Minimal defining sets for full 2-$(v, 3, v-2)$ designs

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Abstract

A t-(v, k, λ) design $D = (X, \mathcal{B})$ with $\mathcal{B} = P_k(X)$ is called a full design. For $t = 2$, $k = 3$ and any v, we give minimal defining sets for these designs. For $v = 6$ and $v = 7$, smallest defining sets are found.

1. Introduction

Let v, k, t, and λ be natural numbers such $v \geq k \geq t > 0$. A t-(v, k, λ) *design* D is an ordered pair $(X, \mathcal{B})$ where X is a v-set, and $\mathcal{B}$ is a collection of k-subsets (called blocks) of X with the property that every t-subset of X appears in exactly λ blocks. A design with no repeated block is called a *simple design.* A t-$\left(v, k, \binom{v-t}{k-t}\right)$ design is called a *quasi-full* design. In this case the number of blocks of the design is $\binom{v}{k}$. If we take all k-subsets of a v-set as blocks, we obtain a simple quasi-full design, which is called a *full design.*

A t-(v, k) *trade* $T = (T_1, T_2)$ consists of two disjoint collections of blocks T_1 and T_2 such that for every t-subset $B \subseteq X$, the number of blocks containing B in T_1 is the same as the number of blocks containing B in T_2. We say a t-(v, k) trade $T = (T_1, T_2)$ is *embedded* in a t-(v, k, λ) design $D = (X, \mathcal{B})$, if T_1 or T_2 or both is contained in $\mathcal{B}$.

In 1990, the concept of a defining set of a t-design was introduced by Gray [1]. A set of blocks which is a subset of a unique t-(v, k, λ) design D is a *defining set*

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of that design, and is denoted by dD. A minimal defining set, denoted by d_mD, is a defining set no proper subset of which is a defining set. A smallest defining set, denoted by d_SD, is a defining set of least cardinality. There is a close relationship between defining sets and trades.

Theorem 1 [1]. Let $D = (X, \mathcal{B})$ be a t-(v, k, λ) design and $S \subseteq \mathcal{B}$. Then S is a defining set of D if and only if $S \cap T \neq \emptyset$ for every embedded trade T of D.

For proving some results we invoke to the following theorem due to Ken Gray [1].

Theorem 2 [1]. If the t-(v, k, λ) design D is a disjoint union of two designs D' and D'' with parameters t-(v, k, λ'), and t-(v, k, λ'') respectively, where $\lambda' + \lambda'' = \lambda$, then $|d_SD| \geq |d_SD'| + |d_SD''|$.

In this paper we obtain a minimal defining set for the families of 2-$(v, 3, v-2)$ full designs.

2. A minimal defining set

In this section, we consider 2-$(v, 3, v-2)$ full designs, and give a set of blocks which is a minimal defining set for these designs. Before starting the proof, let us introduce some notations. We put

$d_3(v) =$ the size of a smallest defining set of the full 2-$(v, 3, v-2)$ design,
$B_v =$ the set of blocks which contain the element v,
$B_{xy} =$ the set of blocks which contain the set $\{x, y\}$.

Theorem 3. If $N = B_{12} \cup B_v$, then the set $S = P_3(X) \backslash N$ is a defining set of the full 2-$(v, 3, v-2)$ design, where $P_3(X)$ denotes the collection of 3-subsets of X.

Proof. Clearly $|N| = \binom{v-1}{2} + v - 3$, and so $|S| = \binom{v}{3} - \binom{v}{2} + 2$.

Suppose that S extends to a quasi-full 2-$(v, 3, v-2)$ design, say $D' = (X, \mathcal{B}')$. We show that D' is a simple design, and hence is the 2-$(v, 3, v-2)$ full design. Let $S' = \mathcal{B}' \backslash S$ and for $1 \leq i \leq n$ suppose that B'_i is the collection of those blocks of S' which contain the element i. Clearly $|B'_v| = \binom{v-1}{2}$ and $|B'_1| = |B'_2| = 2v - 5$. First we show that blocks of S' which contain 1 or 2 must contain both. Next we show that T has no repeated block, then that B'_v has no repeated block, and finally that the design has no repeated block. If $T = B'_1 \backslash B'_v$, then $|T| = v - 3$ since $\{1, v\}$ is not a subset of any block of S. Similarly $|B'_2 \backslash B'_v| = v - 3$. Also we have $S' = B'_v \cup T$, and hence $T = B'_1 \backslash B'_v = B'_2 \backslash B'_v$, and we conclude that every block of T contains the pair $\{1, 2\}$. On the other hand, $|B'_i| = v - 1$ for $3 \leq i \leq v - 1$, and since $|B'_i \cap B'_v| = v - 2$, we conclude that $|T \cap B_i| = 1$ for $3 \leq i \leq v - 1$. Since every block of T contains $\{1, 2\}$, hence T is constructed uniquely and there is no repeated block in T. Now we show that B'_v has no repeated block. Since $\mathcal{B}' = S \cup S' = S \cup B'_v \cup T$, and every pair $\{x, y\}$, $x, y \neq v$ appears in $S \cup T$ $v - 3$ times, hence every pair appears in B'_v exactly once. There are $\binom{v-1}{2}$ pairs from $\{1, 2, \ldots, v-1\}$, and since $|B'_v| = \binom{v-1}{2}$, we conclude that B'_v is constructed uniquely, and hence there are no repeated blocks in $\mathcal{B}'_v$, and in fact D' is the 2-$(v, 3, v-2)$ full design. □

Theorem 4. For $v \geq 6$, the set S which was constructed in the theorem above is indeed a minimal defining set.

