# More on sequences in groups

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**Abstract**

We bring to conclusion the investigation of three problems about sequencings for finite groups: the existence of harmonious sequences in dicyclic groups, the R-sequenceability of dicyclic groups, and the R-sequenceability of the nonabelian groups of order $pq$, where $p$ and $q$ are primes.

## Introduction

Various types of sequences of the elements of a finite group have been studied in connection with questions in combinatorics. In this article we discuss harmonious sequences and $R$-sequences, both of which are connected to complete mappings of a finite group $G$.

**Definition 1** A *complete mapping* of $G$ is a permutation $g \to \theta(g)$ of the elements of $G$ such that $\phi : g \to g\theta(g)$ is again a permutation of the elements of $G$. In this case, the mapping $\phi$ is called an *orthomorphism* of $G$.

The idea of a complete mapping was introduced by H. B. Mann [5] and studied later by L. J. Paige [6]. Results related to this notion and to those that follow are discussed in A. D. Keeedwell's recent survey [4].

**Definition 2** A group $G$ of order $m$ is called **harmonious** if its $m$ elements can be listed in a sequence

$$g_1, g_2, \ldots, g_m$$

such that the products

$$g_1g_2, g_2g_3, \ldots, g_{m-1}g_m, g_mg_1$$

of consecutive elements are all distinct.

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These sequences were introduced in [1]; if $G$ has a harmonious sequence, then $\theta = (g_1, g_2, \ldots, g_m)$ is a complete mapping of $G$, expressed as a single $m$-cycle. Harmonious groups include all groups of odd order, the nontrivial finite abelian groups having noncyclic Sylow 2-subgroups (except for the elementary abelian 2-groups), and the dihedral groups $D_n$ of order $2n$ whenever $n$ is divisible by 4 or $n = 6m$, $m$ odd [1]. In Section 1 we discuss harmoniousness in dicyclic groups.

**Definition 3** A group of order $m$ is called **$R$-sequenceable** if its $m$ elements can be listed in a sequence

$$g_1 = 1, g_2, \ldots, g_m$$

such that the partial products $g_1, g_1g_2, g_1g_2g_3, \ldots, g_1g_2g_3\cdots g_{m-1}$, are all distinct and $g_1g_2g_3\cdots g_{m-1}g_m = 1$.

If $G$ has an $R$-sequencing, then there is a complete mapping $\theta$ of $G$ such that the corresponding orthomorphism $\phi$ fixes one element and permutes the remaining elements in a single cycle. The dihedral group $D_n$ is $R$-sequenceable if and only if $n$ is even [3]. We treat this concept for dicyclic groups in Section 2, and for the nonabelian groups of order $pq$ ($p$ and $q$ prime) in Section 3.

In each of the next three sections, the complete result is designated as a Theorem. Previous results are indicated as Propositions; the new contributions are clearly indicated in our discussion.

# 1 Harmoniousness of dicyclic groups

For an $n \times n$ matrix $M$, the $(i,j)$-entry of $M$ is denoted by $M(i,j)$. For a permutation $\tau$ of degree $n$, the collection of $n$ elements $\{M(i,\tau(i)),\ i = 1, 2, \ldots, n\}$ is denoted by $\tau(M)$. The dicyclic group $Q_{2n}$ of order $4n$ is defined by

$$Q_{2n} = \langle \alpha, \beta : \alpha^{2n} = 1, \beta^2 = \alpha^n, \alpha\beta = \beta\alpha^{-1} \rangle.$$

The following proposition was proved in [7].

**Proposition 1** *Let $A$ and $B$ be two $n \times n$ matrices defined by*

$$A(i,j) = i + j - 2 \mod 2n, \quad i,j = 1, 2, \ldots, n$$
$$B(i,j) = n - i - j + 1 \mod 2n, \quad i,j = 1, 2, \ldots, n.$$

*Then the dicyclic group $Q_{2n}$ is harmonious if there exist two permutations $\pi$ and $\theta$ of degree $n$ such that $\theta \circ \pi$ is an $n$-cycle and $\pi(A) \cup \theta(B)$ is a complete set of residues modulo $2n$.*

**Sketch of the proof.** Let $f = \theta \circ \pi$. Let $c$ be an integer with $1 \leq c \leq n$. For any fixed integer $d$, we define

$$\begin{aligned} b_{2i-1} &= -f^{i-1}(c) + d, \\ b_{2i} &= f^{i-1}(c) + d - 1, \\ a_{2i-1} &= \pi f^{i-1}(c) - 1, \\ a_{2i} &= \pi f^{i-1}(c) + n - 1. \end{aligned}$$

