Doubly dependent sets in graphs

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Abstract

Let G be a graph. A subset X of $V(G)$ is said to be *dependent* if X is not independent. We say that X is *doubly dependent* if both X and $V(G) \setminus X$ are dependent.

Let $d_k(G)$ denote the number of doubly dependent sets in G of cardinality k. In this paper, we show that the sequence $\{d_k(G)\}$ is unimodal and, in particular, if G has r vertices then $\max_k\{d_k(G)\} = d_{\lfloor r/2 \rfloor}(G)$. We also show that the partially ordered set D_G consisting of all doubly dependent sets of G ordered by inclusion does not, in general, have the Sperner property.

1 Introduction

Let G be a graph and let the vertex set of G be denoted by $V(G)$. A subset X of $V(G)$ is called an *independent set* if no two vertices in X are adjacent in G. We will say that a subset X of $V(G)$ is *dependent* if it is not independent. Further, let us define $X \subseteq V(G)$ to be *doubly dependent* if both X and $V(G) \setminus X$ are dependent. Finally, $X \subseteq V(G)$ is said to be *singly dependent* if X is dependent and $V(G) \setminus X$ is independent.

Let $p_k(G)$ be the number of dependent sets in G of cardinality k. Moreover, let $s_k(G)$ and $d_k(G)$ respectively denote the number of singly dependent and doubly dependent sets in G of cardinality k.

The poset consisting of all dependent sets of G ordered by inclusion has $\{p_k(G)\}$ as its sequence of Whitney numbers. Sperner-type results for this poset have been obtained, and several properties of the sequence $\{p_k(G)\}$ have been discovered as we shall see in Section 3. The purpose of this paper is to study the poset of doubly dependent sets and to investigate the sequence $\{d_k(G)\}$.

2 Terminology

Let P be a finite partially ordered set (poset). A subset C of elements of P is called a *chain* if any two elements of C are comparable. The *length* of the chain C is $|C|-1$.

If every maximal chain in P has the same length then P is said to be *graded.* We say that y *covers* x if $x < y$ and there does not exist $z \in P$ such that $x < z < y$. A *rank function* for P is a function $r : P \to \{0, 1, 2, \ldots, \}$ such that $r(y) = r(x) + 1$ whenever y covers x in P. A *ranked poset* consists of a poset together with a rank function.

Let P be a ranked poset with rank function r. The set $P_k = \{x \in P \mid r(x) = k\}$ is called the *k-th rank* of P. The sequence of *rank numbers* or *Whitney numbers* of P is $\{|P_k|\}_{k\geq 0}$. The sequence $a_0, a_1, \ldots, a_n$ of real numbers is said to be *unimodal* if there is an integer k such that

$$a_0 \leq a_1 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_{n-1} \geq a_n.$$

The ranked poset P is called *rank unimodal* if its Whitney numbers form a unimodal sequence. Furthermore, the sequence $a_0, a_1, \ldots, a_n$ is *symmetric* if $a_i = a_{n-i}$ for all i. We say that P is *rank symmetric* if the sequence of its Whitney numbers is symmetric.

An *antichain* is a set of elements of P, no two of which are comparable. A ranked poset P has the *Sperner property* if the maximum size of an antichain in P equals the maximum size of a rank of P. Further terminology regarding the combinatorics of partially ordered sets may be found in [1].

We will also require some basic terminology of graph theory which may be found in [2]. In particular, a graph G is said to be *nontrivial* if its edge set is not empty. For $n \geq 2$, we define the *n-star* S_n to be the complete bipartite graph $K_{1,n-1}$. Let P_n denote the path on n vertices, and let Z_m denote the graph consisting of m vertices and no edges. Finally, for graphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \emptyset$, let $G_1 + G_2$ denote the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

3 Posets from Graphs

3.1 The poset of dependent sets

Let G be a nontrivial graph. We denote by P_G the poset consisting of all dependent sets in G, ordered by set-theoretic inclusion. The poset P_G is graded and also ranked since $r : X \to |X|$ serves as a rank function for P_G. This is a convenient rank function to use even though the minimal elements in P_G have rank 2. Indeed, with this rank function, the Whitney numbers of P_G are $p_2(G), p_3(G), \ldots, p_r(G)$ where $r = |V(G)|$.

