Enumeration of 2-(12, 3, 2) designs

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Abstract

A backtrack search with isomorph rejection is carried out to enumerate the 2-(12, 3, 2) designs. There are 242 995 846 such designs, which have automorphism groups whose size range from 1 to 1536. There are 88 616 310 simple designs. The number of resolvable designs is 62 929; these have 74 700 nonisomorphic resolutions.

1 Introduction

We use the following standard notations. A t-(v, k, λ) design is a family of k-subsets, called blocks, of a v-set such that each t-subset of the v-set is contained in exactly λ blocks. A design is *simple* if it has no repeated blocks. A 2-(v, k, λ) design is called a *balanced incomplete block design* (BIBD). Two more parameters that are related to a design are b, the number of blocks, and r, the number of blocks in which a point occurs. The values of b and r can easily be determined from the values of the other parameters, as

$$vr = bk, r(k-1) = \lambda(v-1).$$

It has for a long time been known that 2-(12, 3, 2) designs exist; see [1]. One construction of such a design is as follows. A 2-(45, 12, 3) design can be constructed from a McFarland difference set [5]. Then we get a desired *derived design* by deleting any block in this design and deleting all points not in this block from the other blocks.

Many nonisomorphic 2-(12, 3, 2) designs are known; Royle [6, 11] quickly found one million with a hill-climbing computer algorithm. A complete enumeration of these designs is, however, yet to be carried out. This is the goal of our work.

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Enumeration of designs is a central topic in design theory, and a lot of studies in this field have been carried out along the years, for example, [2, 3, 4, 9, 12, 13]. Enumeration algorithms are generally based on backtrack search with isomorph rejection. The search algorithms have been continuously improved along the years. Together with the increasing speeds of the computers available, this enables more and more extensive enumerations to be carried out in the future.

The version of backtrack search with isomorph rejection employed in this work is described in Section 2. The results are presented in Section 3. It turns out that the number of nonisomorphic 2-$(12, 3, 2)$ designs is 242 995 846, out of which 62 929 are resolvable with 74 700 nonisomorphic resolutions. There are 88 616 310 simple designs. The sizes of the automorphism groups are further tabulated.

2 Backtrack Search with Isomorph Rejection

To describe our approach we need to define the *incidence matrix* of a design. An incidence matrix is a $v \times b$ (0,1)-matrix with the rows indexed by the points and the columns indexed by the blocks, where a 1 (0) indicates that a block contains (does not contain) a point. In a backtrack search we now try to complete the incidence matrix in a row-by-row manner. The parameters of the design give the following restrictions: every row has r 1s, every column has k 1s, and any two rows have λ common 1s.

As in any backtrack search, it is of utmost importance that isomorphic subconfigurations are detected to reduce the size of the search tree. Two incidence matrices are isomorphic if one can be obtained from the other by permuting the rows and the columns. We here use the common approach that a matrix is rejected in the search if it is not in *canonical form*. Such *orderly algorithms* [10], with several variations, have been used in several earlier studies, including [2, 3, 4, 9, 12, 13].

The canonical form of a $v' \times b$ matrix M with $1 \leq v' \leq v$ is defined in the following way. We define $b(M)$ to be the binary string of length $v'b$ obtained by concatenating the rows of M so that the uppermost, leftmost bit is the most significant. This defines an ordering on matrices, and we say that a matrix M is in canonical form if for every matrix M' obtained by a permutation of the rows and the columns, $b(M') \leq b(M)$.

With one row, $v' = 1$, there is clearly only one matrix in canonical form, and it consists of a row with r 1s in the leftmost positions. In expanding a (0,1)-matrix by adding a new row, we look for rows which have λ points in common with the previous rows, and for all such rows we check whether the matrix is in canonical form. Before constructing the row it is worth checking which of the earlier columns are identical, since if the 1s of the new row in these columns are not in the leftmost positions, then the new matrix is clearly not in canonical form.

