Edge-magic total labelings

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Abstract

Various graph labelings that generalize the idea of a magic square have been discussed. In particular a *magic labeling* on a graph with v vertices and e edges will be defined as a one-to-one map taking the vertices and edges onto the integers $1, 2, \cdots, v+e$ with the property that the sum of the label on an edge and the labels of its endpoints is constant independent of the choice of edge.

Properties of these labelings are surveyed and the question of which families of graphs have magic labelings are addressed.

1 Graph labelings

All graphs in this paper are finite, simple and undirected (although the imposition of directions will cause no complications). The graph G has vertex-set $V(G)$ and edge-set $E(G)$. A general reference for graph-theoretic ideas is [19].

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A *labeling* (or *valuation*) of a graph is a map that carries graph elements to numbers (usually to the positive or non-negative integers). In this paper the domain will usually be the set of all vertices and edges; such labelings are called *total* labelings. Some labelings use the vertex-set alone, or the edge-set alone, and we shall call them *vertex*-labelings and *edge*-labelings respectively. Other domains are possible. The most complete recent survey of graph labelings is [5].

We shall define two labelings of the same graph to be *equivalent* if one can be transformed into the other by an automorphism of the graph.

2 The magic property

Various authors have introduced labelings that generalize the idea of a magic square. Sedláček [17] defined a graph to be magic if it had an edge-labeling, with range the real numbers, such that the sum of the labels around any vertex equalled constant, independent of the choice of vertex. These labelings have been studied by Stewart (see, for example, [18]), who called a labeling *supermagic* if the labels are consecutive integers, starting from 1. Several others have studied these labelings; a recent reference is [6]. Some writers simply use the name "magic" instead of "supermagic" (see, for example, [8]).

Kotzig and Rosa [10] define a magic labeling to be a total labeling in which the labels are the integers from 1 to $|V(G)| + |E(G)|$. The sum of labels on an edge and its two endpoints is constant. In 1996 Ringel and Llado [16] redefined this type of labeling (and called the labelings *edge-magic*, causing some confusion with papers that have followed the terminology of [12], mentioned below); see also [7]. Recently Enomoto *et al* [4] have introduced the name *super edge-magic* for magic labelings in the sense of Kotzig and Rosa, with the added property that the v vertices receive the smaller labels, $\{1, 2, \cdots, v\}$.

In 1983, Lih [13] introduced magic labelings of planar graphs where labels extended to faces as well as edges and vertices, an idea which he traced back to 13th century Chinese roots. Bača (see, for example, [1, 2]) has written extensively on these labelings. A somewhat related sort of magic labeling was defined by Dickson and Rogers in [3].

Lee, Seah and Tan [12] introduced a weaker concept, which they called *edge-magic*, in 1992. The edges are labeled and the sums at the vertices are required to be congruent modulo the number of vertices.

Total labelings have also been studied in which the sum of the labels of all edges adjacent to the vertex x, plus the label of x itself, is constant. A paper on these labelings is in preparation [14].

In order to clarify the terminological confusion defined above, we define a labeling to be *edge-magic* if the sum of all labels associated with an edge equals a constant independent of the choice of edge, and *vertex-magic* if the same property holds for vertices. (This terminology could be extended to other substructures: face-magic, for example.) The domain of the labeling is specified by a modifier on the word "labeling". We shall always require that the labeling is a one-to-one map onto the appropriate set of consecutive integers starting from 1. For example, Stewart studies vertex-magic edge-labelings, and Kotzig and Rosa define edge-magic total labelings. Hypermagic labelings are vertex-magic total labelings.

In this paper we shall study edge-magic total labelings. Two of the early papers on such labelings, [9] and [11], appeared only as research reports. Probably because of this, only a few papers have appeared, but there has recently been a resurgence of interest in these labelings – [16], [7] and [4]. Because of the confusion of terminology, and because several results have not appeared in the open literature, the present paper includes a partial survey of the field, and contains more details of known results than is usual.

3 Edge-magic total labelings

For brevity we shall follow Gallian's lead, and use the term "magic" as was done by Kotzig and Rosa [10] — a "magic labeling" will henceforward mean an edge-magic total labeling.

