Improvements on inequalities for non-negative matrices*

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Abstract

We prove that there is an integer $k \leq (n^2 - 2n + 4)/2$ such that the diagonal entries of A^k are all positive for any non-negative irreducible $n \times n$ matrix A, and that there are integers i, j with $0 \leq i < j \leq 3^{n/2}$ such that $A^i \leq A^j$ for any non-negative $n \times n$ matrix A with no entry in $(0, 1)$ and $n \geq 2$. The results of Wang and Shallit [Linear Algebra Appl. 290 (1999) 135-144] are thus improved.

1. Introduction

In this paper we will be concerned with matrices and vectors with non-negative entrices. For a matrix $A = (a_{ij})$ and scalar c, by the inequality $A > c$ we mean that $a_{ij} > c$ for all i, j, and similarly for the relations $A \geq c$ and $A = c$. For matrices A and B of the same dimensions, by $A \geq B$ we mean the inequality holds entrywise. We adopt similar conventions for vectors.

For an $n \times n$ matrix A, by $\text{diag}(A)$ we mean the vector containing the diagonal entries of A. Let I denote the identity matrix.

A square matrix A is said to be reducible if there is a permutation matrix P such that

$$P^T A P = \begin{pmatrix} B & 0 \\ D & C \end{pmatrix},$$

where the diagonal blocks B and C are square matrices. A is irreducible if it is not reducible.

For an irreducible matrix A, let $\beta(A)$ be the least integer $k \geq 1$ such that $\text{diag}(A^k) > 0$. Define $\beta(n) = \sup \beta(A)$, where the supremum is over all irreducible $n \times n$ matrices. Recently Wang and Shallit [1] proved that $\beta(n) \leq n(n-1)$ for $n \geq 2$. They posed the problem of determining a more precise upper bound for $\beta(n)$.

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For a non-negative $n \times n$ matrix A with no entry in $(0, 1)$, let $\alpha(A)$ be the least positive integer j such that there exists an integer i with $0 \le i < j$ such that $A^i \le A^j$. Define $\alpha(n) = \sup \alpha(A)$, where the supremum is over all non-negative matrices A with no entry in $(0, 1)$. Wang and Shallit [1] have proved that $\alpha(n) \le 2^n$. As is remarked in [1], this inequality is almost surely not best possible.

In this paper we prove more precise bounds for $\beta(n)$ and $\alpha(n)$.

2. Bound for $\beta(n)$

The graph of an $n \times n$ matrix $A = (a_{ij})$ is the directed graph on vertices $v_1, v_2, \cdots, v_n$ such that there is an arc from v_i to v_j if and only if $a_{ij} > 0$. We denote the graph of A by $G(A)$. An s-cycle is a (directed) cycle of length s.

An irreducible matrix A is primitive if there is a positive integer l such that $A^l > 0$. The least such l is called the exponent of A and is denoted $\gamma(A)$.

For an irreducible matrix A, the greatest common divisor of all cycle lengths of $G(A)$ is called the index of imprimitivity of A and is denoted $d(A)$. It is well known (see, e.g., [4]) that a matrix A is irreducible if and only if $G(A)$ is strongly connected and that an irreducible matrix A is primitive if and only if $d(A) = 1$.

We first introduce the following lemmas, which we will use to estimate $\beta(A)$ for an irreducible matrix A.

Lemma 1 [3]. *If A is an $n \times n$ primitive matrix whose graph has at least three distinct cycle lengths, then $\gamma(A) \le \lfloor (n^2 - 2n + 4)/2 \rfloor$.*

Lemma 2 [2]. *Suppose X and Y are $r \times t$ and $t \times r$ non-negative matrices and neither has a zero row or column. Then XY is primitive if and only if YX is, and if XY and YX are primitive, then $\gamma(YX) - 1 \le \gamma(XY) \le \gamma(YX) + 1$.*

Lemma 3 [5]. *If A is an $n \times n$ primitive matrix, then $\gamma(A) \le (n-1)^2 + 1$.*

Our first theorem refines the bound for $\beta(n)$ obtained in [1].

Theorem 1. *Let*

$$f(n) = \left\lfloor \frac{n^2 - 2n + 4}{2} \right\rfloor .$$

Then $\beta(n) \le f(n)$.

Proof. Let A be an irreducible $n \times n$ matrix with $G = G(A)$. Denote by $L(G)$ the set of cycle lengths of G . If G contains an n-cycle, then $\beta(A) \le n \le f(n)$. Suppose in the following that G contains no n-cycle. There are two cases to consider, based on the primitivity of A.

Case 1: A is primitive.