Proof. We show that for any block $B \in S$, the set $N \cup B$ contains a part of a trade with volume four. To prove the theorem we consider two cases:
(i) $1 \in B$ or $2 \in B$. Without loss of generality assume that $1 \in B$, and $B = \{1, x, y\}$ where $x, y \notin \{2, v\}$. Therefore $N \cup B$ contains the trade $\{\{1, x, y\}, \{1, 2, w\}, \{2, x, v\}, \{v, w, y\}\}$ where w is an arbitrary element which is not in the set $\{1, x, y, v, 2\}$.
(ii) $1, 2 \notin B$. In this case $B = \{x, y, z\}$, $B \cap \{1, 2, v\} = \emptyset$. Thus $N \cup B$ contains the trade $\{\{x, y, z\}, \{1, 2, x\}, \{2, z, v\}, \{1, y, v\}\}$. □

3. Determination of $d_3(v)$ for $3 \leq v \leq 7$

Clearly $d_3(3) = d_3(4) = d_3(5) = 0$.

Theorem 5. $d_3(6) = 6$.

Proof. We observe that $P_3(X) = \mathcal{B}_1 \cup \mathcal{B}_2$, where $X = \{1, 2, \ldots, 6\}$ and $D_1 = (X, \mathcal{B}_1)$ and $D_2 = (X, \mathcal{B}_2)$ are 2-$(6, 3, 2)$ designs. Since the smallest defining set of a 2-$(6, 3, 2)$ design has cardinality 3, see [1], by Theorem 2 we conclude that $d_3(v) \geq 6$. Now consider the set $S = \{134, 135, 145, 123, 124, 125\}$. Since the pair 16 must appear in four blocks, hence our design contains the blocks 126, 136, 146, 156. Now if these ten blocks do not uniquely determine our design, then we would have a trade with foundation set with size at most 5, which is a contradiction.

Theorem 6. $d_3(7) \leq 15$.

Proof. Assume that $X = \{1, 2, \ldots, 7\}$. If $S = \{1,3,6\}, \{1,4,5\}, \{1,4,6\}, \{1,5,6\}, \{2,3,4\}, \{2,3,5\}, \{2,3,6\}, \{2,3,7\}, \{2,4,5\}, \{2,4,6\}, \{2,4,7\}, \{2,5,7\}, \{3,4,6\}, \{3,5,6\}, \{4,5,6\}$, we claim that S is a defining set of the 2-$(7, 3, 5)$ full design, and hence $d_3(7) \leq 15$. To prove the claim assume that S can be extended to a quasi-full 2-$(7, 3, 5)$ design $D' = (X, \mathcal{B})$. We show that D' is the 2-$(7, 3, 5)$ full design $D = (X, P_3(X))$. Let $S' = \mathcal{B}' - S$, and let s'_i and s'_{ij} denote the number of blocks of S' which contain the element i and the pair $\{i, j\}$, respectively. We will show that $D = D'$. We divide the proof into three steps.
Step 1. Since 6 appears in 9 blocks of S, hence $s'_6 = 6$. On the other hand $s'_{67} = 5$, $s'_{16} = 2$, and $s'_{26} = 2$, therefore the blocks $\{1, 6, 7\}, \{2, 6, 7\} \in S'$.
Step 2. Since $s'_2 = s'_{12} = 5$, hence the blocks which contain 2 also contain 1. In addition $s'_{2a} = 1$, for $3 \leq a \leq 7$. Hence the block $\{1,2,3\}, \{1,2,4\}, \{1,2,6\}, \{1,2,7\} \in S'$.
Step 3. Consider $S'' = S' \backslash \{\{1,6,7\}, \{2,6,7\}, \{1,2,3\}, \{1,2,4\}, \{1,2,6\}, \{1,2,7\}\}$. Since $s'_{67} = 5$ and $s'_6 = 6$, therefore the element 6 and the pair $\{6, 7\}$ both appear in S'' three times. Also $s'_{46} = s'_{56} = s'_{36} = 1$, so $\{3,6,7\}, \{4,6,7\}, \{5,6,7\} \in S'$.

Now put $S''' = S'' \backslash \{\{3,6,7\}, \{4,6,7\}, \{5,6,7\}\}$. Hence the elements 6 and 2 do not appear in any block of S'''. Therefore only five elements appear in the blocks of S'''. Also $\mathcal{B}\backslash P_3(X) \subseteq S'''$, and hence the blocks of $T = (\mathcal{B}\backslash P_3(X), P_3(X)\backslash\mathcal{B})$ are based on at most five elements. But every trade must be based on at least 6 elements, see [2]. So $\mathcal{B}\backslash P_3(X) = \emptyset$, and thus $D = D'$. □

Theorem 7. $d_3(7) = 15$.

Proof. Suppose that S is a defining set of the 2-$(7,3,5)$ full design. It is well-known that there are 30 distinct 2-$(7,3,1)$ designs in the 2-$(7,3,5)$ full design, see [3]. Let $\mathcal{F} = \{\mathcal{F}_1, \ldots, \mathcal{F}_{30}\}$ be the set of distinct 2-$(7,3,1)$ designs. Every block is contained in six elements of $\mathcal{F}$, and for any i we have $|d_S(\mathcal{F}_i)| = 3$, see [1]. Define $\Gamma = \{(b, \mathcal{F}_i) \mid \mathcal{F}_i \in \mathcal{F}, b \in \mathcal{F}_i \cap S\}$. By counting the pairs of Γ in two ways, we obtain $|S| \times 6 \geq 30 \times 3$. This implies that $|S| \geq 15$. Now by Theorem 6, we conclude that $d_3(7) = 15$ and the proof is complete. □

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