The poset P_G has been studied extensively in connection with a conjecture of Lih (see, for example, [6], [8], [7], and [4]). The following theorem was proved in the case that G has an odd number of vertices by Zhu [8], and for the case of an even number of vertices by Horrocks [4].

Theorem 3.1 *For every nontrivial graph G, the poset P_G has the Sperner property.*

3.2 Matchings in posets

Let P_i and P_j be two ranks in the ranked poset P. We say that there is a *matching from* P_i to P_j if there exists an injection $f : P_i \to P_j$ such that $f(x)$ and x are comparable for all $x \in P_i$. Further, there is a *matching between* two ranks of a ranked poset if there is a matching from the smaller sized rank to the larger one.

The following theorem which shows, in particular, that the rank numbers of P_G are unimodal was obtained by Zha [7].

Theorem 3.2 *Let G be a nontrivial graph on r vertices and let $P_2(G), \ldots, P_r(G)$ be the ranks of P_G. Then P_G is rank unimodal with largest rank $P_{n+1}(G)$ if $r = 2n+1$, and $P_n(G)$ or $P_{n+1}(G)$ if $r = 2n$. Moreover, a matching exists between every pair of adjacent ranks in P_G, except possibly in the case $(P_n(G), P_{n+1}(G))$ when $r = 2n$.*

Using the fact that P_G has the Sperner property for any nontrivial graph G, we may show that, in fact, a matching exists between every pair of adjacent ranks of P_G. To do this will require the classical matching theorem of P. Hall [3].

Theorem 3.3 *Let G be a nontrivial graph. Then there is a matching between every pair of adjacent ranks of P_G.*

Proof: By virtue of Theorem 3.2, we need only consider the ranks $P_n(G)$ and $P_{n+1}(G)$ in P_G where G is a nontrivial graph on $2n$ vertices.

Suppose that $p_n(G) \geq p_{n+1}(G)$. If a matching does not exist between $P_n(G)$ and $P_{n+1}(G)$ then by Hall's theorem there is $S \subseteq P_{n+1}(G)$ such that $|S| > |N(S)|$ where $N(S) = \{X \in P_n(G) \mid X \subseteq Y \text{ for some } Y \in S\}$. Now $A = S \cup (P_n(G) \setminus N(S))$ is an antichain and

$$|A| = |S| + |P_n(G)| - |N(S)| > |P_n(G)|.$$

But this contradicts the fact that P_G has the Sperner property since $P_n(G)$ is the largest rank of P_G. The proof for the case $p_{n+1}(G) \geq p_n(G)$ is similar. □

3.3 The poset of doubly dependent sets

Let D_G be the poset consisting of all doubly dependent sets in G, ordered by inclusion. Like P_G, D_G is graded and ranked. Once again, we use $r : X \to |X|$ as a rank function so the sequence of Whitney numbers of D_G is $\{d_k(G)\}$. In Sections 4 and 5, we prove that this sequence is unimodal and symmetric, and determine the largest Whitney number of D_G. Finally, in Section 6, we show that the poset D_G need not have the Sperner property.

4 Rank Unimodality and Symmetry of D_G

The purpose of this section is to show that the poset D_G is rank unimodal and rank symmetric.

4.1 Rank symmetry

First, we observe that $X \in V(G)$ is doubly dependent if and only if $V(G) \setminus X$ is doubly dependent so D_G is rank symmetric. This fact is recorded in the following theorem.

Theorem 4.1 *Let G be a nontrivial graph on r vertices. Then $d_i(G) = d_{r-i}(G)$ for all $2 \leq i \leq r-2$.*

4.2 Rank unimodality

Theorem 4.2 *Let G be a nontrivial graph on r vertices and let $D_2(G), \ldots, D_{r-2}(G)$ be the ranks of D_G. Then D_G is rank unimodal and the largest rank is $D_n(G)$ (and $D_{n+1}(G)$) if $r = 2n+1$, and $D_n(G)$ if $r = 2n$.*

In order to prove Theorem 4.2, we will use the following lemma which states that a matching in P_G from any rank to the one immediately below it induces a matching between the corresponding ranks in D_G.

