Generators of matrix incidence algebras

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Abstract

Let $n \in \mathbb{Z}^+$ and let K be a field. Let $\preceq$ be a partial order on $\{1, 2, \ldots, n\}$. Let $\mathcal{A}_n(\preceq)$ be the matrix incidence algebra consisting of those $n \times n$ matrices $A = (a_{i,j})$ with entries in K, satisfying $a_{i,j} = 0$ whenever $i \npreceq j$. For a subset $\mathcal{E} \subseteq \mathcal{A}_n(\preceq)$, a necessary and sufficient condition that the algebra generated by $\mathcal{E} \cup \{I\}$ is $\mathcal{A}_n(\preceq)$ is that (i) for every $1 \leq i, j \leq n$ with $i \neq j$, there exists $A \in \mathcal{E}$ such that $a_{i,i} \neq a_{j,j}$ and (ii) for every $i \preceq j$ with j covering i, there exists $B \in \text{span } \mathcal{E}$ such that $b_{i,j} \neq 0$ and $b_{i,i} = b_{j,j}$. If the characteristic of K is zero or $> n$, the algebra $\mathcal{A}_n(=)$ is singly generated and, if $\preceq$ is not equality, $\mathcal{A}_n(\preceq)$ has two generators.

1. PRELIMINARIES

Let $(S, \ll)$ be a locally finite partially ordered set. Here, local finiteness means that every interval $[x, y] = \{u \in S : x \ll u \ll y\}$ is finite. Let K be a field. The incidence algebra $\mathcal{A}(S)$ of S over K is the set of all functions $f : S \times S \to K$ with the property that $f(x, y) = 0$ whenever $x \not\ll y$. $\mathcal{A}(S)$ becomes an associative K-algebra with the pointwise operations of addition and scalar multiplication and with the Dirichlet product :

$$(f * g)(x, y) = \sum_{x \ll u \ll y} f(x, u)g(u, y).$$

The Kronecker delta function is the multiplicative identity of $\mathcal{A}(S)$. In [3], Rota proposed the idea of such algebras as a basis for a unified study of combinatorial theory. In [1] (see also [2]), certain subalgebras of $\mathcal{A}(S)$ were considered. Here we consider generators of $\mathcal{A}(S)$.

If $n \in \mathbb{Z}^+$ and $\preceq$ is a partial order on $\{1, 2, \ldots, n\}$, the corresponding incidence algebra can be identified in a natural way with the algebra of $n \times n$ matrices (with entries in K) $A = (a_{i,j})$ satisfying $a_{i,j} = 0$ whenever $i \not\preceq j$, with the usual matrix operations. (The Dirichlet product becomes matrix multiplication.) We will call such matrix algebras *matrix incidence algebras* and, with a slight change of notation, use $\mathcal{A}_n(\preceq)$ to denote the matrix incidence algebra described immediately above. If $\preceq$ is *consistent with the natural order* on $\{1, 2, \ldots, n\}$ in the sense that $i \preceq j$ implies $i \leq j$, then $\mathcal{A}_n(\preceq)$ is an algebra of upper-triangular matrices. Any incidence algebra arising from a finite partially ordered set is isomorphic to some matrix incidence algebra $\mathcal{A}_n(\preceq)$ where $\preceq$ is consistent with the natural order. Indeeed, if $(S, \ll)$ is a finite partially ordered set and $x_1, x_2, \ldots, x_n$ is an enumeration of S satisfying: $x_i \ll x_j$ implies $i \leq j$, then $\mathcal{A}(S)$ is isomorphic to $\mathcal{A}_n(\preceq)$ (where $\preceq$ is defined by $i \preceq j$ if $x_i \ll x_j$) by the map $f \mapsto (f(x_i, x_j))$. It is not unwise therefore, to concentrate attention on matrix algebras of the type $\mathcal{A}_n(\preceq)$ where $\preceq$ is consistent with the natural order.

Notice that $\mathcal{A}_n(=)$ is the algebra of diagonal matrices, and $\mathcal{A}_n(\leq)$ is the algebra of upper-triangular matrices. Throughout, K will denote a fixed but arbitrary field, and all matrices will be assumed to have entries in K. Also, I will denote the identity matrix and span $\mathcal{E}$ will denote the linear span of $\mathcal{E}$.

2. MAIN THEOREM

First we need a lemma. In what follows, we will use the notation $(A)_{i,j}$ to denote the i, j-entry of a matrix A.

