Some new classes of integral trees with diameters 4 and 6*

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Abstract

In this paper, some new classes of integral trees with diameters 4 and 6 are given. All these classes are infinite. They are different from those in the existing literature.

I. Introduction

The notion of integral graphs was first introduced by F. Harary and A.J. Schwenk in 1974. A graph G is called integral if all the zeros of the characteristic polynomial $P(G, x)$ are integers. The 23rd open problem of reference [2] is about trees with purely integral eigenvalues. All integral trees with diameters less than 4 are given in [2, 5]. Also, some results on integral trees with diameters 4, 5, 6 and 8 can be found in [2-10]. In this paper, some new families of integral trees with diameters 4 and 6 are given. All these classes are infinite. They are different from those of [2-10]. This is a new contribution to the search for integral trees. We believe that it will be useful for constructing other integral trees.

All graphs considered here are simple. For a graph G, we let $V(G)$ denote the vertex set of G and $E(G)$ the edge set. All other notation and terminology can be found in [11].

Lemma 1. (C. Godsil and B. Mckay [1]) If $G \bullet H$ is the graph obtained from G and H by identifying the vertices $v \in V(G)$ and $w \in V(H)$, then

$$P(G \bullet H, x) = P(G, x)P(H_w, x) + P(G_v, x)P(H, x) - xP(G_v, x)P(H_w, x).$$

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where G_v and H_w are the subgraphs of G and H induced by $V(G)\backslash\{v\}$ and $V(H)\backslash\{w\}$, respectively.

Let $S(m,t)$ be the tree of diameter 4 formed by joining the centers of m copies of $K_{1,t}$ to a new vertex v. Let $L(r,m,t)$ be the tree of diameter 6 which is obtained by joining the centers of r copies of $S(m,t)$ to a new vertex u.

Lemma 2. (X. Li and G. Lin [3])
1) $P(K_{1,t},x) = x^{t-1}(x^2-t)$.
2) $P(S(m,t),x) = x^{m(t-1)+1}(x^2-t)^{m-1}[x^2-(m+t)]$.
3) $P(L(r,m,t),x) = x^{rm(t-1)+r-1}(x^2-t)^{r(m-1)}[x^2-(m+t)]^{r-1}$
$\times[x^4-(m+t+r)x^2+rt]$.

II. Integral Trees with Diameter 4

In this section, we shall construct infinitely many new classes of integral trees with diameter 4.

Theorem 1. Let $K_{1,s} \bullet S(m,t)$ be the tree of diameter 4 obtained by identifying the center w of $K_{1,s}$ and the center v of $S(m,t)$. Then $K_{1,s} \bullet S(m,t)$ is integral if and only if t is a perfect square, and $x^4-(m+t+s)x^2+st$ can be factored as $(x^2-a^2)(x^2-b^2)$.

Proof. Because the vertex w is the center of $K_{1,s}$ and the vertex v is the center of the tree $S(m,t)$, if we let $G = K_{1,s}$ and $H = S(m,t)$, then by Lemma 1 we know that

$$P[K_{1,s} \bullet S(m,t),x] = P(K_{1,s},x)P^m(K_{1,t},x) + x^s P(S(m,t),x) - x^{s+1}P^m(K_{1,t},x).$$

By Lemma 2, we have

$$P[K_{1,s} \bullet S(m,t),x] = x^{m(t-1)+(s-1)}(x^2-t)^{m-1}[x^4-(m+t+s)x^2+st].$$

The theorem is thus proved.

Corollary 1. (X. Li and G. Lin [3]) If $s=t$, then the tree $K_{1,s} \bullet S(m,t)$ with diameter 4 is integral if and only if t is a perfect square, and $x^4-(m+2t)x^2+t^2$ can be factored as $(x^2-a^2)(x^2-b^2)$.

Corollary 2. (X. Li and G. Lin [3]) Let a, b and c be positive integers. If $a > b$, $t = a^2b^2c^2$, $m = (a^2-b^2)^2c^2$ then the tree $K_{1,t} \bullet S(m,t)$ with diameter 4 is integral.