Lemma 4.3 *Suppose that $P_{i+1}(G)$ may be matched to $P_i(G)$ in the poset P_G. Then $d_i(G) \geq d_{i+1}(G)$.*

Proof: We will show that $D_{i+1}(G)$ may be matched to $D_i(G)$ from which the result follows immediately.

Accordingly, let $X \in D_{i+1}(G)$. Since $D_{i+1}(G) \subseteq P_{i+1}(G)$, X may be matched to $X \setminus \{s\} \in P_i(G)$ for some $s \in X$. As X is doubly dependent, so also is $V(G) \setminus X$. Thus $V(G) \setminus X \cup \{s\}$ is dependent so $X \setminus \{s\}$ is, in fact, doubly dependent.

Therefore, the matching from $P_{i+1}(G)$ to $P_i(G)$ induces a matching from $D_{i+1}(G)$ to $D_i(G)$ and so $d_i(G) \geq d_{i+1}(G)$. □

4.2.1 Odd number of vertices

We now prove Theorem 4.2 in the case that G is a nontrivial graph on $2n+1$ vertices for some n.

By Theorem 3.2, there is a matching in P_G from $P_{i+1}(G)$ to $P_i(G)$ for, in particular, all $n+1 \leq i \leq 2n-2$. Therefore,

$$d_{n+1}(G) \geq d_{n+2}(G) \geq \cdots \geq d_{2n-2}(G) \geq d_{2n-1}(G)$$

by Lemma 4.3. This, together with the rank symmetry, establishes Theorem 4.2.

4.2.2 Even number of vertices

Suppose that G is a nontrivial graph on $2n$ vertices for some n.

As above, by Theorem 3.2, there is a matching in P_G from $P_{i+1}(G)$ to $P_i(G)$ for, in particular, all $n+1 \leq i \leq 2n-3$. Thus

$$d_{n+1}(G) \geq d_{n+2}(G) \geq \cdots \geq d_{2n-3}(G) \geq d_{2n-2}(G)$$

and therefore, by Theorem 4.1,

$$d_2(G) \leq d_3(G) \leq \cdots \leq d_{n-2}(G) \leq d_{n-1}(G).$$

In order to establish Theorem 4.2 it remains to show that $d_n(G) \geq d_{n-1}(G)$ which is the purpose of Section 5.

5 The Largest Rank in D_G when $|V(G)|$ is even

This section is devoted to proving the following result.

Theorem 5.1 *Let G be a nontrivial graph on $2n$ vertices. Then*

$$d_n(G) \geq d_{n-1}(G).$$

The proof of Theorem 5.1, the details of which are presented in Section 5.4, proceeds as follows. First, in Section 5.1, we show that if G contains a component of a particular form then $d_n(G) \geq d_{n-1}(G)$. Otherwise, we claim that G is either connected or that each component of G is "large". In either case, it is shown (Sections 5.2 and 5.3 respectively) that $p_n(G) \geq p_{n+1}(G)$ from which Theorem 5.1 follows.

5.1 Some Direct Sums

Let G be a nontrivial graph on $2n$ vertices. In this section, we show that $d_n(G) \geq d_{n-1}(G)$ provided that G has a component having a particular form. The particular forms that we consider are a single vertex, a triangle, an edge, and a star graph.

5.1.1 An Isolated Vertex

Lemma 5.2 *Let G be a graph on $2n$ vertices. If $G = K_1 + H$ and H has at least one edge then $d_n(G) \geq d_{n-1}(G)$.*

Proof: Let x be an isolated vertex in G. The number of sets X in $D_k(G)$ which contain x is $d_{k-1}(H)$ since $X \setminus x$ is a doubly dependent $(k-1)$-set in H. Similarly, the number of sets in $D_k(G)$ which do not contain x is $d_k(H)$.