LEMMA. *Let $n \in \mathbb{Z}^+$ and let $T_1, T_2, \ldots, T_n$ be $n \times n$ upper-triangular matrices satisfying $(T_i)_{i,i} = 0, i = 1, 2, \ldots, n$. Then $T_1 T_2 \ldots T_n = 0$.*

Proof. The result is true when $n = 1$. Assume that it is true for n and let $T_1, T_2, \ldots, T_{n+1}$ be $(n+1) \times (n+1)$ upper-triangular matrices satisfying $(T_i)_{i,i} = 0, i = 1, 2, \ldots, n+1$. By the assumption, $T_1 T_2 \ldots T_n = \begin{pmatrix} 0 & X \\ 0 & \lambda \end{pmatrix}$, for some $n \times 1$ matrix X and $\lambda \in K$. Since $T_{n+1} = \begin{pmatrix} Y & Z \\ 0 & 0 \end{pmatrix}$, for some $n \times n$ matrix Y and $n \times 1$ matrix Z we have

$$T_1 T_2 \ldots T_n T_{n+1} = \begin{pmatrix} 0 & X \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} Y & Z \\ 0 & 0 \end{pmatrix} = 0.$$

The proof is completed by induction.∎

In the above lemma the order of the factors is important. For example, over any field of characteristic zero, only one product of the following three matrices is zero :

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

THEOREM. *Let $n \in \mathbb{Z}^+$ and let K be a field. Let $\preceq$ be a partial order on $\{1, 2, \ldots, n\}$ and let $\mathcal{A}_n(\preceq)$ be the corresponding matrix incidence algebra over K.*

Let $\mathcal{E}$ be a subset of $\mathcal{A}_n(\preceq)$. The algebra generated by $\mathcal{E} \cup \{I\}$ is $\mathcal{A}_n(\preceq)$ if and only if

(i) for every $1 \leq i, j \leq n$ with $i \neq j$, there exists $A \in \mathcal{E}$ such that $(A)_{i,i} \neq (A)_{j,j}$, and

(ii) for every $1 \leq i, j \leq n$ with $i \preceq j$ and with j covering i, there exists $B \in$ span $\mathcal{E}$ such that $(B)_{i,j} \neq 0$ and $(B)_{i,i} = (B)_{j,j}$.

Proof. Suppose first that the partial order $\preceq$ is consistent with the natural order. Then, as noted earlier, $\mathcal{A}_n(\preceq)$ is an algebra of upper-triangular matrices.

Let $\mathcal{B}$ denote the algebra generated by $\mathcal{E} \cup \{I\}$ and put $\mathcal{A} = \mathcal{A}_n(\preceq)$.

Suppose that conditions (i) and (ii) are satisfied. We proceed by induction. The result is true for $n = 1$. Assume that the result is true for n. Consider the situation for $n+1$. Temporarily, for any $(n+1) \times (n+1)$ matrix A let $\hat{A}$ denote the $n \times n$ matrix occurring in the top left-hand corner of A. Since $\widehat{\text{span}\mathcal{E}} =$ span $\hat{\mathcal{E}}$, the induction assumption gives that the algebra generated by $\hat{\mathcal{E}} \cup \{\hat{I}\}$ is $\hat{\mathcal{A}}$. But the algebra generated by $\hat{\mathcal{E}} \cup \{\hat{I}\}$ is $\hat{\mathcal{B}}$, so $\hat{\mathcal{A}} = \hat{\mathcal{B}}$. Thus, for every $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $\hat{A} = \hat{B}$.

By condition (i), for every $1 \leq i \leq n$, there exists $A_i \in \mathcal{B}$ such that $(A_i)_{i,i} = 0$ and $(A_i)_{n+1,n+1} = 1$. Then A_0 defined by $A_0 = A_1 A_2 \ldots A_n$ belongs to $\mathcal{B}$ and, by the lemma, satisfies $\hat{A}_0 = 0$ and $(A_0)_{n+1,n+1} = 1$.

If $n+1$ covers no element of $\{1, 2, \ldots, n\}$, every entry in the last column of any element of $\mathcal{A}$ is zero, except possibly for the $n+1, n+1$- entry. In this case it easily follows that $\mathcal{B} = \mathcal{A}$. For then, if $A \in \mathcal{A}$ and $B \in \mathcal{B}$ satisfy $\hat{A} = \hat{B}$, then $B + \mu A_0 = A$, for some $\mu \in K$. So $A \in \mathcal{B}$.

On the other hand, suppose that $n+1$ covers at least one element of $\{1, 2, \ldots, n\}$. Let $E_{k,l}$ denote the $(n+1) \times (n+1)$ matrix having k, l-th entry equal to 1 and all other entries zero. We show that $E_{i,n+1} \in \mathcal{B}$, for every i satisfying $i \preceq n+1$. From this, together with the fact that $\hat{\mathcal{A}} = \hat{\mathcal{B}}$, it readily follows that $\mathcal{A} = \mathcal{B}$.