Remark 1. Note that Corollaries 1 and 2 are obtained directly from Theorem 1. They are Theorem 3 and Corollary 3 of [3], respectively.

Theorem 2. For positive integers a and b, let $a > b$, $t = 4a^2b^2$, $s = (a^2 + b^2)^2$ and $m = (a^2 - b^2)^2$. If $2(a^2 + b^2)$ is a perfect square, that is, there exists an integer c satisfying $2(a^2 + b^2) = c^2$, then the tree $K_{1,s} \bullet S(m,t)$ with diameter 4 is integral.

Proof. Because $a > b$, $t = 4a^2b^2$, $s = (a^2 + b^2)^2$ and $2(a^2 + b^2) = c^2$, we have that

$$\begin{aligned} x^4 - (m + t + s)x^2 + st &= x^4 - 2(a^2 + b^2)^2x^2 + 4a^2b^2(a^2 + b^2)^2 \\ &= [x^2 - 2a^2(a^2 + b^2)][x^2 - 2b^2(a^2 + b^2)] \\ &= (x^2 - a^2c^2)(x^2 - b^2c^2). \end{aligned}$$

From Theorem 1 the theorem follows.

Lemma 3. (Z. Cao [5]) All solutions of the diophantine equation (1)

$$x^2 + y^2 = 2z^2. \tag{1}$$

are given by

$$x = |2ab + (a^2 - b^2)|c, \quad y = |2ab - (a^2 - b^2)|c, \quad z = (a^2 + b^2)c,$$

where $(a, b) = 1$, $2 \nmid (a + b)$ and c is a positive integer.

Corollary 3. For any positive integers a, b and c, let $s = 4(a^2 + b^2)^4c^4$, $m = 64a^2b^2(a^2 - b^2)^2c^4$ and $t = 4(a^4 + b^4 - 6a^2b^2)^2c^4$, where $(a, b) = 1$ and $2 \nmid (a + b)$. Then the tree $K_{1,s} \bullet S(m,t)$ with diameter 4 is integral.

Proof. This follows directly from Theorems 1 and 2 and Lemma 3.

Lemma 4. (L. Wang, X. Li and R. Liu [8]) There exist positive integers $N = 2^l p_1^{l_1} p_2^{l_2} \cdots p_s^{l_s}$, where $l = 0$ or 1, $s \geq 2$, and p_i are primes of the form $p_i \equiv 1(mod\, 4)$, for $i = 1, 2, \cdots, s$, such that N can be expressed as

$$a^2 + b^2 = c^2 + d^2 \tag{2}$$

satisfying $a|cd$ or $b|cd$, where a, b, c and d are positive integers with $c > a$, $b > d$, $(a, b) = 1$ and $(c, d) = 1$. In particular, there are such N's with $N = (p_1p_2 \cdots p_s)^2$.

For Lemma 4, we simply list the following examples.