We notice that

$$\begin{aligned} b_{2i} + a_{2i-1} &= A(f^{i-1}(c), \pi f^{i-1}(c)) + d \text{ and} \\ b_{2i+1} - a_{2i} &= B(\pi f^{i-1}(c), f^{i}(c)) + d. \end{aligned}$$

Direct calculation shows that

$$\{b_{2i} - b_{2i-1} + n : i = 1, 2, \ldots, n\} \cup \{a_{2i} + a_{2i-1} : i = 1, 2, \ldots, n\}$$

is a complete set of residues modulo $2n$ and

$$\{b_{2i+1} - a_{2i} : i = 1, 2, \ldots, n\} \cup \{b_{2i} + a_{2i-1} : i = 1, 2, \ldots, n\} = \pi(A) \cup \theta(B) + d$$

which is also a complete set of residues modulo $2n$ by our hypothesis. Therefore, the following sequence is a harmonious sequence of $Q_{2n}$:

$$\beta\alpha^{b_1}, \beta\alpha^{b_2}, \alpha^{a_1}, \alpha^{a_2}, \beta\alpha^{b_3}, \beta\alpha^{b_4}, \alpha^{a_3}, \alpha^{a_4}, \ldots, \beta\alpha^{b_{2n-1}}, \beta\alpha^{b_{2n}}, \alpha^{a_{2n-1}}, \alpha^{a_{2n}}.$$

∎

By using Proposition 1 the following result was proved in [7].

**Proposition 2** *If $n$ is a multiple of 4 or 6, the dicyclic group $Q_{2n}$ is harmonious.*

It remains to deal with the case of $n = 4k + 2$. We define two permutations $\pi$ and $\theta$ of degree $n$ by

$$\pi(x) = \begin{cases} x + 2k + 1 & \text{if } 1 \le x \le k, \\ x + 2k + 2 & \text{if } k + 1 \le x \le 2k, \\ x - 2k & \text{if } 2k + 1 \le x \le 3k + 1, \\ 3k + 2 & \text{if } x = 3k + 2, \\ x - 2k - 1 & \text{if } 3k + 3 \le x \le 4k + 2, \end{cases}$$

$$\theta(y) = \begin{cases} 1 & \text{if } y = 1, \\ 4k + 2 & \text{if } y = 2, \\ y + 2k - 1 & \text{if } 3 \le y \le k + 2, \\ k + 1 & \text{if } y = k + 3, \\ y + 2k - 2 & \text{if } k + 4 \le y \le 2k + 3, \\ y - 2k - 2 & \text{if } 2k + 4 \le y \le 3k + 2, \\ y - 2k - 1 & \text{if } 3k + 3 \le y \le 4k + 2. \end{cases}$$

By the definitions of matrices $A$ and $B$ in Proposition 1, we obtain

$$\pi(A) = \{2k, 2k + 1, 2k + 2, 2k + 3, \ldots, 4k, 4k + 2, 4k + 3, 4k + 4, 4k + 5, \ldots, 6k, 6k + 1, 6k + 2\}$$

and

$$\theta(B) = \{0, 1, 2, 3, \ldots, 2k - 3, 2k - 2, 2k - 1, 4k + 1, 6k + 3, 6k + 4, \ldots, 8k + 1, 8k + 2, 8k + 3\}.$$

Therefore, $\pi(A) \cup \theta(B)$ is a complete set of residues modulo $2n$.

Considering $k$ modulo 3, we find that in two cases $\theta \circ \pi$ is a cycle of length $n$. If $k = 3t$, by direct calculation, we have $\theta \circ \pi = (1\ 4k\ 4k-3\ 4k-6\ 4k-9\ \dots\ 3k+6\ 3k+3\ 3k+1\ 3k\ 3k-1\ 3k-2\ \dots\ 2k+3\ 2k+2\ 4k+2\ 4k-1\ 4k-4\ 4k-7\ \dots\ 3k+5\ 3k+2\ k\ k-1\ k-2\ k-3\ \dots\ 3\ 2\ 4k+1\ 4k-2\ 4k-5\ \dots\ 3k+7\ 3k+4\ k+1\ k+2\ k+3\ \dots\ 2k\ 2k+1)$ which is an $n$-cycle.