Thus $d_k(G) = d_{k-1}(H) + d_k(H)$ so

$$\begin{aligned} d_n(G) - d_{n-1}(G) &= d_{n-1}(H) + d_n(H) - (d_{n-2}(H) + d_{n-1}(H)) \\ &= d_n(H) - d_{n-2}(H) \\ &= d_n(H) - d_{n+1}(H) \geq 0 \end{aligned}$$

by Lemma 4.3, since $P_{n+1}(H)$ may be matched to $P_n(H)$ in the poset P_H by Theorem 3.2. □

5.1.2 A Disjoint Triangle

Lemma 5.3 *Let G be a graph on $2n$ vertices. If $G = K_3 + H$ and H has at least one edge then $d_n(G) \geq d_{n-1}(G)$.*

Proof: We enumerate the number of sets in $D_k(G)$ by considering the cardinality of $X \cap V(K_3)$ for each $X \in D_k(G)$.

First, suppose that $X \in D_k(G)$ is such that $X \cap V(K_3) = \emptyset$. Then X is a dependent k-set in H, the number of which is $p_k(H)$.

Secondly, suppose that $|X \cap V(K_3)| = 1$. Then $X \cap V(K_3)$ is a dependent $(k-1)$-set in H. Since $|X \cap V(K_3)| = 1$ may occur in 3 ways, the number of possibilities for X is $3p_{k-1}(H)$.

Thirdly, suppose that $|X \cap V(K_3)| = 2$. Then $(V(G) \setminus X) \cap V(H)$ is a dependent $(2n-k-1)$-set in H. Since $|X \cap V(K_3)| = 2$ may occur in 3 ways, the number of possibilities in this case is $3p_{2n-k-1}(H)$.

Finally, if $|X \cap V(K_3)| = 3$ then we must select a dependent $(2n-k)$-set in H for $V(G) \setminus X$ which may be done in $p_{2n-k}(H)$ ways.

Thus

$$d_k(G) = p_k(H) + 3p_{k-1}(H) + 3p_{2n-k-1}(H) + p_{2n-k}(H)$$

so after routine simplification

$$d_n(G) - d_{n-1}(G) = -p_{n+1}(H) - p_n(H) + 5p_{n-1}(H) - 3p_{n-2}(H) \geq 0$$

since, by Theorem 3.2 and the fact that H has $2n-3$ vertices, $P_{n-1}(H)$ is the rank of largest size in P_H. □

5.1.3 A Disjoint Edge

In this section, we show that Theorem 5.1 holds if G contains a disjoint edge. We will require the following two lemmas.

Lemma 5.4 *Suppose that G is a graph on $2n$ vertices which has at least one edge. Then, in the poset P_G, $p_n \geq p_{n+2}$ and $2p_{n+1} \geq p_n + p_{n+2}$ where $p_i = p_i(G)$.*

Proof: In the poset P_G, let $E_i = \{(X,Y) \mid X \in P_i(G), Y \in P_{i+1}(G), X < Y\}$. We now enumerate $|E_i|$ in two different ways. First, as each $X \in P_i(G)$ is covered by $2n-i$ elements of $P_{i+1}(G)$, we have $|E_i| = (2n-i)p_i$. On the other hand, each $Y \in P_{i+1}(G)$ covers $i+1$, i, or $i-1$ elements of $P_i(G)$ so $(i-1)p_{i+1} \leq |E_i| \leq (i+1)p_{i+1}$. Therefore,

$$(i-1)p_{i+1} \leq (2n-i)p_i \leq (i+1)p_{i+1}. \tag{1}$$

Setting $i = n$ in (1), we obtain, in particular, $np_n \geq (n-1)p_{n+1}$. For $i = n+1$, we have, in particular, $(n-1)p_{n+1} \geq np_{n+2}$. Combining these inequalities gives $p_n \geq p_{n+2}$.

Furthermore, $np_n \leq (n+1)p_{n+1}$ is obtained by taking $i = n$ in (1). Adding this inequality to $(n-1)p_{n+1} \geq np_{n+2}$ yields $2p_{n+1} \geq p_n + p_{n+2}$. □

Lemma 5.5 *Suppose that $P_{i-1}(G)$ may be matched to $P_i(G)$ in the poset P_G. Then $s_i(G) \geq s_{i-1}(G)$.*

Proof: Let $X \in P_{i-1}(G)$ be singly dependent. Since $P_{i-1}(G)$ may be matched to $P_i(G)$, X may be matched to $X \cup \{s\}$ for some $s \in V(G) \setminus X$. The result now follows upon showing that $X \cup \{s\}$ is also singly dependent.