(i) For $N = 2^l p_1^{l_1} p_2^{l_2} \cdots p_s^{l_s}$, we have

1) $5 \times 13 = 7^2 + 4^2 = 8^2 + 1^2$,
2) $5 \times 17 = 7^2 + 6^2 = 9^2 + 2^2$,
3) $5 \times 41 = 13^2 + 6^2 = 14^2 + 3^2$,
4) $5 \times 53 = 12^2 + 11^2 = 16^2 + 3^2$,
5) $5 \times 101 = 19^2 + 12^2 = 21^2 + 8^2$,
6) $13 \times 17 = 11^2 + 10^2 = 14^2 + 5^2$,
7) $13 \times 37 = 16^2 + 15^2 = 20^2 + 9^2$,
8) $13 \times 53 = 20^2 + 17^2 = 25^2 + 8^2$,
9) $13 \times 97 = 30^2 + 19^2 = 35^2 + 6^2$,
10) $13 \times 113 = 37^2 + 10^2 = 38^2 + 5^2$,
11) $13 \times 181 = 47^2 + 12^2 = 48^2 + 7^2$,
12) $13 \times 313 = 62^2 + 15^2 = 63^2 + 10^2$,
13) $13 \times 317 = 61^2 + 20^2 = 64^2 + 5^2$,
14) $13 \times 337 = 59^2 + 30^2 = 66^2 + 5^2$,
15) $13 \times 613 = 87^2 + 20^2 = 88^2 + 15^2$,
16) $13 \times 733 = 77^2 + 60^2 = 85^2 + 48^2$,
17) $13 \times 757 = 79^2 + 60^2 = 96^2 + 25^2$,
18) $17 \times 37 = 23^2 + 10^2 = 25^2 + 2^2$,
19) $17 \times 53 = 26^2 + 15^2 = 30^2 + 1^2$,
20) $17 \times 257 = 63^2 + 20^2 = 65^2 + 12^2$,
21) $17 \times 73 = 29^2 + 20^2 = 35^2 + 4^2$,
22) $17 \times 137 = 40^2 + 27^2 = 48^2 + 5^2$,
23) $17 \times 193 = 41^2 + 40^2 = 55^2 + 16^2$,
24) $29 \times 37 = 28^2 + 17^2 = 32^2 + 7^2$,
25) $29 \times 41 = 30^2 + 17^2 = 33^2 + 10^2$,
26) $29 \times 61 = 37^2 + 20^2 = 40^2 + 13^2$,
27) $29 \times 89 = 41^2 + 30^2 = 50^2 + 9^2$,
28) $29 \times 281 = 57^2 + 70^2 = 90^2 + 7^2$,
29) $29 \times 389 = 84^2 + 65^2 = 105^2 + 16^2$,
30) $41 \times 61 = 49^2 + 10^2 = 50^2 + 1^2$,
31) $5 \times 13 \times 17 = 24^2 + 23^2 = 32^2 + 9^2$,
32) $5 \times 13 \times 17 = 31^2 + 12^2 = 32^2 + 9^2$,
33) $5 \times 13 \times 17 = 31^2 + 12^2 = 33^2 + 4^2$,
34) $5 \times 13 \times 17 \times 37 = 167^2 + 114^2 = 194^2 + 57^2$,
35) $257 \times 65537 = 4095^2 + 272^2 = 4097^2 + 240^2$.

(ii) For $N = (p_1 p_2 \cdots p_s)^2$, we have

1) $(5 \times 13)^2 = 56^2 + 33^2 = 63^2 + 16^2$,
2) $(5 \times 29)^2 = 143^2 + 24^2 = 144^2 + 17^2$,
3) $(13 \times 17)^2 = 171^2 + 140^2 = 220^2 + 21^2$,
4) $(17 \times 37)^2 = 460^2 + 429^2 = 621^2 + 100^2$,
5) $(41 \times 61)^2 = 2301^2 + 980^2 = 2499^2 + 100^2$.

Remark 2. We found the above positive integers by checking $5p_1$, $13p_2$, $17p_3$, $29p_4$, where each prime $p_i \equiv 1(mod\, 4)$, for $i = 1, 2, 3, 4$ such that $13 \le p_1 \le 1009$, $17 \le p_2 \le 1009$, $29 \le p_3 \le 229$ and $37 \le p_4 \le 557$; while other positive integers are obtained from one by one checking. In addition, we note that some of them are Fermat primes $F_n = 2^{2^n} + 1$, for $n = 1, 2, 3, 4$.

From Theorem 1 and Lemma 4, we shall construct infintely many new classes of integral trees with diameter 4.

Theorem 3. Let m_1, t_1, s_1, a, b, c and d be positive integers satisfying the following conditions

$$m_1 + t_1 + s_1 = a^2 + b^2 = c^2 + d^2,$$

where $c > a$, $b > d$, $(a, b) = 1$, $(c, d) = 1$ and $a|cd$ or $b|cd$. For the tree $K_{1,s} \bullet S(m, t)$ of Theorem 1, we have

(1) If $a|cd$, for any positive integer n, let $m = m_1 n^2$, $m_1 = b^2 - (cd/a)^2$, $t = t_1 n^2$, $t_1 = (cd/a)^2$, $s = s_1 n^2$ and $s_1 = a^2$, then $K_{1,s} \bullet S(m,t)$ is an integral tree with diameter 4.