Definition. A *magic labeling* on G will mean a one-to-one map λ from $V(G) \cup E(G)$ onto the integers $1, 2, \cdots, v+e$, where $v = |V(G)|$ and $e = |E(G)|$, with the property that, given any edge (x, y),

$$\lambda(x) + \lambda(x, y) + \lambda(y) = k$$

for some constant k. It will be convenient to call $\lambda(x) + \lambda(x, y) + \lambda(y)$ the *edge sum* of (x, y), and k the (constant) *magic sum* of G. A graph is called *magic* if it admits any magic labeling.

The basic requirements in order that $\{x_1, x_2, \cdots, x_v\} = \lambda(V(G))$, where λ is a magic labeling of a graph G with magic sum k, are

(i) $x_h + x_i + x_j = k$ cannot occur if any two of $\lambda^{-1}(x_i), \lambda^{-1}(x_j), \lambda^{-1}(x_k)$ are adjacent;

(ii) the sums $x_i + x_j$, where $(\lambda^{-1}(x_i), \lambda^{-1}(x_j))$ is an edge, are all distinct;

(iii) $0 < k - (x_i + x_j) \leq v + e$ when $\lambda^{-1}(x_i)$ is adjacent to $\lambda^{-1}(x_j)$.

Suppose λ is a magic labeling of a given graph. If x and y are adjacent vertices, then edge (x, y) has label $k - \lambda(x) - \lambda(y)$. Since the sum of all these labels plus the sum of all the vertex labels must equal the sum of the first $v + e$ positive integers, k is determined. So the vertex labels specify the complete labeling.

Of course, not every possible assignment will result in a magic labeling: the above process may give a non-integral value for k, or give repeated labels.

4 Some elementary counting

As a standard notation, assume the graph G has v vertices and e edges. It will be convenient to write $M = v + e + 1$. For notational convenience, we always say vertex v_i has degree d_i and receives label x_i.

Among the labels, write S for the set $\{x_i : 1 \le i \le v\}$ of vertex labels, and s for the sum of elements of S. Then S can consist of the v smallest labels, the v largest labels, or somewhere in between, so

$$\sum_{i=1}^{v} i \le s \le \sum_{i=1+e}^{v+e} i,$$

$$\binom{v+1}{2} \le s \le ve + \binom{v+1}{2}. \quad (1)$$

Clearly, $\sum_{xy \in E}(\lambda(x,y) + \lambda(x) + \lambda(y)) = ek$. This sum contains each label once, and each vertex label x_i an additional $d_i - 1$ times. So

$$ke = \binom{M}{2} + \sum (d_i - 1)x_i. \quad (2)$$

If e is even, every d_i is odd and $v + e \equiv 2 (\mathrm{mod}\, 4)$ then (2) is impossible, as noted in [16]. We have

Theorem 1 [16] *If G has e even and $v + e \equiv 2(\mathrm{mod}\, 4)$, and every vertex of G has odd degree, then G is not magic.* □

Corollary 1.1 *The complete graph K_n is not magic when $n \equiv 4(\mathrm{mod}\, 8)$. The n-spoke wheel W_n is not magic when $n \equiv 3(\mathrm{mod}\, 4)$.* □

(We shall see in Section 7 that K_n is never magic for $n > 6$, so the first part of the Corollary really only eliminates K_4.)

In particular, suppose G is regular of degree d. Then (2) becomes

$$ke = (d-1)s + \binom{M}{2} = (d-1)s + \tfrac{1}{2}(v+e)(v+e+1) \quad (3)$$

or, since $e = \frac{1}{2}dv$,

$$kdv = 2(d-1)s + (v+e)(v+e+1). \quad (4)$$

5 Duality

Given a labeling λ, its dual labeling λ' is defined by

$$\lambda'(v_i) = M - \lambda(v_i),$$

and for any edge x,

$$\lambda'(x) = M - \lambda(x).$$

It is easy to see that if λ is a magic labeling with magic sum k then λ' is a magic labeling with magic sum $k' = 3M - k$. The sum of vertex labels is $s' = vM - s$.

Either s or s' will be less than or equal to $\frac{1}{2}vM$. This means that, in order to see whether a given graph is magic, it suffices to check either all cases with $s \le \frac{1}{2}vM$ or all cases with $s \ge \frac{1}{2}vM$ (equivalently, $k \le \frac{3}{2}M$ or $k \ge \frac{3}{2}M$).

6 Graphs with complete subgraphs

A *well-spread sequence* $A = (a_1, a_2, \cdots, a_n)$ of length n is a sequence with the following properties:

1. $0 < a_1 < a_2 < \cdots < a_n$;
2. $a_i + a_j \neq a_k + a_\ell$ whenever $i \neq j$ and $k \neq \ell$ (except, of course, when $\{a_i, a_j\} = \{a_k, a_\ell\}$).

