The first BSTS with different upper and lower chromatic numbers*

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Abstract

In this paper the first BSTS in which the upper chromatic number is different from the lower chromatic number is determined. It is a BSTS(19) and its order is the lowest for which this property holds. In addition, all the possible strict colourings and their upper and lower chromatic number for systems of triples of the type BSTS(15), BSTS(19) and BSTS(21) are also determined.

1 Introduction

A *mixed hypergraph* [12, 13] is defined by the triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is a finite set whose elements are called *vertices*, while $\mathcal{C}$ and $\mathcal{D}$ are two families of subsets of X called $\mathcal{C}$*-edges* and $\mathcal{D}$*-edges* respectively.

If $\mathcal{C} = \emptyset$, then $\mathcal{H}$ is called a $\mathcal{D}$-hypergraph. If $\mathcal{D} = \emptyset$, then $\mathcal{H}$ is called a $\mathcal{C}$-hypergraph. A *strict k-colouring* is a colouring of the vertices of a $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$

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which uses k colours, such that in each *$\mathcal{C}$-edge* there are at least two vertices coloured with the same colour, while in each *$\mathcal{D}$-edge* there are at least two vertices coloured with different colours. If it is not necessary to know the number of colours used in a strict k-colouring then we will call this colouring a *strict colouring.*

A mixed hypergraph is called *uncolourable* if it cannot be coloured by a strict colouring.

The minimum number of colours in a strict colouring of $\mathcal{H}$ is called the *lower chromatic number* and is denoted by $\chi(\mathcal{H})$. The maximum number of colours for which there is a strict colouring is called the *upper chromatic number* and is denoted by $\bar{\chi}(\mathcal{H})$.

A $\mathcal{D}$-hypergraph coincides with the classic hypergraph as defined in [1] and the lower chromatic number coincides with the chromatic number introduced by Erdös and Hajnal in 1966 [3].

Two strict colourings are *different* if two vertices are coloured with two different colours in one strict colouring and with the same colour in the other one. Let us indicate with r_k the number of different strict colourings that colour $\mathcal{H}$ using k colours. Let us call the vector $R(\mathcal{H}) = (r_1, r_2, \cdots, r_{|X|})$ the *chromatic spectrum* of $\mathcal{H}$. When in a chromatic spectrum of $\mathcal{H}$ it is possible to find a $r_m = 0$ and two $r_l \neq 0$ and $r_n \neq 0$ with $l < m < n$ then this spectrum is called *broken* [6].

A Steiner Triple System, STS($\mid X \mid$), is defined by the pair $(X, \mathcal{B})$, where X is a finite set of v vertices and $\mathcal{B}$ is a family of subsets of X whose elements are called *blocks.* The following properties hold for this system:

1. each block contains only 3 vertices;

2. every two distinct vertices of X belong to one and only one block of $\mathcal{B}$.

For STS(v), it is known that v cannot be arbitrary: $v \equiv 1 \; or \; 3 \; (mod \; 6 \;)$. For a Steiner triple system of order v, a subset S of X is called a subsystem, if the blocks determined from any two distinct vertices of S are contained entirely in S. $\mid S \mid = s$ is the order of the subsystem.

A subset $L \subseteq X$ which does not contain $\mathcal{C}$-edges and $\mathcal{D}$-edges is called a *bi-stable set.*

In [7] STSs(v) were studied as particular mixed hypergraphs, called BSTSs(v) (Bi-Steiner Triple Systems), in which each block is simultaneously a $\mathcal{C}$-edge (also called a *co-edge*) and a $\mathcal{D}$-edge.

In [2] important results about strict 3, 4, 5-colourings were studied for small BSTSs.

In this paper all the possible strict colourings for all BSTSs(15), BSTSs(19) and BSTSs(21) are determined, together with the first BSTS in which the upper chromatic number is different from the lower chromatic number and other important information regarding the chromatic spectrum of these systems. We also determine an infinite class of BSTSs that can be coloured with at least two strict colourings using a different number of colours.

2 Preliminary results for BSTSs

In this section we present a technique for the construction of systems of triples, along with already obtained results which we be of use in the following sections. Here we will illustrate a technique for the recursive construction of STS(v). It is the well-known "doubling plus one construction" and makes it possible to obtain an STS($2v+1$) from an STS(v).