(2) If $a|cd$, for any positive integer n, let $m = m_1 n^2$, $m_1 = b^2 - (cd/a)^2$, $s = s_1 n^2$, $s_1 = (cd/a)^2$, $t = t_1 n^2$ and $t_1 = a^2$, then $K_{1,s} \bullet S(m,t)$ is an integral tree with diameter 4.

(3) If $b|cd$, for any positive integer n, let $m = m_1 n^2$, $m_1 = a^2 - (cd/b)^2$, $t = t_1 n^2$, $t_1 = (cd/b)^2$, $s = s_1 n^2$ and $s_1 = b^2$, then $K_{1,s} \bullet S(m,t)$ is an integral tree with diameter 4.

(4) If $b|cd$, for any positive integer n, let $m = m_1 n^2$, $m_1 = a^2 - (cd/b)^2$, $s = s_1 n^2$, $s_1 = (cd/b)^2$, $t = t_1 n^2$ and $t_1 = b^2$, then $K_{1,s} \bullet S(m,t)$ is an integral tree with diameter 4.

Proof. This follows directly from Theorem 1 and Lemma 4.

Example 1. Note that $5 \times 13 = 7^2 + 4^2 = 8^2 + 1^2$. From Theorem 3 we have two cases for constructing such integral trees.

(1) If we let $t = 4n^2$, $s = 16n^2$ and $m = 45n^2$ for any positive integer n, then the tree $K_{1,s} \bullet S(m,t)$ is an integral one with diameter 4. Its spectrum is

$$Spec[K_{1,16n^2} \bullet S(45n^2, 4n^2)] = \begin{pmatrix} 0 & \pm n & \pm 2n & \pm 8n \\ 180n^4 - 29n^2 - 1 & 1 & 45n^2 - 1 & 1 \end{pmatrix}.$$

If $n = 1$, we know that the tree $K_{1,16} \bullet S(45,4)$ is an integral one with diameter 4, the order of which is 242.

(2) If we let $t = 16n^2$, $s = 4n^2$ and $m = 45n^2$ for any positive integer n, then the tree $K_{1,s} \bullet S(m,t)$ is an integral one with diameter 4. Its spectrum is

$$Spec[K_{1,4n^2} \bullet S(45n^2, 16n^2)] = \begin{pmatrix} 0 & \pm n & \pm 4n & \pm 8n \\ 720n^4 - 41n^2 - 1 & 1 & 45n^2 - 1 & 1 \end{pmatrix}.$$

If $n = 1$, we know that the tree $K_{1,4} \bullet S(45,16)$ is an integral one with diameter 4, the order of which is 770.

In fact, by the same methods as in Example 1, we can construct a family of integral trees with diameter 4 from every identity in the list of our Lemma 4. The family of integral trees given in Example 1 is obtained exactly from the first identity in the list of Lemma 4.

III. Integral Trees with Diameter 6

In this section, we shall construct infinitely many new integral trees with diameter 6.

Theorem 4. Let $K_{1,s} \bullet L(r,m,t)$ be the tree of diameter 6 obtained by identifying the center w of $K_{1,s}$ and the center u of $L(r,m,t)$. Then $K_{1,s} \bullet L(r,m,t)$ is integral if and only if t and $m+t$ are perfect squares, and $x^4-(m+t+r+s)x^2+rt+s(m+t)$ can be factored as $(x^2-a^2)(x^2-b^2)$.