Since X is singly dependent, $V(G) \setminus X$ is independent. Therefore, $V(G) \setminus (X \cup \{s\})$ is also independent so $X \cup \{s\}$ is singly dependent. □

Lemma 5.6 *Let G be a graph on $2n$ vertices. If $G = K_2 + H$ and H is a nontrivial graph then $d_n(G) \geq d_{n-1}(G)$.*

Proof: Let (x, y) be a disjoint edge in G. As in Lemma 5.3, we enumerate the number of sets in $D_k(G)$ by considering the intersection of X with $\{x, y\}$ for each $X \in D_k(G)$.

First, if $X \cap \{x, y\} = \emptyset$ then X is a dependent k-set in H, the number of which is $p_k(H)$.

Secondly, if $X \cap \{x, y\} = \{x\}$ then $X \cap V(H)$ and $(V(G) \setminus X) \cap V(H)$ are both dependent sets in H. The number of such sets X is therefore $d_{k-1}(H)$. Similarly, if $X \cap \{x, y\} = \{y\}$ then there are $d_{k-1}(H)$ possibilities.

Finally, if $X \cap \{x, y\} = \{x, y\}$ then $V(G) \setminus X$ is a dependent $(2n - k)$-set in H, the number of which is $p_{2n-k}(H)$.

Therefore $d_k(G) = p_k(H) + p_{2n-k}(H) + 2d_{k-1}(H)$ so

$$d_n(G) - d_{n-1}(G) = [2p_n(H) - p_{n-1}(H) - p_{n+1}(H)] + 2(d_{n-1}(H) - d_{n-2}(H)). \quad (2)$$

As H has $2n - 2$ vertices, the largest rank in P_H is either $P_{n-1}(H)$ or $P_n(H)$, by Theorem 3.2. We consider two cases accordingly.

First, suppose that $p_{n-1}(H) \geq p_n(H)$. Then $P_n(H)$ may be matched to $P_{n-1}(H)$ by Theorem 3.3 and so $d_{n-1}(H) \geq d_n(H) = d_{n-2}(H)$ by Lemma 4.3 and Theorem 4.1. Moreover, by Lemma 5.4, $2p_n(H) \geq p_{n-1}(H) + p_{n+1}(H)$. Thus both terms on the right hand side of (2) are nonnegative and the result follows.

Conversely, suppose that $p_n(H) > p_{n-1}(H)$. By Lemma 5.4, $p_{n-1}(H) \geq p_{n+1}(H)$ and so from (2)

$$\begin{aligned} d_n(G) - d_{n-1}(G) &\geq 2p_n(H) - 2p_{n-1}(H) + 2(d_{n-1}(H) - d_{n-2}(H)) \\ &= 2[(p_n(H) - d_n(H)) - (p_{n-1}(H) - d_{n-1}(H))] \\ &= 2[s_n(H) - s_{n-1}(H)] \geq 0 \end{aligned}$$

by Lemma 5.5, since $P_{n-1}(H)$ may be matched to $P_n(H)$ by Theorem 3.3. □

5.1.4 A Disjoint Star

Recall that the n-star S_n is isomorphic to $K_{1,n-1}$. The following lemma may be shown to hold for all star graphs. A general proof, however, is complicated and, as we require the lemma only for 3-stars and 4-stars, we opt to prove it only in these special cases.

Lemma 5.7 *Let G be a graph on $2n$ vertices. If $G = S_{r+1} + H$ where $r = 2$ or 3 then $d_n(G) \geq d_{n-1}(G)$.*

Proof: In S_{r+1}, let x be the vertex of degree r and let $y_1, y_2, \ldots, y_r$ be the other vertices. We will obtain an expression for $d_k(G)$ by considering how $X \in D_k(G)$ intersects $\{x, y_1, \ldots, y_r\}$.