Construction 1 *Let us assume an STS(v), represented by the pair $(X', \mathcal{B}')$, with $\mid X' \mid = v$, and let X'' be a set of vertices disjoint from X' with a cardinality of $\mid X'' \mid = v+1$. It is possible to consider a 1-factorization $\mathcal{F} = \{F_1, F_2, \ldots, F_v\}$ of a complete graph K_{v+1} ($v+1$ is even) on the set of vertices X''. Let us now define the family of triple $\mathcal{B}$ on the set $X' \cup X''$ as follows:*

1. *each triple belonging to $\mathcal{B}'$ belongs to $\mathcal{B}$;*
2. *if $z_i \in X'$ $(i = 1, 2, \cdots, v)$ and $y_1,\ y_2 \in X''$ then $\{z_i, y_1, y_2\} \in \mathcal{B}$ if and only if $\{y_1, y_2\} \in F_i$.*

It is easy to prove that $(X, \mathcal{B})$ is an STS($2v+1$) and that $(X', \mathcal{B}')$ is a subsystem of it, whereas X'' is a bi-stable set.

In the following sections we will use the following theorem, in more generality in [4]:

Theorem 1 *In S(2,3,v), if $H \subseteq X$, $\mid H \mid = h$, T_h is the set of blocks included in H, and T_{v-h} is the set of blocks included in $X - H$, then*

$$\mid T_h \mid = r_0 - \binom{v-h}{1} r_1 + \binom{v-h}{2} - \mid T_{v-h} \mid$$

where $b_i = \binom{v-i}{2-i} / \binom{3-i}{2-i}$ *for* $0 \leq i \leq 2$. □

In [7] the following three results were proved.

Theorem 2 *If $\mathcal{H}$ is a BSTS of order $v \leq 2^h - 1$ $(h \in N)$, then $\bar{\chi}(\mathcal{H}) \leq h$.* □

Corollary 1 *If $\mathcal{H}$ is a BSTS(v) with $v \leq 2^h - 1$ and $\bar{\chi}(\mathcal{H}) = h$ then:*

1. $v = 2^h - 1$;
2. *in any strict coloring of $\mathcal{H}$ with h colors, the color classes have cardinalities of*

$$2^0,\ 2^1,\ 2^2,\ \cdots,\ 2^{h-1},$$

and all of them are bi-stable sets;

3. $\mathcal{H}$ is obtained from the STS(3) by repeated application of doubling plus one constructions. $\square$

Theorem 3 *The upper chromatic number of a BSTS of order $2^h - 1$ is equal to h if and only if it is obtained from the STS(3) by a sequence of doubling plus one constructions.* $\square$

Theorem 3 makes it possible to identify an infinite class of BSTSs with a cardinality of $2^h - 1$ and an upper chromatic number of $\bar{\chi} = h$.
In [8] the following Lemma,

Lemma 4 *Let $1 \leq n_1 \leq n_2 \leq ... \leq n_h \leq ... \leq n_k$ be an increasing sequence of k natural numbers and denote $s_i = n_1 + n_2 + ... + n_i$; if*

$$s_h(s_h - 1) = 3\sum_{j=1}^{h} n_j(n_j - 1) \qquad (h)$$

$$s_i(s_i - 1) \leq 3\sum_{j=1}^{i} n_j(n_j - 1) \qquad (i)$$

where $h < k$ and $h < i \leq k$, then

$$n_i \geq 2^{i-h-1} \cdot (s_h + 1),$$

$\square$

allows us to determine two infinite clases of STSs with a certain upper chromatic number. In fact from [8, 10] we can obtain the Theorem:

Theorem 5 *All BSTSs(v'), where $v' = 10 \cdot 2^k - 1$ or $14 \cdot 2^k - 1$, obtained from a sequence of doubling plus one construction which starts from STS(9) or STS(13), are colourable and $\bar{\chi}(v') = 3 + k$.* $\square$

3 Upper chromatic number for small BSTSs(v)

For any BSTS(v), if $\mathcal{P}$ is a strict colouring of the system using h colours, then let X_i denote the set of vertices coloured with the colour *(i)*, and let X_i be called the *i-th colour class* whose cardinality is $| X_i |= n_i$. Hence forward we will characterise strict colourings by means of vectors, the components of which indicate the increasing sequence of cardinalities of the colour classes. Let $\mathcal{S}_I$ be the union of $| I |$ colour classes, where $I \subseteq \{1, 2, \cdots, h\}$ define any subset of colours used in $\mathcal{P}$. If $| \mathcal{S}_I |= s_I$, it is possible to prove the following Theorems (see also [9]):

Theorem 6 *If $\mathcal{P}$ is a strict colouring for a BSTS, then the inequalities*

$$s_I(s_I - 1) \leq 3 \sum_{j \in I} n_j(n_j - 1) \tag{1}$$

are true for $2 \leq\mid I \mid\leq h$.

