A q–analogue of a formula of Hernandez obtained by inverting a result of Dilcher

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Abstract

We prove a q–analogue of the formula

$$\sum_{1\le k\le n}\binom{n}{k}(-1)^{k-1}\sum_{1\le i_1\le i_2\le\cdots\le i_m=k}\frac{1}{i_1 i_2\dots i_m}=\sum_{1\le k\le n}\frac{1}{k^m}$$

by inverting a formula due to Dilcher.

1 The identities

Hernández in [6] proved the following identity:

$$\sum_{1\le k\le n}\binom{n}{k}(-1)^{k-1}\sum_{1\le i_1\le i_2\le\cdots\le i_m=k}\frac{1}{i_1 i_2\dots i_m}=\sum_{1\le k\le n}\frac{1}{k^m}. \tag{1}$$

However this identity does not really require a proof, since we will show that it is just an inverted form of an identity of Dilcher [2];

$$\sum_{1\le k\le n}\binom{n}{k}(-1)^{k-1}\frac{1}{k^m}=\sum_{1\le i_1\le i_2\le\cdots\le i_m\le n}\frac{1}{i_1 i_2\dots i_m}. \tag{2}$$

For $k\ge 1$, define

$$a_k:=-\sum_{1\le i_1\le i_2\le\cdots\le i_m=k}\frac{1}{i_1 i_2\dots i_m}\qquad\text{and}\qquad b_k:=\frac{1}{k^m},$$

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then the identities are

$$\begin{aligned}\sum_{1\le k\le n}\binom{n}{k}(-1)^k a_k &= \sum_{1\le k\le n} b_k,\\ \sum_{1\le k\le n}\binom{n}{k}(-1)^k b_k &= \sum_{1\le k\le n} a_k.\end{aligned} \tag{3}$$

These are *inverse relations*, as can be seen by introducing ordinary generating functions $A(z)=\sum a_n z^n$ and $B(z)=\sum b_n z^n$. Then (3) gives immediately

$$\begin{aligned}A\Big(\frac{z}{z-1}\Big) &= B(z),\\ B\Big(\frac{z}{z-1}\Big) &= A(z).\end{aligned}$$

However

$$w=\frac{z}{z-1}\longleftrightarrow z=\frac{w}{w-1},$$

and the proof is finished. An alternative argument that will be useful in the sequel when we do the q–analogue, is as follows. We take differences in (3) of the lines indexed with n resp. $n-1$; then we have to prove that

$$b_n=\sum_{1\le k\le n}\binom{n-1}{k-1}(-1)^k a_k \quad\Longleftrightarrow\quad a_n=\sum_{1\le k\le n}\binom{n-1}{k-1}(-1)^k b_k.$$

Now in this form this is a traditional inverse relation; see e. g. [7]. An explicit argument will follow in the next section for the q–instance.
We note that Dilcher's sum appears also in disguised form in [3].

2 A q–analogue

Dilcher's formula (2) is a corollary of his elegant q–version;

$$\sum_{1\le k\le n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^{k-1}\frac{q^{\binom{k+1}{2}+(m-1)k}}{(1-q^k)^m}=\sum_{1\le i_1\le i_2\le\cdots\le i_m\le n}\frac{q^{i_1}}{1-q^{i_1}}\cdots\frac{q^{i_m}}{1-q^{i_m}}.$$

Here, $\begin{bmatrix}n\\k\end{bmatrix}_q$ denotes the Gaussian polynomial

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

with

$$(x;q)_n := (1-x)(1-xq)\dots(1-xq^{n-1}).$$

Apart from Dilcher's paper [2], the article [1] is also of some relevance in this context. Therefore it is a natural question to find a q–analogue of Hernández' formula, or, what amounts to the same, to find the appropriate inverse relations for the q–analogues. We state them in the following lemma.

Lemma 1.

$$\begin{aligned}\sum_{1\le k\le n} b_k &= \sum_{1\le k\le n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} a_k,\\ \sum_{1\le k\le n} q^{-k} a_k &= \sum_{1\le k\le n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-kn+\binom{k}{2}} b_k.\end{aligned} \tag{4}$$

Proof. Again, taking differences in (4), we have to prove that

$$b_n = \sum_{1\le k\le n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} a_k \iff a_n = \sum_{1\le k\le n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^k q^{-(n-1)k+\binom{k}{2}} b_k. \tag{5}$$

However, after trivial modifications, this is the inverse pair reported in [5], exercise (2.6.6 (b)). Credits for it are given to Carlitz, Szegő, and Rogers; compare the references in [5].

After a first version of this note was circulated, O. Warnaar kindly informed me that this lemma would also follow from results in [4]. □

We would like to remark that an alternative formulation can be given in terms of matrices of *connection coefficients.*

This can be done in terms of the original formulæ (4), but looks much nicer when referring to (5):

Define matrices

$$T := \left[\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^k q^{\binom{k}{2}}\right]_{n,k}, \qquad U := \left[\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^k q^{-(n-1)k+\binom{k}{2}}\right]_{n,k},$$

then

$$TU = I.$$

Theorem 2. [q–analogue of Hernández' formula]

$$\sum_{1\le k\le n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{k-1} q^{-kn+\binom{k}{2}} \sum_{1\le i_1\le i_2\le\dots\le i_m=k} \frac{q^{i_1}}{1-q^{i_1}} \cdots \frac{q^{i_m}}{1-q^{i_m}} = \sum_{1\le k\le n} \frac{q^{k(m-1)}}{(1-q^k)^m}.$$

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