First, suppose that $x \in X$. If $X \cap \{y_1, \ldots, y_r\} = \emptyset$ then both $X \cap V(H)$ and $(V(G) \setminus X) \cap V(H)$ are dependent sets in H. The number of such sets X is $d_{k-1}(H)$. Otherwise, $|X \cap \{y_1, \ldots, y_r\}| = i$ for some $1 \leq i \leq r$. In this case, $V(G) \setminus X$ is a dependent $[2n - k - (r - i)]$-set in H, the number of which is $p_{2n-k-(r-i)}(H)$. Since $|X \cap \{y_1, \ldots, y_r\}| = i$ may occur in $\binom{r}{i}$ ways, the number of possibilities for X is $\binom{r}{i} p_{2n-k-(r-i)}(H)$.

Conversely, suppose that $x \notin X$. If $(V(G) \setminus X) \cap \{y_1, \ldots, y_r\} = \emptyset$ then there are $d_{k-r}(H)$ possibilities for X. Otherwise, $|(V(G) \setminus X) \cap \{y_1, \ldots, y_r\}| = i$ for some $1 \leq i \leq r$ and there are $\binom{r}{i} p_{k-(r-i)}(H)$ ways to select X.

Thus we have

$$d_k(G) = d_{k-1}(H) + d_{k-r}(H) + \sum_{i=1}^{r} \binom{r}{i} [p_{2n-k-(r-i)}(H) + p_{k-(r-i)}(H)].$$

First, suppose that $r = 2$. After routine simplification, we obtain

$$d_n(G) - d_{n-1}(G) = (d_{n-1}(H) - d_{n-3}(H)) + (3p_{n-1}(H) - 2p_{n-2}(H) - p_{n+1}(H)).$$

Since H has $2n - 3$ vertices, by Theorem 3.2 $P_{n-1}(H)$ is the largest rank in P_H so $3p_{n-1}(H) - 2p_{n-2}(H) - p_{n+1}(H) \geq 0$. Moreover, $d_{n-3}(H) = d_n(H)$ so

$$d_{n-1}(H) - d_{n-3}(H) = d_{n-1}(H) - d_n(H) \geq 0$$

by Theorem 4.2. Therefore, $d_n(G) - d_{n-1}(G) \geq 0$.

Secondly, for $r = 3$, we have

$$\begin{aligned}
d_n(G) - d_{n-1}(G) &= d_{n-1}(H) + d_{n-3}(H) - d_{n-2}(H) - d_{n-4}(H) \\
&\quad -3p_{n-3}(H) + 3p_{n-2}(H) + 2p_{n-1}(H) - p_n(H) - p_{n+1}(H) \\
&= (d_{n-1}(H) - d_{n-4}(H)) \\
&\quad +[(p_{n-2}(H) - d_{n-2}(H)) - (p_{n-3}(H) - d_{n-3}(H))] \\
&\quad +(-2p_{n-3}(H) + 2p_{n-2}(H) + 2p_{n-1}(H) - p_n(H) - p_{n+1}(H)).
\end{aligned}$$

The largest rank in P_H is either $P_{n-2}(H)$ or $P_{n-1}(H)$. By the unimodality of the Whitney numbers in P_H, we have $-2p_{n-3}(H) + 2p_{n-2}(H) + 2p_{n-1}(H) - p_n(H) - p_{n+1}(H) \geq 0$. Moreover, as H has $2n - 4$ vertices,

$$d_{n-1}(H) - d_{n-4}(H) = d_{n-1}(H) - d_n(H) \geq 0$$

by Theorem 4.2. Finally,

$$[(p_{n-2}(H) - d_{n-2}(H)) - (p_{n-3}(H) - d_{n-3}(H))] = s_{n-2}(H) - s_{n-3}(H) \geq 0$$

by Lemma 5.5, since $P_{n-3}(H)$ may be matched to $P_{n-2}(H)$ in P_H. For $r = 3$ then, $d_n(G) - d_{n-1}(G) \geq 0$. $\square$

5.2 Connected Graphs

The following lemma, found in [5], will be used in Section 5.4 as part of the proof of Theorem 5.1 for connected graphs.