Proof. The set $\mathcal{S}_I$ contains no more than $|B_I| = [s_I(s_I - 1)/6]$ blocks and at least $(1/4)[s_I(s_I - 1) - \sum_{j \in I} n_j(n_j - 1)]$ blocks, corresponding to half the number of pairs of vertices coloured in $\mathcal{S}_I$ with two different colours. So $|B_I| \geq (1/4)[s_I(s_I - 1) - \sum_{j \in I} n_j(n_j - 1)]$ and (1) is demonstrated. □

Corollary 2 *If $\mathcal{P}$ is a strict colouring for a BSTS(v) and if for the same $\mathcal{S}_I$ we have:*

$$s_I(s_I - 1) \ = \ 3 \sum_{j \in I} n_j(n_j - 1), \tag{2}$$

then $\mathcal{S}_I$ is a subsystem of the BSTS(v).

Proof. We have $\mid B_I \mid$ blocks inside $\mathcal{S}_I$, and any pair of vertices of $\mathcal{S}_I$ coloured with two different colours or with one colour is inside a block which is contained in $\mathcal{S}_I$. □

Corollary 3 *If $\mathcal{P}$ is a strict colouring for BSTS(v), then*

$$v(v - 1) = 3 \sum_{i=1}^{h} n_i(n_i - 1).$$

Proof. This is evident because $\mid B \mid= (1/4)[v(v - 1) - \sum_{i=1}^{h} n_i(n_i - 1)]$. □

Lemma 7 *If $\mathcal{P}$ is a strict colouring for a BSTS(v), then for each pair of colour classes X_j and X_h the number of blocks in $X_h \bigcup X_k$ is $(n_h n_k)/2$.*

Proof. Each pair of elements containing one element from X_h and one element from X_k belongs to a single block contained in $X_h \bigcup X_k$. More precisely, each block contained in $X_h \bigcup X_k$ contains exactly two bichromatic pair of vertices. □

Theorem 8 *If $\mathcal{P}$ is a strict colouring for BSTS(v), then there is only one odd sized coloring class.*

Proof. By the previous Lemma there cannot exist two colouring classes with an odd cardinality. The existence of a single colouring class with an odd cardinality follows from the spectrum of STSs. □

Using Corollary 3 it is possible to obtain the following result.

Theorem 9 *A BSTS(15) is colourable only if it can be obtained by a sequence of double colourings starting from BSTS(3).*

Proof. In fact the only strict colouring possible is $(1,2,4,8)$ and so the Theorem follows from Corollary 1. □

Therefore all BSTSs(15) which contain the system STS(7) as a subsystem are strictly colourable with a single strict colouring and $\chi = \bar{\chi} = 4$.

For BSTSs(19) we have the following result.

Theorem 10 *If a BSTS(19) is strictly colourable it can be coloured only with one of the following three strict colourings: $(1,4,4,10)$, $(1,2,8,8)$ and $(4,6,9)$. In addition, there exist uncolourable BSTSs(19).*

Proof. By Theorem 6 and Corollary 3, the only possible colourings for a BSTS(19), if they exist, are: $\alpha = (1,4,4,10)$, $\beta = (1,2,8,8)$, $\gamma = (4,6,9)$.

From Corollary 2, the strict colouring $\alpha = (1,4,4,10)$ can be obtained for any BSTS(19) that contains a STS(9) as a subsystem.

There exists a BSTS(19) that can be coloured by means of the strict colouring $\beta = (1,2,8,8)$; see Table 1.