Let

$$\rho(A) = a_n + a_{n-1} - a_2 - a_1 + 1$$
$$\rho^*(n) = \min \rho(A)$$

where the minimum is taken over all well-spread sequences A of length n. Well-spread sequences were defined in [9]. The value of $\rho^*(n)$ is discussed in [9] (see also [15]); for our purposes we need to know that

$$\rho^*(7) = 30, \ \rho^*(8) = 43, \tag{5}$$

and

$$\rho^*(n) \geq n^2 - 5n + 14 \text{ when } n > 8. \tag{6}$$

Suppose G has a magic labeling λ with magic sum k, and suppose G contains a complete subgraph H with n vertices. (The usual parameters v and e refer to G, not to H.) Write $x_1, x_2, \cdots, x_n$ for the vertices of H, $a_i = \lambda(x_i)$, and suppose the vertices have been ordered so that $a_1 < a_2 \cdots < a_n$. Then obviously $A = (a_1 < a_2 \cdots < a_n)$ is a well-spread sequence. Then

$$\lambda(x_n x_{n-1}) = k - a_n - a_{n-1},$$

and since $\lambda(x_n x_{n-1})$ is a label,

$$k - a_n - a_{n-1} \geq 1. \tag{7}$$

Similarly

$$\lambda(x_2 x_1) = k - a_2 - a_1,$$

and since $\lambda(x_2 x_1)$ is a label,

$$k - a_2 - a_1 \leq v + e. \tag{8}$$

Combining (7) and (8) we have

$$v + e \geq a_n + a_{n-1} - a_2 - a_1 + 1 = \rho(A) \geq \rho^*(n).$$

Theorem 2 [11] *If the magic graph G contains a complete subgraph with n vertices, then the number of vertices and edges in G is at least $\rho^*(n)$.* □

7 Complete graphs

Theorem 3 [11] *No complete graph with more than 6 vertices is magic.*

Proof. Suppose a magic labeling of K_n existed. From Theorem 2,

$$n + \binom{n}{2} \geq \rho^*(n).$$

For $n = 7$, this says $28 \geq 30$, and for $n = 8$, it says $36 \geq 43$: see (5). Both are false. And for $n > 8$, (6) yields

$$\tfrac{1}{2}n(n-1) + n \geq n^2 - 5n + 14$$

or equivalently

$$n^2 - 11n + 28 \leq 0,$$

which is false for $n > 8$. □

7.1 All magic labelings of complete graphs

Here are all magic labelings for complete graphs. Notice that in every case the solution for a given k is unique (if one exists).

K_2 Trivially possible.

K_3 Sum values to be considered are $k = 9, 10, 11, 12$.

$k = 9$, $s = 6$, $S = \{1,2,3\}$.
$k = 10$, $s = 9$, $S = \{1,3,5\}$.
$k = 11$, $s = 12$, $S = \{2,4,6\}$.
$k = 12$, $s = 15$, $S = \{4,5,6\}$.

K_4 No solutions, by Corollary 1.1.

K_5 Sum values to be considered are $k = 18, 21, 27, 30, 33$.

$k = 18$, $s = 20$, $S = \{1,2,3,5,9\}$.
$k = 21$, $s = 30$, no solutions.
$k = 24$, $s = 40$, $S = \{1,8,9,10,12\}$.
$k = 24$, $s = 40$, $S = \{4,6,7,8,15\}$.
$k = 27$, $s = 50$, no solutions.
$k = 30$, $s = 60$, $S = \{7,11,13,14,15\}$.

K_6 Sum values to be considered are $k = 21, 25, 29, 33, 37, 41, 45$.

$k = 21$, $s = 21$, no solutions.
$k = 25$, $s = 36$, $S = \{1,3,4,5,9,14\}$.
$k = 29$, $s = 51$, $S = \{2,6,7,8,10,18\}$.
$k = 33$, $s = 66$, no solutions.
$k = 37$, $s = 81$, $S = \{4,12,14,15,16,20\}$.
$k = 41$, $s = 96$, $S = \{8,11,17,18,19,21\}$.
$k = 45$, $s = 111$, no solutions.

