$$

mod 21, where $a_0, a_1, \ldots, a_6$ are fixed points. We then obtain 9 holey parallel classes on $\mathbb{Z}_{21} \times \{1,2\}$ by constructing a resolvable 3-GDD of type 3^7 (i.e. a Kirkman Triple System KTS(21)) on each of $\mathbb{Z}_{21} \times \{1\}$ and $\mathbb{Z}_{21} \times \{2\}$, in which the groups are represented by the pairs at pure difference ± 7 in each of the two orbits.

Now the leave of the EKPD(58)–EKPD(16) consists of the edges at pure difference ± 9 on orbit 1 and the edges at mixed difference 8 between orbits 1 and 2. The excess of the EKCD(58)–EKCD(16) consists of the edges at pure difference ± 6 on orbit 1 and the edges at mixed difference 19 between orbits 1 and 2. These two graphs are isomorphic.

EKPD(70)–EKPD(22)

Point Set $(\mathbb{Z}_{24} \times \{1,2\}) \cup \{a_0, a_1, \ldots, a_7\} \cup \{\infty_1, \infty_2, \ldots, \infty_{14}\}$.

Near-Parallel Classes:

We get 24 near-parallel classes by developing the triples

$8_1 0_2 4_2$	$a_7 19_1 2_2$	$\infty_8 7_1 7_2$
$a_0 16_1 9_2$	$\infty_1 11_1 13_2$	$\infty_9 5_1 17_2$
$a_1 21_1 12_2$	$\infty_2 18_1 21_2$	$\infty_{10} 14_1 22_2$
$a_2 13_1 23_2$	$\infty_3 3_1 16_2$	$\infty_{11} 23_1 10_2$
$a_3 10_1 8_2$	$\infty_4 22_1 3_2$	$\infty_{12} 4_1 1_2$
$a_4 6_1 20_2$	$\infty_5 0_1 19_2$	$\infty_{13} 2_1 11_2$
$a_5 1_1 5_2$	$\infty_6 12_1 6_2$	$\infty_{14} 15_1 14_2$
$a_6 9_1 15_2$	$\infty_7 17_1 18_2$	(20_1)

mod 24, where $a_0, a_1, \ldots, a_7$ are fixed points. We then obtain 10 holey parallel classes on $\mathbb{Z}_{24} \times \{1, 2\}$ by constructing a resolvable 3-GDD of type 6^4 [10] on each of $\mathbb{Z}_{24} \times \{1\}$ and $\mathbb{Z}_{24} \times \{2\}$, aligning the groups on differences $\pm 4, \pm 8$, and ± 12, and using the edges at difference ± 8 to generate the tenth holey parallel class.

EKPD(70)–EKPD(22)

Point Set $(\mathbb{Z}_{24} \times \{1, 2\}) \cup \{a_0, a_1, \ldots, a_7\} \cup \{\infty_1, \infty_2, \ldots, \infty_{14}\}$.

Near-Parallel Classes:

We get 24 near-parallel classes by developing the triples

$8_1 0_2 4_2$	$a_7 19_1 2_2$	$\infty_8 7_1 7_2$
$a_0 16_1 9_2$	$\infty_1 11_1 13_2$	$\infty_9 5_1 17_2$
$a_1 21_1 12_2$	$\infty_2 18_1 21_2$	$\infty_{10} 14_1 22_2$
$a_2 13_1 23_2$	$\infty_3 3_1 16_2$	$\infty_{11} 23_1 10_2$
$a_3 10_1 8_2$	$\infty_4 22_1 3_2$	$\infty_{12} 4_1 1_2$
$a_4 6_1 20_2$	$\infty_5 0_1 19_2$	$\infty_{13} 2_1 11_2$
$a_5 1_1 5_2$	$\infty_6 12_1 6_2$	$\infty_{14} 15_1 14_2$
$a_6 9_1 15_2$	$\infty_7 17_1 18_2$	(20_1)

mod 24, where $a_0, a_1, \ldots, a_7$ are fixed points. We then obtain 10 holey parallel classes on $\mathbb{Z}_{24} \times \{1, 2\}$ by constructing a resolvable 3-GDD of type 6^4 [10] on each of $\mathbb{Z}_{24} \times \{1\}$ and $\mathbb{Z}_{24} \times \{2\}$, aligning the groups on differences $\pm 4, \pm 8$, and ± 12, and using the edges at difference ± 8 to generate the tenth holey parallel class.

EKCD(70)–EKCD(22)

Point Set $(\mathbb{Z}_{24} \times \{1, 2\}) \cup \{a_0, a_1, \ldots, a_7\} \cup \{\infty_1, \infty_2, \ldots, \infty_{14}\}$.

Near-Parallel Classes:

We get 24 near-parallel classes by developing the triples

$0_1 4_1 7_2$	$a_7 2_1 6_1$	$\infty_8 18_1 22_2$
$a_0 1_2 5_2$	$\infty_1 22_1 10_2$	$\infty_9 8_1 0_2$
$a_1 3_1 11_2$	$\infty_2 12_1 21_2$	$\infty_{10} 21_1 18_2$
$a_2 2_2 14_2$	$\infty_3 20_1 13_2$	$\infty_{11} 10_1 16_2$
$a_3 17_1 17_2$	$\infty_4 19_1 8_2$	$\infty_{12} 7_1 3_2$
$a_4 1_1 13_1$	$\infty_5 11_1 12_1$	$\infty_{13} 16_1 15_2$
$a_5 9_1 20_2$	$\infty_6 15_1 6_2$	$\infty_{14} 5_1 23_2$
$a_6 14_1 9_2$	$\infty_7 23_1 4_2$	(19_2)

mod 24, where the subscripts on a are evaluated mod 8. We then obtain 9 holey parallel classes on $\mathbb{Z}_{24} \times \{1,2\}$ by constructing a resolvable 3-GDD of type 6^4 [10] on each of $\mathbb{Z}_{24} \times \{1\}$ and $\mathbb{Z}_{24} \times \{2\}$, aligning the groups on differences $\pm 4, \pm 8$, and ± 12. We then get 3 further holey parallel classes on $\mathbb{Z}_{24} \times \{1,2\}$ by developing the triples $0_1 2_2 10_2$ and $2_1 10_1 0_2$ mod 24.