	$\{1,3,16\}$	$\{2,3,5\}$	$\{3,4,6\}$
	$\{4,5,7\}$	$\{5,6,8\}$	$\{6,7,9\}$
1)	$\{7,8,10\}$	$\{8,9,11\}$	$\{9,10,12\}$
	$\{10,11,13\}$	$\{11,12,14\}$	$\{12,13,15\}$
	$\{13,14,16\}$	$\{1,14,15\}$	$\{2,15,16\}$
	$\{1,2,4\}$		
	– – ––	– – ––	– – ––
	$\{3,7,12\}$	$\{4,8,13\}$	$\{5,9,14\}$
	$\{6,10,15\}$	$\{7,11,16\}$	$\{1,8,12\}$
2)	$\{2,9,13\}$	$\{3,10,14\}$	$\{4,11,15\}$
	$\{5,12,16\}$	$\{1,6,13\}$	$\{2,7,14\}$
	$\{3,8,15\}$	$\{4,9,16\}$	$\{2,6,11\}$
	$\{1,5,10\}$		
	$\{3,13,17\}$	$\{5,11,17\}$	$\{9,15,17\}$
	$\{2,8,17\}$	$\{4,14,17\}$	$\{6,12,17\}$
	$\{10,16,17\}$	$\{1,7,17\}$	$\{3,11,18\}$
3)	$\{5,15,18\}$	$\{7,13,18\}$	$\{2,10,18\}$
	$\{4,12,18\}$	$\{6,16,18\}$	$\{8,14,18\}$
	$\{1,9,18\}$	$\{3,9,19\}$	$\{5,13,19\}$
	$\{7,15,19\}$	$\{2,12,19\}$	$\{4,10,19\}$
	$\{6,14,19\}$	$\{8,16,19\}$	$\{1,11,19\}$
	$\{17,18,19\}$		

Table 1

In this system the colour classes are identified by the sets $\mathcal{A} = \{17\}$, $\mathcal{B} = \{18, 19\}$ and $\mathcal{C} = \{1, 3, 5, 7, 9, 11, 13, 15\}$ $\mathcal{D} = \{2, 4, 6, 8, 10, 12, 14, 16\}$.

There exists a BSTS(19) that can be coloured by means of the strict colouring $\gamma = (4, 6, 9)$. One of these is shown, in fact, in the following Table:

1)	$\{11,1,2\}$	$\{12,1,3\}$	$\{13,1,4\}$
	$\{14,2,3\}$	$\{15,2,4\}$	$\{16,3,4\}$
	$\{17,5,6\}$	$\{18,7,8\}$	$\{19,9,10\}$
	– – –	– – –	– – –
2)	$\{1,14,15\}$	$\{1,16,17\}$	$\{1,18,19\}$
	$\{2,12,13\}$	$\{2,16,18\}$	$\{2,17,19\}$
	$\{3,11,13\}$	$\{3,15,19\}$	$\{3,14,18\}$
	$\{4,11,19\}$	$\{4,14,18\}$	$\{4,12,17\}$
	$\{5,11,18\}$	$\{5,12,14\}$	$\{5,13,15\}$
	$\{5,16,19\}$	$\{6,11,14\}$	$\{6,13,19\}$
	$\{6,15,18\}$	$\{6,12,16\}$	$\{7,11,17\}$
	$\{7,12,19\}$	$\{7,15,16\}$	$\{7,13,14\}$
	$\{8,11,16\}$	$\{8,14,19\}$	$\{8,12,15\}$
	$\{8,13,17\}$	$\{9,11,12\}$	$\{9,13,18\}$
	$\{9,15,17\}$	$\{9,14,16\}$	$\{10,11,15\}$
	$\{10,12,18\}$	$\{10,13,16\}$	$\{10,14,17\}$
	– – –	– – –	– – –
3)	$\{1,5,10\}$	$\{1,6,7\}$	$\{1,8,9\}$
	$\{2,5,9\}$	$\{2,6,8\}$	$\{2,7,10\}$
	$\{3,5,8\}$	$\{3,6,10\}$	$\{3,7,9\}$
	$\{4,5,7\}$	$\{4,6,9\}$	$\{4,10,8\}$

Table 2

In this system the colour classes are identified by the sets $\mathcal{A} = \{1, 2, 3, 4\}$, $\mathcal{B} = \{5, 6, 7, 8, 9, 10\}$ and $\mathcal{C} = \{11, 12, 13, 14, 15, 16, 17, 18, 19\}$.