Lemma 5.8 *Let $n \geq 3$ be a positive integer. If G is a connected graph on $2n$ vertices then*

$$p_n(G) \geq p_{n+1}(G).$$

5.3 Spanning Subgraphs

A graph G on $2n$ vertices may contain a spanning subgraph H such that $p_n(H) \geq p_{n+1}(H)$. The following lemma, found in [4], shows that provided H satisfies an additional condition then $p_n(G) \geq p_{n+1}(G)$. Therefore, should G contain such a subgraph H, it will be shown in Section 5.4 that Theorem 5.1 holds for G.

Lemma 5.9 *Let G be a graph on $2n$ vertices, and let H be a spanning subgraph of G. If*

1. *$p_n(H) \geq p_{n+1}(H)$, and*
2. *for any two isolated vertices x and y of H, $H \setminus \{x, y\}$ has no more than $\sum_{i=2}^{n-1} \binom{2i-1}{i}$ independent sets of size $n-1$, and*
3. *P_H has the Sperner property,*

then P_G has the Sperner property, and $p_n(G) \geq p_{n+1}(G)$.

There are three particular subgraphs of G that will be of interest in Section 5.4 and we now show that each of these subgraphs satisfies the hypotheses of Lemma 5.9. To do this will require the *independent set generating function* for the graph H which is defined to be the polynomial $f(H) = \sum_{i \geq 0} a_i x^i$ where a_i is the number of independent sets in H of cardinality i.

First, let $H_1 = 2P_4 + Z_{2n-8}$. (Recall that P_n denotes the path on n vertices, S_n is the complete bipartite graph $K_{1,n-1}$, and Z_n denotes the graph consisting of n vertices and no edges.) The independent set generating function for H_1 is

$$\begin{aligned} f(H_1) &= (1 + 4x + 3x^2)^2 (1 + x)^{2n-8} \\ &= (1 + 8x + 22x^2 + 24x^3 + 9x^4)(1 + x)^{2n-8} \end{aligned}$$

and so

$$p_k(H_1) = \binom{2n}{k} - \binom{2n-8}{k} - 8\binom{2n-8}{k-1} - 22\binom{2n-8}{k-2} - 24\binom{2n-8}{k-3} - 9\binom{2n-8}{k-4}.$$

By expanding the binomial coefficients, it may be shown that the inequality $p_n(H_1) \geq p_{n+1}(H_1)$ is equivalent to

$$(2n - 7)(n - 2) \geq 0$$

which holds for $n \geq 4$.

If x and y are any two isolated vertices in H_1 then

$$\begin{aligned} a_{n-1}(H_1 \setminus \{x,y\}) &= [x^{n-1}](1+8x+22x^2+24x^3+9x^4)(1+x)^{2n-10} \\ &= \binom{2n-10}{n-1} + 8\binom{2n-10}{n-2} + 22\binom{2n-10}{n-3} + 24\binom{2n-10}{n-4} \\ &\quad +9\binom{2n-10}{n-5}. \end{aligned}$$

We wish to show that $a_{n-1}(H_1 \setminus \{x,y\}) \leq \sum_{i=2}^{n-1} \binom{2i-1}{i}$. For $n = 5$, we have

$$a_4(H_1 \setminus \{x,y\}) = 9 \leq \binom{7}{4}$$

and for $n = 6$,

$$a_5(H_1 \setminus \{x,y\}) = 24\binom{2}{0} + 9\binom{2}{1} \leq \binom{9}{5}.$$

Finally, by the unimodality of the binomial coefficient,

$$a_{n-1}(H_1 \setminus \{x,y\}) \leq 64\binom{2n-10}{n-5} \leq \binom{2n-3}{n-1}$$

for all $n \geq 7$.

By Theorem 3.1, P_{H_1} has the Sperner property and thus H_1 satisfies all three hypotheses of Lemma 5.9.

Secondly, consider $H_2 = P_4 + S_5 + Z_{2n-9}$. We have $f(H_2) = (1+4x+3x^2)[(1+x)^4 + x](1+x)^{2n-9}$ and the inequality $p_n(H_2) \geq p_{n+1}(H_2)$ may be seen to be equivalent to

$$5n^3 + 24n^2 - 269n + 420 \geq 0$$

which holds for $n \geq 5$.