Case 1.1: $|L(G)| = 2$. Suppose $L(G) = \{p, q\}$ with $p < q \le n-1$. If $p+q \ge n+1$, then every p-cycle interects every q-cycle, and hence $\beta(A) \le p+q \le (n-2)+(n-1) = 2n - 3 \le f(n)$, while if $p + q \le n$, then $\beta(A) \le pq \le ((p+q)/2)^2 \le n^2/4 \le f(n)$.

Case 1.2: $|L(G)| \ge 3$. In this case, we have $n \ge 4$. By Lemma 1 we have $\beta(A) \le \gamma(A) \le \lfloor (n^2 - 2n + 4)/2 \rfloor = f(n)$.

Case 2: A is not primitive. Suppose $d(A) = d \geq 2$. By classical results on imprimitive matrices (see [4, pp.71-73]), there is a permutation matrix P such that

$$P^T AP = \begin{pmatrix} 0 & A_1 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & A_{d-1} \\ A_d & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where the diagonal zero blocks are square and each block A_i has no zero row or column; furthermore, if A_i is of dimension $n_i \times n_{i+1}$ ($n_{d+1} = n_1$), and we put $B_i = A_i A_{i+1} \cdots A_d A_1 \cdots A_{i-1}$, then

$$P^T A^d P = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_d \end{pmatrix},$$

where B_i is an $n_i \times n_i$ primitive matrix for each i with $1 \leq i \leq d$.

If $d = n$, then clearly $\beta(A) = n \leq f(n)$. If $n = 3$ and $d = 2$, then $\beta(A) = 2 \leq f(3) = 3$. Suppose $2 \leq d \leq n-1$ and $n \geq 4$.

Let $n_m = \min_{1 \leq i \leq d} n_i$ where $1 \leq m \leq d$ and $\gamma(B_t) = \max_{1 \leq i \leq d} \gamma(B_i)$ where $1 \leq t \leq d$.

We claim that $\gamma(B_t) \leq \gamma(B_m) + 1$. This is obvious if $t = m$. Suppose without loss of generality that $1 \leq t < m \leq d$. Let $X = A_t A_{t+1} \cdots A_{m-1}$ and $Y = A_m A_{m+1} \cdots A_d A_1 \cdots A_{t-1}$. Then $B_t = XY$ and $B_m = YX$. By Lemma 2, we have $\gamma(B_t) = \gamma(XY) \leq \gamma(YX) + 1 = \gamma(B_m) + 1$, as desired.

Note that $n_1 + n_2 + \cdots + n_d = n$. We have $n_m \leq n/d$. It follows from Lemma 3 that

$$\begin{aligned} \max_{1 \leq i \leq d} \gamma(B_i) &= \gamma(B_t) \leq \gamma(B_m) + 1 \\ &\leq (n_m - 1)^2 + 1 + 1 \\ &\leq (\tfrac{n}{d} - 1)^2 + 2. \end{aligned}$$

Hence

$$\begin{aligned} \beta(A) &\leq d \max_{1 \leq i \leq d} \gamma(B_i) \\ &\leq d(\tfrac{n}{d} - 1)^2 + 2d \\ &= \tfrac{(n-d)^2}{d} + 2d. \end{aligned}$$

The function $h(d) = (n-d)^2/d + 2d$ is a decreasing function of d in $[2, n/\sqrt{3}]$ and an increasing function in $[n/\sqrt{3}, n-1]$. Hence it assumes its largest value either for $d = 2$ or $d = n-1$. We have

$$h(2) = (n-2)^2/2 + 2, \quad h(n-1) = 2(n-1) + 1/(n-1).$$

It is easy to see that $\lfloor h(n-1) \rfloor \leq \lfloor h(2) \rfloor \leq f(n)$ for $n \geq 6$, and $\lfloor h(2) \rfloor \leq \lfloor h(n-1) \rfloor \leq f(n)$ for $n = 4$ or 5. Hence

$$\beta(A) \leq h(d) \leq \max\{\lfloor h(2) \rfloor, \lfloor h(n-1) \rfloor\} \leq f(n). \quad \square$$

3. Bound for $\alpha(n)$

For a non-negative $n \times n$ matrix A with no entry in $(0,1)$, Wang and Shallit [1] proved that $\alpha(n) \leq 2^n$ for all $n \geq 1$, and this bound cannot be replaced by $e^{\sqrt{n \log n}}$. We are going to improve this result. First we give a lemma that will be used.

Lemma 4 [1]. *Suppose $A \geq 0$ is an $n \times n$ matrix of the form*

$$A = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$

where B, D are square matrices with $D \geq I$. For integers $l \geq 0$, define the matrices C_l by

$$A^l = \begin{pmatrix} B^l & 0 \\ C_l & D^l \end{pmatrix}.$$

Then for all $l \geq 0$, we have $C_l \leq C_{l+1}$ and $D^l \leq D^{l+1}$.

An easily verified fact is that $f(n) = \lfloor (n^2 - 2n + 4)/2 \rfloor \leq 3^{n/2}$ for all $n \geq 2$.