It is possible to determine uncolourable BSTSs(19), in fact Mathon, Phelps and Rosa in [11] found two cyclic STSs(19) with maximum bi-stable set having a cardinality seven. □

The structure of systems coloured with the colouring α is determined, thanks to Theorem 5 and Corollary 2, by the concept of double construction illustrated above.

In the system coloured with β, $\mathcal{A} \bigcup \mathcal{B}$ is obviously a block. By Theorem 1 we get $| T_{|\mathcal{C}\cup\mathcal{D}|} | + | T_{|\mathcal{A}\cup\mathcal{B}|} |= 33$ and therefore $| T_{|\mathcal{C}\cup\mathcal{D}|} |= 32$. $T_{|\mathcal{C}\cup\mathcal{D}|}$, in parts 1) and 2) of *Table* 1, is obtained cyclically from blocks $\{1, 2, 4\}$ and $\{1, 5, 10\}$ in such a way that each block always intersects $\mathcal{C}$ and $\mathcal{D}$. The pairs of $\mathcal{C}$ and $\mathcal{D}$ not present in any block of $T_{|\mathcal{C}\cup\mathcal{D}|}$ are partitioned into 3 1-factors that are to be found in part 3) of *Table* 1.

All systems that are coloured with γ have the following structure: from Theorem 1 we have $| T_{10} | - | T_9 |= 12$, so in the set $\mathcal{A} \bigcup \mathcal{B}$, as $\mathcal{C}$ is a bi-stable set, there are exactly 12 blocks. As $| T_{11} | - | T_8 |= 13$, each vertex of $\mathcal{C}$ can form one and only one block containing two vertices of $\mathcal{A} \bigcup \mathcal{B}$. In *Table* 2 the blocks formed by one vertex of $\mathcal{C}$ and two of $\mathcal{A}$ or $\mathcal{B}$ are represented in part 1). The pairs of $\mathcal{A} \bigcup \mathcal{B}$ not present in 1) are in part 3) and define the only 12 blocks in $\mathcal{A} \bigcup \mathcal{B}$. In part 2) there are all the pairs of the K_9 identified by $\mathcal{C}$. It is evident that such a system cannot contain a STS(9) as a subsystem, because the maximum cardinality of its bi-stable sets is less than ten.

Theorem 10 allows us to determine the colourings of colourable BSTSs(19). It is interesting to have information about the chromatic spectrum of these systems, and it is above all important to demonstrate that there can exist a BSTS(19) in which the upper chromatic number is different from the lower chromatic number. This result is important because all BSTSs studied up to now have a spectrum in which the upper chromatic number is the same as the lower one.

Numerical analysis of the bi-stable sets shows that in the systems in *Table* 1 and *Table* 2 the upper chromatic number coincides with the lower one, while in *Table* 3 below a BSTS(19) that can be coloured with both three and four colours is determined.

$\{1,2,13\}$	$\{1,3,12\}$	$\{1,4,11\}$
$\{5,6,19\}$	$\{6,7,16\}$	$\{6,8,15\}$
$\{7,8,17\}$	$\{7,9,18\}$	$\{8,10,14\}$

$\{5,7,1\}$	$\{5,8,4\}$	$\{5,9,3\}$
$\{5,10,2\}$	$\{6,9,2\}$	$\{6,10,1\}$
$\{6,3,4\}$	$\{7,10,3\}$	$\{7,2,4\}$
$\{8,9,1\}$	$\{8,2,3\}$	$\{4,9,10\}$
$\{1,15,18\}$	$\{1,14,17\}$	$\{1,16,19\}$
$\{6,11,12\}$	$\{6,13,14\}$	$\{6,17,18\}$
$\{8,11,16\}$	$\{8,12,13\}$	$\{8,18,19\}$
$\{7,11,13\}$	$\{7,12,14\}$	$\{7,15,19\}$
$\{2,11,14\}$	$\{2,12,15\}$	$\{2,16,18\}$
$\{2,17,19\}$	$\{3,13,18\}$	$\{3,11,19\}$
$\{3,14,15\}$	$\{3,16,17\}$	$\{4,12,17\}$
$\{4,14,18\}$	$\{4,13,19\}$	$\{4,15,16\}$
$\{5,11,17\}$	$\{5,12,18\}$	$\{5,13,15\}$
$\{5,16,14\}$	$\{9,11,15\}$	$\{9,12,16\}$
$\{9,13,17\}$	$\{9,14,19\}$	$\{10,11,18\}$
$\{10,12,19\}$	$\{10,13,16\}$	$\{10,17,15\}$