Proof. Because the vertex w is the center of $K_{1,s}$ and the vertex u is the center of the tree $L(r,m,t)$, if we let $G=K_{1,s}$ and $H=L(r,m,t)$, then by Lemma 1 we know that

$$\begin{aligned} P[K_{1,s} \bullet L(r,m,t),x] = \ & P(K_{1,s},x)P^r[S(m,t),x]+x^sP[L(r,m,t),x] \\ & -x^{s+1}P^r[S(m,t),x]. \end{aligned}$$

By Lemma 2, we have

$$\begin{aligned} P[K_{1,s} \bullet L(r,m,t),x] = \ & x^{rm(t-1)+r+(s-1)}(x^2-t)^{r(m-1)}[x^2-(m+t)]^{r-1} \\ & \times[x^4-(m+t+r+s)x^2+rt+s(m+t)]. \end{aligned}$$

The theorem is thus proved.

Corollary 5. If s=t, then the tree $K_{1,t} \bullet L(r,m,t)$ of diameter 6 is integral if and only if t, $m+t$ and $m+t+r$ are perfect squares.

From Theorem 4, we shall construct infintely many new classes of integral trees with diameter 6. They are different from those ones of [2-10].

Theorem 5. For the tree $K_{1,r} \bullet L(s,m,t)$ of diameter 6, let the numbers m,t,s,m_1, t_1, s_1, a, b, c and d be as in (1) or (3) in Theorem 3, and let $r=t$ and $m_1+t_1+s_1$ be perfect squares. Then $K_{1,t} \bullet L(s,m,t)$ is an integral tree with diameter 6.

Proof. This follows from Corollary 5.

Example 2. Note that $(5\times 13)^2=56^2+33^2=63^2+16^2$. From Theorem 5, if we let $r=t=(18n)^2, m=765n^2$ and $s=(56n)^2$ for any positive integer n, then the tree $K_{1,t} \bullet L(s,m,t)$ is an integral one with diameter 6. Its spectrum is

$$Spec[K_{1,324n^2} \bullet L(3136n^2,765n^2,324n^2)] = \begin{pmatrix} 0 & \pm 18n & \pm 33n & \pm 65n \\ a & b & c & 1 \end{pmatrix},$$

where $a=777288960n^6-2399040n^4+3460n^2-1$, $b=2399040n^4-3136n^2+1$ and $c=3136n^2-1$. By setting $n=1$, we get a minimal integral tree $K_{1,324} \bullet L(3136,765,324)$ with diameter 6 in this class, the order of which is 779,691,461.

In fact, by the same methods as in Example 2, we can construct a family of integral trees with diameter 6 from every identity in the second half of the list in our Lemma 4.

References

[1] C. D. Godsil and B. D. Mckay, Constructing cospectral graphs, Aequationes Math., 25(1982) 257-268.

[2] M. Capobianco, S. Maurer, D. McCarthy and J. Molluzzo, A collection of open problems, Annals New York Academy of Sciences, (1980) 582-583.

[3] Xueliang Li and Guoning Lin, On the problem of integral trees, Kexue Tongbao, 32:11(1987) 813-816.

[4] Ruying Liu, Integral trees of diameter 5, J. Systems Sci. & Math. Sci., 8:4(1988) 357-360.

[5] Zhenfu Cao, On the integral trees of diameter R when $3 \leq R \leq 6$, J. Heilongjiang University, 2(1988) 1-3.

[6] Ruying Liu, Some new families of trees with integral eigenvalues, J. Qinghai Normal University, 3(1988) 1-5.

[7] Zhenfu Cao, Some new classes of integral trees with diameter 5 or 6, J. Systems Sci. & Math. Sci., 11:1(1991) 20-26.

[8] Ligong Wang, Xueliang Li and Ruying Liu, Integral trees with diameter 6 or 8, Electronic Notes on Discrete Math. (1999).

[9] Pingzhi Yuan, Integral trees of diameter 4, J. Systems Sci. & Math. Sci., 18:2(1998) 177-181.

[10] Dashen Xu, Some new classes of the integral trees with diameter 4, J. Hebei University, 17:1(1997) 12-16.

[11] D. Cvetković, M. Doob and H. Sachs, Spectra of Graphs, Academic Press, New York, Francisco, London, (1980) 59-60.

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