Let the elements of $\{1, 2, \ldots, n\}$ covered by $n+1$ be $i_1, i_2, \ldots, i_q$, where $1 \leq q \leq n$ and $i_1 < i_2 < \ldots < i_q < n+1$. By condition (ii), for every $1 \leq p \leq q$, there exists $B_p \in \mathcal{B}$ such that $(B_p)_{i_p,n+1} = 1$ and $(B_p)_{i_p,ip} = (B_p)_{n+1,n+1} = 0$. For every $1 \leq p \leq q$, there exists $F_p \in \mathcal{B}$ such that $\hat{E}_{i_p,i_p} = \hat{F}_p$. All the rows of $F_p B_p$ are zero except the i_p-th and this is the same as the i_p-th row of B_p. Note that, for every $1 \leq p \leq q$ and $X, Y \in \mathcal{A}$ we have $\sum_{j=1}^{n} (X)_{i_p,j}(Y)_{j,n+1} = (X)_{i_p,i_p}(Y)_{i_p,n+1}$ since $n+1$ covers i_p. Using this (recalling that $(B_p)_{i_p,i_p} = 0$) gives $F_p B_p A_0 = E_{i_p,n+1}$, so $E_{i_p,n+1} \in \mathcal{B}$ for $1 \leq p \leq q$.

Next, suppose that $1 \leq i \leq n$ and $i \preceq n+1$, but $n+1$ does not cover i. Then $i \preceq i_p$, for some $1 \leq p \leq q$ and there exists $G_{i,i_p} \in \mathcal{B}$ such that $\hat{E}_{i,i_p} = \hat{G}_{i,i_p}$. Then $G_{i,i_p} F_p B_p A_0 = E_{i,n+1}$, so $E_{i,n+1} \in \mathcal{B}$.

Since $A_0 \in \mathcal{B}$ and $E_{i,n+1} \in \mathcal{B}$ whenever $1 \leq i \leq n$ and $i \preceq n+1$, we get that $E_{n+1,n+1} \in \mathcal{B}$. It now follows that $\mathcal{A} = \mathcal{B}$ and the proof is completed by induction.

Conversely, let $\mathcal{A} = \mathcal{B}$. Since $\{A \in \mathcal{A} : (A)_{i,i} = (A)_{j,j}\}$ is a proper subalgebra of $\mathcal{A}$, for every $1 \leq i, j \leq n$ with $i \neq j$, condition (i) holds.

Suppose that condition (ii) does not hold. Then there exist $1 \le i, j \le n$ with $i \preceq j$ and with j covering i, such that, for every $B \in \operatorname{span} \mathcal{E}$, $(B)_{i,j} = 0$ or $(B)_{i,i} \ne (B)_{j,j}$. This conclusion is, in fact, valid for every $B \in \operatorname{span}(\mathcal{E} \cup \{I\})$. Let $\mathcal{F} = \operatorname{span}(\mathcal{E} \cup \{I\})$. Temporarily, for any $n \times n$ matrix A, let $\widetilde{A}$ be the 2×2 matrix given by $\widetilde{A} = \begin{pmatrix} (A)_{i,i} & (A)_{i,j} \\ (A)_{j,i} & (A)_{j,j} \end{pmatrix}$. Then $\widetilde{\mathcal{F}} = \operatorname{span}(\widetilde{\mathcal{F}} \cup \{\widetilde{I}\})$ is a subspace of the vector space of 2×2 upper-triangular matrices, of dimension < 3. It cannot be the case that $(B)_{i,j} = 0$, for every $B \in \mathcal{E}$ since $\{A \in \mathcal{A} : (A)_{i,j} = 0\}$ is a proper subalgebra of $\mathcal{A}$ containing I. (Note that, since j covers i, for every $X, Y \in \mathcal{A}$, we have $(XY)_{i,j} = (X)_{i,i}(Y)_{i,j} + (X)_{i,j}(Y)_{j,j}$.) Hence there exists $B_0 \in \mathcal{E}$ such that $\widetilde{B} = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}$ with $\gamma \ne 0$. Since $\alpha \ne \beta$, $\begin{pmatrix} 0 & \delta \\ 0 & 1 \end{pmatrix} \in \widetilde{\mathcal{F}}$, for some $\delta \ne 0$. It follows that $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \delta \\ 0 & 1 \end{pmatrix} \right\}$ is a basis for $\widetilde{\mathcal{F}}$. But the linear span of $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \delta \\ 0 & 1 \end{pmatrix} \right\}$ is an algebra (since $\begin{pmatrix} 0 & \delta \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 0 & \delta \\ 0 & 1 \end{pmatrix}$). Thus $\widetilde{\mathcal{F}}$ is a proper subalgebra of the algebra of all 2×2 upper-triangular matrices. The set of matrices $\{A \in \mathcal{A} : \widetilde{A} \in \widetilde{\mathcal{F}}\}$ is therefore a proper subalgebra of $\mathcal{A}$ containing $\mathcal{E} \cup \{I\}$. (Note that $\widetilde{XY} = \widetilde{X}\widetilde{Y}$, for every $X, Y \in \mathcal{A}$.) This contradicts the fact that $\mathcal{B} = \mathcal{A}$ and the proof for the case where $\preceq$ is consistent with the natural order is complete.