Table 3

This BSTS can be coloured by means of a colouring β with the following colour classes: $\mathcal{A} = \{6\}$, $\mathcal{B} = \{11,12\}$, $\mathcal{C} = \{1,3,4,5,10,17,18,19\}$ and $\mathcal{D} = \{2,7,8,9,13, 14,15,16\}$ and a colouring γ with the following classes: $\mathcal{A}' = \{1,2,3,4\}$, $\mathcal{B}' = \{5,6,7,8,9,10\}$ and $\mathcal{C}' = \{11,12,13,14, 15,16,17,18,19\}$.

Bearing in mind the results obtained here and in [9], the following Theorems can be stated.

Theorem 11 *The smallest order for which there exists a BSTS with an upper chromatic number different from its lower chromatic number is* 19. $\square$

Theorem 12 *There exists a BSTS(*19*) for which* $\chi = \bar{\chi} = 3$*, a BSTS(*19*) for which* $\chi = \bar{\chi} = 4$*, and a BSTS(*19*) for which* $\chi = 3$ *and* $\bar{\chi} = 4$. $\square$

Let us now consider BSTSs(21). The following result holds for these systems.

Theorem 13 *There exists a BSTS(*21*) that can be coloured by the two strict colourings* $(4,8,9)$ *and* $(5,6,10)$*, and in addition there exist uncolourable BSTSs(*21*).*

Proof. The colourings $(4,8,9)$ and $(5,6,10)$ are the only ones that meet the hypotheses of Theorem 6 and Corollaries 2 and 3. In addition, from *Table* 5 it can be seen that there exist BSTSs(21) that can be coloured with these colourings.

$BSTS_1$			$BSTS_2$		
$\{13,1,2\}$	$\{14,1,3\}$	$\{15,1,4\}$	$\{12,1,2\}$	$\{13,2,3\}$	$\{14,3,4\}$
$\{16,2,3\}$	$\{17,2,4\}$	$\{18,3,4\}$	$\{15,4,5\}$	$\{16,5,1\}$	$\{17,1,3\}$
$\{19,5,6\}$	$\{20,6,7\}$	$\{21,7,8\}$	$\{18,3,5\}$	$\{19,5,2\}$	$\{20,2,4\}$
$\{13,8,9\}$	$\{14,9,10\}$	$\{15,10,11\}$	$\{21,4,1\}$	$\{1,13,14\}$	$\{1,15,18\}$
$\{16,11,12\}$	$\{17,5,7\}$	$\{18,6,8\}$	$\{1,19,20\}$	$\{2,14,15\}$	$\{2,16,17\}$
$\{19,9,11\}$	$\{20,10,12\}$	$\{21,12,5\}$	$\{2,18,21\}$	$\{3,12,15\}$	$\{3,16,19\}$
$\{1,16,17\}$	$\{1,18,19\}$	$\{1,20,21\}$	$\{3,20,21\}$	$\{4,12,13\}$	$\{4,16,18\}$
$\{2,14,15\}$	$\{2,18,20\}$	$\{2,19,21\}$	$\{4,17,19\}$	$\{5,12,14\}$	$\{5,13,20\}$
$\{3,13,20\}$	$\{3,17,19\}$	$\{3,15,21\}$	$\{5,17,21\}$	$\{6,12,16\}$	$\{6,13,17\}$
$\{4,13,21\}$	$\{4,14,20\}$	$\{4,16,19\}$	$\{6,14,20\}$	$\{6,15,21\}$	$\{6,18,19\}$
$\{5,13,14\}$	$\{5,15,20\}$	$\{5,16,18\}$	$\{7,12,17\}$	$\{7,13,21\}$	$\{7,14,18\}$
$\{6,13,15\}$	$\{6,14,16\}$	$\{6,17,21\}$	$\{7,15,19\}$	$\{7,16,20\}$	$\{8,12,21\}$
$\{7,13,16\}$	$\{7,14,18\}$	$\{7,15,19\}$	$\{8,13,19\}$	$\{8,15,20\}$	$\{8,14,16\}$
$\{8,19,20\}$	$\{8,15,16\}$	$\{8,14,17\}$	$\{8,17,18\}$	$\{9,12,20\}$	$\{9,13,18\}$
$\{9,15,17\}$	$\{9,16,20\}$	$\{9,18,21\}$	$\{9,19,21\}$	$\{9,14,17\}$	$\{9,15,16\}$
$\{10,13,19\}$	$\{10,16,21\}$	$\{10,17,18\}$	$\{10,12,18\}$	$\{10,16,21\}$	$\{10,13,15\}$
$\{11,13,18\}$	$\{11,14,21\}$	$\{11,17,20\}$	$\{10,17,20\}$	$\{10,14,19\}$	$\{11,12,19\}$
$\{12,15,18\}$	$\{12,13,17\}$	$\{12,14,19\}$	$\{11,13,16\}$	$\{11,14,21\}$	$\{11,15,17\}$
$\{1,5,8\}$	$\{1,6,10\}$	$\{1,7,11\}$	$\{11,18,20\}$	$\{1,6,7\}$	$\{1,8,9\}$
$\{1,12,9\}$	$\{2,6,11\}$	$\{2,7,12\}$	$\{1,10,11\}$	$\{2,6,11\}$	$\{2,7,8\}$
$\{2,5,9\}$	$\{2,8,10\}$	$\{3,5,10\}$	$\{2,9,10\}$	$\{3,6,10\}$	$\{3,7,9\}$
$\{3,6,12\}$	$\{3,7,9\}$	$\{3,8,11\}$	$\{3,8,11\}$	$\{4,6,9\}$	$\{4,7,11\}$
$\{4,5,11\}$	$\{4,6,9\}$	$\{4,7,10\}$	$\{4,8,10\}$	$\{5,6,8\}$	$\{5,7,10\}$
$\{4,8,12\}$			$\{5,11,9\}$		