8 Cycles

The cycle C_v is regular of degree 2 and has v edges. So (1) becomes

$$v(v+1) \leq 2s \leq 2v^2 + v(v+1) = v(3v+1),$$

and (3) is

$$kv = s + v(2v+1),$$

whence v divides s; in fact $s = (k - 2v - 1)v$. When v is odd, s has $v+1$ possible values $\frac{1}{2}v(v+1), \frac{1}{2}v(v+3), \cdots, \frac{1}{2}v(v+2i-1), \cdots, \frac{1}{2}v(3v+1)$, with corresponding magic sums $\frac{1}{2}(5v+3), \frac{1}{2}(5v+5), \cdots, \frac{1}{2}(5v+2i+1), \cdots, \frac{1}{2}(7v+3)$. For even v, there are v values $s = \frac{1}{2}v^2 + v, \frac{1}{2}v^2 + 2v, \cdots, \frac{1}{2}v^2 + iv, \cdots, \frac{3}{2}v^2$, with corresponding magic sums $\frac{5}{2}v + 2, \frac{5}{2}v + 3, \cdots, \frac{5}{2}v + i + 1, \cdots, \frac{7}{2}v + 1$.

All odd cycles are super edge-magic as it is shown in [4]. Kotzig and Rosa [10] proved that all cycles are magic, producing examples with $k = 3v + 1$ for v odd, $k = \frac{5}{2}v + 2$ for $v \equiv 2 \pmod 4$ and $k = 3v$ for $v \equiv 0 \pmod 4$. In [7], labelings are exhibited for the minimum values of k in all cases. For convenience we give proofs for all cases, not exactly the same as the proofs in the papers cited. In each case the proof consists of exhibiting a labeling. If vertex-names need to be cited, we assume the cycle to be $(u_1, u_2, \cdots, u_v)$.

Theorem 4 *Every odd cycle has a magic labeling with* $k = \frac{1}{2}(5v+3)$.

Proof. Say $v = 2n+1$. Consider the cyclic vertex labeling $(1, n+1, 2n+1, n, \cdots, n+2)$, where each label is derived from the preceding one by adding $n \pmod{2n+1}$. The successive pairs of vertices have sums $n+2, 3n+2, 3n+1, \cdots, n+3$, which are all different. If $k = 5n+4$, the edge labels are $4n+2, 2n+2, 2n+3, \cdots, 4n+1$, as required. We have a magic labeling with $k = 5n+4 = \frac{1}{2}(5v+3)$ and $s = \frac{1}{2}v(v+1)$ (the smallest possible values). □

By duality, we have:

Corollary 4.1 *Every odd cycle has a magic labeling with* $k = \frac{1}{2}(7v+3)$. □

Theorem 5 *Every odd cycle has a magic labeling with* $k = 3v+1$.

Proof. Again write $v = 2n+1$. Consider the cyclic vertex labeling $(1, 2n+1, 4n+1, 2n-1, \cdots, 2n+3)$; in this case each label is derived from the preceding one by adding $2n \pmod{4n+2}$. The construction is such that the second, fourth, $\cdots$, $2n$-th vertices receive labels between 2 and $2n+1$ inclusive, while the third, fifth, $\cdots$, $(2n+1)$-th receive labels between $2n+2$ and $4n+1$. The successive pairs of vertices have sums $2n+2, 6n+2, 6n, 6n-2, \cdots, 2n+4$; if $k = 3v+1 = 6n+4$, the edge labels are $4n+2, 2, 4, \cdots, 4n$. We have a magic labeling with $k = 3v+1$ and $s = v^2$ (the case $i = \frac{1}{2}(v+1)$ in the list). □

Corollary 5.1 *Every odd cycle has a magic labeling with* $k = 3v+2$. □

Theorem 6 *Every even cycle has a magic labeling with* $k = \frac{1}{2}(5v+4)$.