In constructing the new row, a column with k 1s must clearly get a 0. The leftmost 1 is further placed only in the first possible column [2].

The search as described above is fairly straightforward, and the only more complicated part is to efficiently test whether the matrix is in canonical form. This is a central task which will use most of the CPU time. Our approach follows that of [2].

For a given matrix we proceed in a row-by-row manner to find out whether the matrix is in canonical form or not. If not, we may stop the investigation immediately. By definition, if the matrix M'' formed by taking some v'' $(1 \leq v'' \leq v')$ rows of the matrix M in some order and permuting the columns so that they are in lexicographic order—with the largest column first—has $b(M'') > b(M')$, where M' consists of the first v'' rows of the matrix M, then M is not in canonical form.

We then search for such a matrix M'' using a depth-first backtrack search, where we on level i, $1 \leq i \leq v'$, choose a row to become row i in the permuted matrix. In every node of this tree, we compare $b(M'')$ from the permuted partial matrix and $b(M')$ from the original partial matrix. If $b(M'') > b(M')$, then the matrix M is not in canonical form; if $b(M'') < b(M')$, then we backtrack; and if $b(M'') = b(M')$, then we continue the search. If the matrix is in canonical form, the full automorphism group is given by the number of times level v' was reached with $b(M'') = b(M')$ on that level.

3 The Results

The approach described in the previous section was implemented in a C program and run in a network with fifteen 233–500 MHz PCs. The search was distributed in the following way using the program *autoson* [7].

For up to eight rows, all incidence matrices in canonical form were constructed (with the additional requirement that the leftmost 1 is in the first possible column). The numbers of such matrices are 1 (one row), 1, 2, 4, 12, 73, 849, and 42397 (eight rows), respectively. The matrices with eight rows were divided into blocks of 100 matrices, which were used as starting points when the problem was distributed among the computers.

After two weeks and about half a year of CPU time, the enumeration was ready, and the number of 2-$(12, 3, 2)$ designs turned out to be 242 995 846, settling a parameter for design number 55 in [6]. Due to the multitude of designs, only a very short time was available to study various properties. Saving the designs is possible—if several gigabytes of external memory is available—but not practical. Therefore, all calculations were done on-the-fly. For every design found, we saved the size of its automorphism group, which we get automatically in checking its canonical form, and checked whether the design is simple. Moreover, it was checked whether the design is resolvable, and if so, the number of nonisomorphic resolutions were counted; this was done using a clique-finding approach described in [8].

A total of 88 616 310 simple designs were found in the search. There are 62 929 nonisomorphic resolvable 2-$(12, 3, 2)$ designs; these have 74 700 nonisomorphic resolutions (the bound ≥ 2 is given in [6]). The sizes of the automorphism groups are displayed in Table 1. These sizes range from 1 to 1536. The column Nd gives the total number of designs and Ns gives the number of simple designs.

Since we get 2-$(12, 3, 2)$ derived designs from symmetric 2-$(45, 12, 3)$ designs, the former could be used as starting configurations in an enumeration of the latter, as discussed in [12]. The multitude of 2-$(12, 3, 2)$ designs, however, renders this

$\|Aut(D)\|$	Nd	Ns
1	242 885 893	88 593 070
2	106 395	22 010
3	1 845	716
4	1 220	384
6	254	47
8	133	51
9	3	0
11	5	5
12	27	6
16	28	8
18	5	0
24	7	2
32	9	4
36	2	0
48	6	2
54	1	0
64	2	1
72	2	0
128	2	2
144	1	0
192	3	1
432	1	0
576	1	0
1536	1	1
Total	242 995 846	88 616 310

Table 1: Automorphism group sizes.

particular approach infeasible at least for the time being.

We would finally like to point out that as the results are obtained in a computer search, they are correct if the computer program is correct. For a computer search of this magnitude it is difficult, if not impossible, to get certainty regarding correctness of results. Confidence in the program was acquired by testing it against many earlier enumeration results in [6].

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