To verify the second condition in Lemma 5.9, we must show that

$$\begin{aligned} &\binom{2n-11}{n-1} + 9\binom{2n-11}{n-2} + 29\binom{2n-11}{n-3} + \\ &43\binom{2n-11}{n-4} + 35\binom{2n-11}{n-5} + 16\binom{2n-11}{n-6} + 3\binom{2n-11}{n-7} \leq \sum_{i=2}^{n-1}\binom{2i-1}{i} \end{aligned}$$

for $n \geq 6$. This may be checked directly for $n = 6$ and, for $n \geq 7$, it may be shown that

$$136\binom{2n-11}{n-5} \leq \binom{2n-3}{n-1}$$

which implies the desired inequality.

Like P_{H_1}, P_{H_2} also has the Sperner property so H_2 satisfies all three hypotheses of Lemma 5.9.

Finally, let $H_3 = 2S_5 + Z_{2n-10}$. The independent set generating function for H_3 is $f(H_3) = [(1+x)^4 + x]^2(1+x)^{2n-10}$ and, like H_1 and H_2, H_3 may be shown to satisfy the hypotheses of Lemma 5.9. The details are omitted.

5.4 Proof of Theorem 5.1

We are now ready to present a proof of Theorem 5.1. Let G be a nontrivial graph on $2n$ vertices.

We note first of all that the result holds trivially if $n = 1$ or 2 since, in either case, $d_{n-1}(G) = 0$. We will therefore assume that $n \geq 3$.

First, if G is connected, then $p_n(G) \geq p_{n+1}(G)$ by Lemma 5.8. By Theorem 3.3, there is a matching from $P_{n+1}(G)$ to $P_n(G)$ in P_G and therefore, by Lemma 4.3, $d_n(G) \geq d_{n+1}(G) = d_{n-1}(G)$.

Otherwise, G has at least two components. If one of the components is K_1, K_2, K_3, or S_3 then the result follows from Lemma 5.2, 5.6, 5.3, or 5.7 respectively.

Otherwise, each component in G contains at least 4 vertices. Let C_1 and C_2 be two of the components. If either C_1 or C_2 is isomorphic to S_4 then the result follows from Lemma 5.7.

Otherwise, both C_1 and C_2 contain either P_4 (the path on 4 vertices), or S_5 as a subgraph. Equivalently, G contains either H_1, H_2, or H_3 as a subgraph where $H_1 = 2P_4 + Z_{2n-8}$, $H_2 = P_4 + S_5 + Z_{2n-9}$, and $H_3 = 2S_5 + Z_{2n-10}$. In any event, we have $p_n(G) \geq p_{n+1}(G)$ (by Lemma 5.9) and the result follows upon applying Theorem 3.3 and Lemma 4.3.

6 Spernerity and D_G

Let G be a nontrivial graph. The poset P_G is known to have the Sperner property. It is also rank unimodal but not, in general, rank symmmetric.

As D_G is both rank unimodal and rank symmetric, it is perhaps surprising to discover that D_G does not have the Sperner property in general. In fact, one need not search far to find a counterexample. Let $V(G) = \{1, 2, 3, 4, 5\}$ and $E(G) = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}\}$. Then $d_2(G) = d_3(G) = 3$ but there is an antichain of size 4, namely $\{\{1, 2\}, \{2, 5\}, \{1, 3, 4\}, \{3, 4, 5\}\}$.

We have been unable, however, to find a counterexample with an even number of vertices. Indeed, that D_G has the Sperner property is trivially true if $|V(G)| = 2$ or 4, and may be verified to be true also when $|V(G)| = 6$. Moreover, if $p_n(G) \geq p_{n+1}(G)$ then $P_{n+1}(G)$ may be matched to $P_n(G)$ in P_G by virtue of Theorem 3.3. This matching then induces a matching from $D_{n+1}(G)$ to $D_n(G)$ in D_G (by Lemma 4.3) and it follows that D_G has the Sperner property. In closing then, we make the following conjecture.

Conjecture 6.1 *Let G be a nontrivial graph on $2n$ vertices. Then D_G has the Sperner property.*

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