Table 4

where the colour classes for the system $BSTS_1$ are $\mathcal{A} = \{1,2,3,4\}$, $\mathcal{B} = \{5,6,7,8,9,10,11,12\}$ and $\mathcal{C} = \{13,14,15,16,17,18,19,20,21\}$, while for the system $BSTS_2$

they are $\mathcal{A}' = \{1,2,3,4,5\}$, $\mathcal{B}' = \{6,7,8,9,10,11\}$ and $\mathcal{C}' = \{12,13,14,15,16,17,18, 19,20,21\}$. Given the results obtained in [11], there exist eight STSs(21) where the maximum cardinality of a bi-stable set is eight. □

Theorem 14 *For colourable BSTSs(21), we have $\bar{\chi} = \chi = 3$.* □

From the previous results, and using the same techniques as in [8] it is possible to determine infinite classes of BSTSs with a certain chromatic number. More specifically, there exist classes of BSTSs($20 \cdot 2^k - 1$) with $\bar{\chi} = 3 + k$, others with $\bar{\chi} = 4 + k$, and classes of BSTSs($22 \cdot 2^k - 1$) with $\bar{\chi} = 3 + k$. In addition, all BSTSs that contain an uncolourable BSTS(15), BSTS(19), or BSTS(21) as a subsystem are themselves uncolourable BSTSs. It is important to note that there can exist infinite classes of BSTSs($20 \cdot 2^k - 1$) in which the upper chromatic number is not the same as the lower one.

Theorem 15 *If a BSTS($20 \cdot 2^k - 1$) is obtained by a sequence of k double constructions and from a BSTSs(19) in which $\chi = 3$ and $\bar{\chi} = 4$, then this system is $3 + k$ and $4 + k$ colourable, and also $\bar{\chi} = 4 + k$.*

Proof. The proof is easily obtained from Lemma 4 and the fact that the system is obtained by a sequence k of double constructions. □

4 Concluding remarks

This paper is the first to study upper and lower chromatic numbers of BSTSs and gives the first information on the chromatic spectrum of these systems.

We have determined the smallest BSTS (order 19) in which the upper chromatic number is different from the lower one. On the basis of this result we have determined an infinite class of BSTSs that can be coloured with strict colourings using a different number of colours.

For all colourable BSTSs(15) BSTSs(19) and BSTSs(21), all the possible strict colourings have been determined. In [7] it is demonstrated that there exist uncolourable BSTSs; in this paper the uncolourable BSTSs of the order of 15, 19, and 21 are characterised.

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