Proof. Write $v = 2n$. If n is even,

$$\lambda(u_i) = \begin{cases} (i+1)/2 & \text{for } i = 1, 3, \cdots, n+1 \\ 3n & \text{for } i = 2 \\ (2n+i)/2 & \text{for } i = 4, 6, \cdots, n \\ (i+2)/2 & \text{for } i = n+2, n+4, \cdots, 2n \\ (2n+i-1)/2 & \text{for } i = n+3, n+5, \cdots, 2n-1, \end{cases}$$

while if n is odd,

$$\lambda(u_i) = \begin{cases} (i+1)/2 & \text{for } i = 1, 3, \cdots, n \\ 3n & \text{for } i = 2 \\ (2n+i+2)/2 & \text{for } i = 4, 6, \cdots, n-1 \\ (n+3)/2 & \text{for } i = n+1 \\ (i+3)/2 & \text{for } i = n+2, n+4, \cdots, 2n-1 \\ (2n+i)/2 & \text{for } i = n+3, n+5, \cdots, 2n-2 \\ n+2 & \text{for } i = 2n. \end{cases}$$

□

Corollary 6.1 *Every cycle of length divisible by 4 has a magic labeling with* $k = \frac{1}{2}(7v+2)$. □

Theorem 7 *Every cycle of length divisible by 4 has a magic labeling with* $k = 3v$.

Proof. For $v = 4$ the result is given by Theorem 6. So assume $v \geq 8$, write $v = 4n, n > 1$. The required labeling is

$$\lambda(u_i) = \begin{cases} i & \text{for } i = 1, 3, \cdots, 2n-1 \\ 4n+i+1 & \text{for } i = 2, 4, \cdots, 2n-2 \\ i+1 & \text{for } i = 2n, 2n+2, \cdots, 4n-2 \\ 4n+i & \text{for } i = 2n+1, 2n+3, \cdots, 4n-3 \\ 2 & \text{for } i = 4n-1 \\ 2v-2 & \text{for } i = 4n. \end{cases}$$

□

Corollary 7.1 *Every cycle of length divisible by 4 has a magic labeling with* $k = 3v+3$. □

8.1 Small cycles

We list all magic labelings of cycles up to C_6.

There are four labelings of C_3: see under K_3, in Section 7.1.

For C_4, the possibilities are $k = 12, 13, 14, 15$, with $s = 12, 16, 20, 24$ respectively. The unique solution for $k = 12$ is the cyclic vertex-labeling $(1, 3, 2, 6)$. For $k = 13$

there are two solutions: $(1, 5, 2, 8)$ and $(1, 4, 6, 5)$. The other cases are duals of these two.

For C_5, one must consider $k = 14, 15, 16 (s = 15, 20, 25)$ and their duals. The unique solution for $k = 14$ is $(1, 4, 2, 5, 3)$ (the solution from Theorem 4). There are no solutions for $k = 15$. For $k = 16$, one obtains $(1, 5, 9, 3, 7)$ (the solution from Theorem 5) and also $(1, 7, 3, 4, 10)$. Many other possible sets S must be considered when $k = 15$ or 16, but all can be eliminated using the following observation. The set S cannot contain three labels that add to k: for, in C_5, some pair of the corresponding vertices must be adjacent (given any three vertices of C_5, at least two must be adjacent), and the edge joining them would require the third label.

C_6 has possible sums $k = 17, 18, 19$ $(s = 24, 30, 36)$ and duals. For $k = 17$ there are three solutions: $(1, 5, 2, 3, 6, 7)$, $(1, 6, 7, 2, 3, 5)$ and $(1, 5, 4, 3, 2, 9)$. Notice that two non-isomorphic solutions have the same set of vertex labels. There is one solution for $k = 18$, $(1, 8, 4, 2, 5, 10)$, and six for $k = 19$, namely $(1, 6, 11, 3, 7, 8)$, $(1, 7, 3, 12, 5, 8)$, $(1, 8, 7, 3, 5, 12)$, $(1, 8, 9, 4, 3, 11)$, $(2, 7, 11, 3, 4, 9)$ and $(3, 4, 5, 6, 11, 7)$.

In the case of C_7, the possible magic sums run from 19 to 26, and Godbold and Slater [7] found that all can be realized; there are 118 labelings up to isomorphism. The corresponding numbers for C_8, C_9 and C_{10} are 282, 1540 and 7092 [7].

8.2 Generalizations of cycles

Paths

A path is a simplest caterpillar and those are known to be magic [10, 16].

Alternatively, the path P_n can be viewed as a cycle C_n with an edge deleted.

Say λ is a magic labeling of C_n with the property that label $2n$ appears on an edge. If that edge is deleted, the result is a P_n with a magic labeling.

For every n, there is a labeling of C_n in which $2n$ appears on an edge - the labelings in Theorem 4 and 6 have this property.

Suns

An n-sun is a cycle C_n with an edge terminating in a vertex of degree 1 attached to each vertex.