Finally, let $\preceq$ be any partial order defined on $\{1, 2, \ldots, n\}$. Choose a permutation τ of $\{1, 2, \ldots, n\}$ satisfying $\tau(i) \preceq \tau(j)$ imples $i \le j$. The partial order $\ll$, defined on $\{1, 2, \ldots, n\}$ by $i \ll j$ if $\tau(i) \preceq \tau(j)$, is consistent with the natural order. Let V be the $n \times n$ matrix having all its $i, \tau(i)$-entries equal to 1 ($i = 1, 2, \ldots, n$) and all other entries zero. Then V is invertible and V^{-1} is the transpose of V (it has all its $\tau(i), i$- entries equal to 1 and zeros elsewhere). Then $A \mapsto VAV^{-1}$ is an algebra isomorphism of $\mathcal{A}_n(\preceq)$ onto $\mathcal{A}_n(\ll)$. Let $\mathcal{E}$ be as in the statement of the theorem. Then $\mathcal{E} \cup \{I\}$ generates $\mathcal{A}_n(\preceq)$ if and only if $V\mathcal{E}V^{-1} \cup \{I\}$ generates $\mathcal{A}_n(\ll)$. It is easy to check that $V\mathcal{E}V^{-1}$ satisfies conditions (i) and (ii), with $\preceq$ replaced by $\ll$, if and only if $\mathcal{E}$ satisfies them as they stand. This completes the proof. ∎

The above theorem includes results about generating sets for the algebra of diagonal matrices and for the algebra of upper-triangular matrices: take $\preceq$ to be, respectively, equality or the natural order.

COROLLARY 1. *If $\mathcal{E}$ is a set of diagonal $n \times n$ matrices, then the algebra generated by $\mathcal{E} \cup \{I\}$ is the algebra of all diagonal matrices if and only if, for every $1 \le i, j \le n$ with $i \ne j$, there exists $A \in \mathcal{E}$ such that $(A)_{i,i} \ne (A)_{j,j}$. If the characteristic of the field K is zero or $> n$, any diagonal matrix with non-zero distinct diagonal entries generates the algebra of $n \times n$ diagonal matrices over K.*

Proof. The first part of the statement of this corollary follows from the theorem. Suppose that the characteristic of K is zero or $> n$, and let A be any diagonal matrix with non-zero distinct diagonal entries. Then $\{A, I\}$ generates the algebra of diagonal matrices by the theorem. But $p(A) = I$ for some polynomial

$p(x)$ satisfying $p(0) = 0$. It follows that $\{A\}$ generates the algebra of diagonal matrices.∎

The proof of the following corollary is obvious.

COROLLARY 2. *If $\preceq$ is different from equality, and if the characteristic of the field K is zero or $> n$, then $\mathcal{A}(\preceq)$ has two generators. Indeed, in this case, any diagonal matrix with non-zero distinct diagonal entries, and any matrix having a non-zero i,j-entry whenever j covers i and all its other entries equal to zero, together generate $\mathcal{A}(\preceq)$.*

REMARKS. 1. Clearly '$\mathcal{E}$' can be replaced by 'span $\mathcal{E}$' in condition (i) of the statement of the theorem. However, 'span $\mathcal{E}$' cannot be replaced by '$\mathcal{E}$' in condition (ii). For example, if $\mathcal{E} = \left\{ \begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \right\}$, then $\mathcal{E} \cup \{I\}$ generates the algebra of upper-triangular 2×2 matrices since $\begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{span } \mathcal{E}$, but $\mathcal{E}$ itself does not satisfy condition (ii) of the theorem.
2. In the statement of the theorem '$\mathcal{E} \cup \{I\}$' cannot be replaced by '$\mathcal{E}$'. For example,

$$\mathcal{E} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

does not generate the algebra of 3×3 upper-triangular matrices.

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