Theorem 2. *For all $n \geq 2$, we have $\alpha(n) \leq 3^{n/2}$.*

Proof. Let A be a non-negative $n \times n$ matrix with no entry in $(0,1)$. We use induction on n to prove the theorem. For $n = 2$, if A is irreducible, then clearly $A^0 = I \leq A^2$, while if A is reducible, then we have either $A = A^2$ or $A^2 = A^3 = 0$. Hence $\alpha(A) \leq 3$ for $n = 2$.

Assume $n \geq 3$ and the result holds for all m with $2 \leq m < n$. The proof is now divided into the following two cases.

Case 1: A is irreducible. By Theorem 1, there is an integer k, $1 \leq k \leq f(n)$, such that $\text{diag}(A^k) > 0$. Note that every positive diagonal entry of A^k is ≥ 1. We have $I = A^0 \leq A^k$. Hence $\alpha(A) \leq k \leq f(n) \leq 3^{n/2}$.

Case 2: A is reducible. There is a permutation matrix P such that

$$P^T A P = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{t1} & A_{t2} & \cdots & A_{tt} \end{pmatrix},$$

where $A_{11}, A_{22}, \cdots, A_{tt}$ are square matrices that are either 0 or irreducible.

Case 2.1: $A_{tt} = 0$. The last column of A is 0. We write

$$A = \begin{pmatrix} B & 0 \\ x & 0 \end{pmatrix},$$

where x is a vector of dimension $n-1$. Note that $n - 1 \geq 2$. By induction, $\alpha(B) \leq 3^{(n-1)/2}$, i.e., there are integers i, j with $0 \leq i < j \leq 3^{(n-1)/2}$ such that $B^i \leq B^j$. It follows that

$$A^{i+1} = \begin{pmatrix} B^{i+1} & 0 \\ xB^i & 0 \end{pmatrix} \leq \begin{pmatrix} B^{j+1} & 0 \\ xB^j & 0 \end{pmatrix} = A^{j+1},$$

and $1 \leq i+1 < j+1 \leq 3^{(n-1)/2}+1 \leq 3^{n/2}$. Hence $\alpha(A) \leq 3^{n/2}$.

Case 2.2: A_{tt} is irreducible. Suppose A_{tt} is of dimension $m \times m$ with $1 \leq m \leq n-1$. By Theorem 1, there is an integer k with $1 \leq k \leq f(m) \leq 3^{m/2}$ such that $A_{tt}^k \geq I$. We write

$$A = \begin{pmatrix} B & 0 \\ C & A_{tt} \end{pmatrix}.$$

Case 2.2.1: B is 0 of dimension 1×1. Then C is a column vector of dimension $n-1$. By similar arguments as in Case 2.1, we have

$$A^{i+1} = \begin{pmatrix} 0 & 0 \\ A_{tt}^i C & A_{tt}^{i+1} \end{pmatrix} \leq \begin{pmatrix} 0 & 0 \\ A_{tt}^j & A_{tt}^{j+1} \end{pmatrix} = A^{j+1},$$

and $1 \leq i+1 < j+1 \leq 3^{(n-1)/2}+1 \leq 3^{n/2}$. Hence $\alpha(A) \leq 3^{n/2}$.

Case 2.2.2: B is not 0 of dimension 1×1. Then we have either $m \leq n-2$ or B is of dimension 1×1 but not 0. In the former case, we know by the induction hypothesis applied to B^k that there are integers i, j with $0 \leq i < j \leq 3^{(n-m)/2}$ such that $(B^k)^i \leq (B^k)^j$, while in the later case we have $(B^k)^i \leq (B^k)^j$ where $i = 0$ and $j = 1$. Note that

$$A^k = \begin{pmatrix} B^k & 0 \\ C_k & A_{tt}^k \end{pmatrix}$$

for some C_k. By Lemma 4, $(A^k)^i \leq (A^k)^j$ and $0 \leq ki < kj \leq 3^{m/2}3^{(n-m)/2} = 3^{n/2}$. Hence $\alpha(A) \leq 3^{n/2}$.

The proof is now completed. □

References

[1] M. Wang and J. Shallit, An inequality for non-negative matrices, Linear Algebra Appl. 290 (1999) 135-144.

[2] D.A. Gregory, S.J. Kirkland and N.J. Pullman, A bound on the exponent of a primitive matrix using Boolean rank, Linear Algebra Appl. 217 (1995) 101-116.

[3] M. Lewin and Y. Vitek, A system of gaps in the exponent set of primitive matrices, Illinois J. Math. 25 (1981) 87-98.

[4] R.A. Brualdi and H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, Cambridge, 1991.

[5] H. Wieldndt, Unzerlegbare, nicht negative matrizen, Math. Zeitschrift 52 (1950) 642-648.

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