Theorem 8 *All suns are magic.* □

Proof. First we treat the odd case. Denote by λ the magic labeling of C_n given in Theorem 4. We construct a labeling μ which has $\mu(u) = \lambda(u) + n$ whenever u is a vertex or edge of the cycle. If a vertex has label x then the new vertex attached to it has label a_x, where $a_x \equiv x - \frac{1}{2}(n-1) (\mathrm{mod}\, n)$ and $1 \leq a_x \leq n$, and the edge joining them has label b_x, where $b_x \equiv n + 1 - 2x (\mathrm{mod}\, n)$ and $3n + 1 \leq b_x \leq 4n$. Then μ is a magic labeling with $k = \frac{1}{2}(11n + 3)$.

In the even case, λ is the magic labeling of C_n given in Theorem 6. The labeling μ again has $\mu(u) = \lambda(u) + n$ whenever u is an element of the cycle. The vertex with label x is adjacent to a new vertex with label a_x, and the edge joining them has label b_x, where:

- if $1 \leq x \leq \frac{1}{2}n$ then

- $a_x \equiv x + \frac{1}{2}n (\mathrm{mod}\, n)$ and $1 \le a_x \le n$,
- $b_x \equiv 2 - 2x (\mathrm{mod}\, n)$ and $3n+1 \le b_x \le 4n$;

• if $1 + \frac{1}{2}n \le x < n$ then
- $a_x \equiv x + \frac{1}{2}n + 1 (\mathrm{mod}\, n)$ and $1 \le a_x \le n$,
- $b_x \equiv 1 - 2x (\mathrm{mod}\, n)$ and $3n+1 \le b_x \le 4n$;

• $a_{\frac{3n}{2}} = b_{\frac{3n}{2}} = 1$.

Then μ is a magic labeling with $k = \frac{1}{2}(11n+4)$. □

Kites

An (n,t)-kite consists of a cycle of length n with a t-edge path (the *tail*) attached to one vertex. We write its labeling as the list of labels for the cycle (ending on the attachment point), separated by a semicolon from the list of labels for the path (starting at the vertex nearest the cycle).

Theorem 9 *An $(n,1)$-kite (a kite with tail length 1) is magic.*

Proof. For convenience, suppose the tail vertex is y and its point of attachment is z.

First, suppose n is odd. Denote by λ the magic labeling of C_n given in Theorem 4, with the vertices arranged so that $\lambda(z) = \frac{1}{2}(n+1)$. Define a labeling μ by $\mu(x) = \lambda(x)+1$ whenever x is an element of the cycle, $\mu(y) = 2v+2)$ and $\mu(y,z) = 1$. Then μ is a magic labeling with $k = \frac{1}{2}(5n+9)$.

If v is even, λ is the magic labeling of Theorem 6, with $\lambda(z) = \frac{1}{2}(v+2)$. Define a labeling μ by $\mu(x) = \lambda(x)+1$ whenever x is an element of the cycle, $\mu(y) = 2v+2)$ and $\mu(y,z) = 1$. Then μ is a magic labeling with $k = \frac{1}{2}(5v+10)$. □

9 Complete bipartite graphs

A magic labeling of a complete bipartite graph can be specified by giving two sets S_1 and S_2 of vertex labels.

Theorem 10 [10] *The complete bipartite graph $K_{m,n}$ is magic for any m and n.*

Proof. The sets $S_1 = \{n+1, 2n+2, \cdots, m(n+1)\}, S_2 = \{1, 2, \cdots, n\}$, define a magic labeling with $k = (m+2)(n+1)$. □

9.1 Small cases

A computer search has been carried out for magic labelings of $K_{2,3}$. The usual considerations show that $14 \le k \le 22$, with cases $k = 19, 20, 21, 22$ being the duals of cases $k = 17, 16, 15, 14$. The solutions up to $k = 18$ are

$k = 14$, no solutions
$k = 15$, $S_1 = \{1, 2\}$, $S_2 = \{3, 6, 9\}$
$k = 16$, $S_1 = \{1, 2\}$, $S_2 = \{5, 8, 11\}$
$S_1 = \{1, 3\}$, $S_2 = \{5, 6, 11\}$
$S_1 = \{4, 6\}$, $S_2 = \{1, 2, 7\}$
$S_1 = \{4, 8\}$, $S_2 = \{1, 2, 3\}$
$k = 17$, $S_1 = \{1, 8\}$, $S_2 = \{5, 6, 7\}$
$S_1 = \{5, 6\}$, $S_2 = \{1, 4, 9\}$
$k = 18$, $S_1 = \{1, 5\}$, $S_2 = \{9, 10, 11\}$
$S_1 = \{7, 11\}$, $S_2 = \{1, 2, 3\}$.
(The last two are of course duals.)