The leave of the EKPD(70)–EKPD(22) consists of the edges at pure difference ± 4 and ± 12 on $\mathbb{Z}_{24} \times \{1\}$, together with the edges at pure difference ± 12 on $\mathbb{Z}_{24} \times \{2\}$. The excess of the EKCD(70)–EKCD(22) also consists of the edges at pure difference ± 4 and ± 12 on $\mathbb{Z}_{24} \times \{1\}$ together with the edges at pure difference ± 12 on $\mathbb{Z}_{24} \times \{2\}$.

EKPD(82)–**EKPD**(22)

Point Set $(\mathbb{Z}_{30} \times \{1,2\}) \cup \{a_0, a_1, \infty_1, \infty_2, \ldots, \infty_{20}\}$.

Near-Parallel Classes:

We get 30 near-parallel classes by developing the triples

$4_1 0_2 3_2$	$\infty_1 21_1 14_2$	$\infty_8 26_1 12_2$	$\infty_{15} 18_1 25_2$
$19_1 1_2 7_2$	$\infty_2 9_1 28_2$	$\infty_9 8_1 5_2$	$\infty_{16} 14_1 22_2$
$0_1 3_1 6_2$	$\infty_3 6_1 23_2$	$\infty_{10} 16_1 17_2$	$\infty_{17} 13_1 24_2$
$12_1 4_2 16_2$	$\infty_4 25_1 20_2$	$\infty_{11} 10_1 8_2$	$\infty_{18} 15_1 15_2$
$1_1 7_1 21_2$	$\infty_5 29_1 9_2$	$\infty_{12} 17_1 19_2$	$\infty_{19} 28_1 13_2$
$a_0 2_1 11_1$	$\infty_6 27_1 10_2$	$\infty_{13} 5_1 26_2$	$\infty_{20} 22_1 27_2$
$a_1 2_2 11_2$	$\infty_7 20_1 29_2$	$\infty_{14} 24_1 18_2$	(23_1)

mod 30, where the subscripts on a are evaluated mod 2. We then obtain 10 holey parallel classes on $\mathbb{Z}_{30} \times \{1,2\}$ by constructing a resolvable TD $(3,10)$ on each of $\mathbb{Z}_{30} \times \{1\}$ and $\mathbb{Z}_{30} \times \{2\}$, aligning the groups on differences $\pm 3, \pm 6, \pm 9, \pm 12$ and ± 15.

EKCD(82)–**EKCD**(22)

Point Set $(\mathbb{Z}_{30} \times \{1,2\}) \cup \{a_0, a_1, \infty_1, \infty_2, \ldots, \infty_{20}\}$.

Near-Parallel Classes:

We get 30 near-parallel classes by developing the triples

$0_1 5_1 21_2$	$\infty_1 25_1 27_2$	$\infty_8 4_1 28_2$	$\infty_{15} 8_1 26_2$
$1_1 11_1 24_2$	$\infty_2 26_1 25_2$	$\infty_9 19_1 8_2$	$\infty_{16} 13_1 17_2$
$3_1 9_1 14_2$	$\infty_3 24_1 20_2$	$\infty_{10} 20_1 0_2$	$\infty_{17} 21_1 5_2$
$16_1 1_2 6_2$	$\infty_4 10_1 10_2$	$\infty_{11} 7_1 13_2$	$\infty_{18} 6_1 9_2$
$15_1 2_2 12_2$	$\infty_5 12_1 19_2$	$\infty_{12} 29_1 7_2$	$\infty_{19} 28_1 29_2$
$a_0 2_1 17_1$	$\infty_6 27_1 22_2$	$\infty_{13} 18_1 16_2$	$\infty_{20} 14_1 23_2$
$a_1 3_2 18_2$	$\infty_7 23_1 15_2$	$\infty_{14} 22_1 4_2$	(11_2)

mod 30, where the subscripts on a are evaluated mod 2. We then obtain 12 holey parallel classes on $\mathbb{Z}_{30} \times \{1,2\}$ by constructing a resolvable 3-GDD of type 6^5 [10] on each of $\mathbb{Z}_{30} \times \{1\}$ and $\mathbb{Z}_{30} \times \{2\}$, aligning the groups on differences $\pm 5, \pm 10$, and ± 15.

The leave of the EKPD(82)–EKPD(22) consists of the edges at pure difference ± 12 and ± 15 on $\mathbb{Z}_{30} \times \{1\}$, together with the edges at pure difference ± 15 on $\mathbb{Z}_{30} \times \{2\}$. The excess of the EKCD(82)–EKCD(22) consists of the edges at pure difference ± 6 and ± 15 on $\mathbb{Z}_{30} \times \{1\}$, and the edges at pure difference ± 15 on $\mathbb{Z}_{30} \times \{2\}$. These two graphs are isomorphic.

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