For $K_{3,3}$ one has $18 \leq k \leq 30$, and k must be even. Cases $k = 26, 28, 30$ are dual to cases $k = 22, 20, 18$. The solutions are
$k = 18$, no solutions
$k = 20$, $S_1 = \{1, 2, 3\}$, $S_2 = \{4, 8, 12\}$
$S_1 = \{1, 2, 9\}$, $S_2 = \{4, 6, 8\}$
$k = 22$, $S_1 = \{1, 2, 3\}$, $S_2 = \{7, 11, 15\}$
$S_1 = \{1, 3, 5\}$, $S_2 = \{7, 8, 15\}$
$S_1 = \{1, 5, 12\}$, $S_2 = \{6, 7, 8\}$
$k = 24$, no solutions.

9.2 Stars

A star is also a caterpillar and the fact that caterpillars are magic was given in [10, 16]. Here we present an alternative magic labeling of the star $K_{1,n}$.

Lemma 11 *In any magic labeling of a star, the center receives label 1, $n+1$ or $2n+1$.*

Proof. Suppose the center receives label x. Then

$$kn = \binom{2n+2}{2} + (n-1)x. \tag{9}$$

Reducing (9) modulo n we find

$$x \equiv (n+1)(2n+1) \equiv 1$$

and the result follows. □

Theorem 12 *There are $3 \cdot 2^n$ magic labelings of $K_{1,n}$, up to equivalence.*

Proof. Denote the center of a $K_{1,n}$ by c, the peripheral vertices by $v_1, v_2, \cdots, v_n$ and edge (c, v_i) by e_i. From Lemma 11 and (9), the possible cases for a magic labeling are $\lambda(c) = 1, k = 2n + 4$, $\lambda(c) = n + 1, k = 3n + 3$ and $\lambda(c) = 2n + 1, k = 4n + 2$. As the labeling is magic, the sums $\lambda(v_i) + \lambda(e_i)$ must all be equal to $M = k - \lambda(c)$

(so $M = 2n+3, 2n+2$ or $2n+1$). Then in each case there is exactly one way to partition the $2n+1$ integers $1, 2, \cdots, 2n+1$ into $n+1$ sets

$$\{\lambda(c)\}, \{a_1, b_1\}, \{a_2, b_2\}, \cdots, \{a_n, b_n\}$$

where every $a_i + b_i = M$. For convenience, choose the labels so that $a_i < b_i$ for every i and $a_1 < a_2 < \cdots < a_n$. Then up to isomorphism, one can assume that $\{\lambda(v_i), \lambda(e_i)\} = \{a_i, b_i\}$. Each of these n equations provides two choices, according as $\lambda(v_i) = a_i$ or b_i, so each of the three values of $\lambda(c)$ gives 2^n magic labelings of $K_{1,n}$. □

10 Odds, ends and conjectures

Trees

It is conjectured ([10], also [16]) whether all trees are magic. Kotzig and Rosa [10] proved that all caterpillars are magic. (A caterpillar is a graph derived from a path by hanging any number of pendant vertices from the vertices of the path.)

Enomoto *et al* [4] checked that all trees with less than 16 vertices are magic.

The Petersen graph

If the standard representation is used, with an ordinary cycle outside and a step-two cycle inside, the vertex labels

outside vertices (around the cycle) 13694;

inside vertices 105872 (1 adjacent to 10, 3 to 5, ...).

define a magic labeling.

Wheels

As was noted in Corollary 1.1, the n-spoke wheel W_n has no magic labeling when $n \equiv 3 (\bmod 4)$. Enomoto *et al* [4] have checked all wheels up to $n = 29$ and found that the graph is magic if $n \not\equiv 3 (\bmod 4)$. It is conjectured that W_n is magic whenever $n \not\equiv 3 (\bmod 4)$.

Disconnected graphs

Kotzig and Rosa show tK_4 is not magic for t odd, and the same is obviously true of the union of odd numbers of copies of K_n when $n \equiv 4 (\bmod 8)$. No results are known for even numbers of copies.

The one-factor F_{2n}, consisting of n independent edges, is magic if and only if n is odd [10].

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