If $k = 3t+2$, we have $\theta \circ \pi = (1\ 4k\ 4k-3\ 4k-6\ 4k-9\ \dots\ 3k+5\ 3k+2\ k\ k-1\ k-2\ k-3\ \dots\ 3\ 2\ 4k+1\ 4k-2\ 4k-5\ \dots\ 3k+6\ 3k+3\ 3k+1\ 3k\ 3k-1\ 3k-2\ \dots\ 2k+3\ 2k+2\ 4k+2\ 4k-1\ 4k-4\ 4k-7\ \dots\ 3k+7\ 3k+4\ k+1\ k+2\ k+3\ \dots\ 2k\ 2k+1)$ which is a cycle of length $n$.

Therefore, by Proposition 1 we can state

**Proposition 3** *If $n = 12t+2$ or $n = 12t+10$, the dicyclic group $Q_{2n}$ is harmonious.*

In the remaining we have case $n = 12t+6$, and this is covered by Proposition 2. It is shown in [1] that $Q_{2n}$ is not harmonious if $n$ is an odd integer or $n = 2$. Hence, by Propositions 2 and 3 the following is true:

**Theorem 1** *$Q_{2n}$ is harmonious if and only if $n$ is an even integer greater than 2.*

It is obvious that a harmonious group may have many harmonious sequences. We can, for example, give an alternative construction for the case of $n = 8k+2$ as follows.

Let $\pi$ and $\theta$ be permutations of degree $n$ defined by

$$\pi(x) = \begin{cases} 4k+2+x & \text{if } 1 \le x \le 2k-1, \\ 4k+4+x & \text{if } 2k \le x \le 4k-2 \text{ and } x \text{ is even}, \\ 2k+1 & \text{if } x = 2k+1, \\ 4k+x & \text{if } 2k+3 \le x \le 4k+1 \text{ and } x \text{ is odd}, \\ 4k+2 & \text{if } x = 4k, \\ x-4k-1 & \text{if } 4k+2 \le x \le 8k+2 \text{ and } x \ne 6k+2, \\ 6k+2 & \text{if } x = 6k+2, \end{cases}$$

$$\theta(y) = \begin{cases} y & \text{if } y = 1, \\ 4k+y & \text{if } 2 \le y \le 4k+2, \\ y-4k-1 & \text{if } 4k+3 \le y \le 8k+2. \end{cases}$$

By the definitions of matrices A and B in Proposition 1, we have

$$\pi(A) = \{4k, 4k+1, 4k+2, \dots, 8k-1, 8k, 8k+2, 8k+3, \dots, 12k+2\} \bmod 2n \text{ and}$$

$$\theta(B) = \{0, 1, 2, \dots, 4k-1, 8k+1, 12k+3, 12k+4, \dots, 16k+3\} \bmod 2n.$$

Therefore $\pi(A) \cup \theta(B)$ is a complete set of residues modulo $2n$.

By direct calculation, we have $\theta \circ \pi = (1\ 2\ 3\ \dots\ 2k-1\ 2k\ 2k+3\ 2k+2\ 2k+5\ 2k+4\ 2k+7\ 2k+6\ \dots\ 4k+1\ 4k\ 8k+2\ 8k+1\ 8k\ 8k-1\ \dots\ 6k+3\ 6k+2\ 2k+1\ 6k+1\ 6k\ 6k-1\ \dots\ 4k+2)$ which is an $n$-cycle.

## 2 The R-sequenceability of dicyclic groups

The following two propositions were proved in [8].

**Proposition 4** *$Q_{2n}$ is R-sequenceable if there are integers $a_2, a_3, \ldots, a_{2n-1}$ and $b_1$, $b_2, \ldots, b_{2n}$ satisfying*

*(1)* $0, a_2, a_3, \ldots, a_{2n-1}$ *are distinct* mod $2n$,

*(2)* $b_1, b_2, \ldots, b_{2n}$ *are distinct* mod $2n$,

*(3)* $0, a_2, a_3 - a_2, \ldots, a_n - a_{n-1}, b_{n+1} - b_n, b_{n+2} - b_{n+1}, \ldots, b_{2n} - b_{2n-1}$ *are distinct* mod $2n$,

*(4)* $b_1 + a_n, b_1 + a_{n+1} + n, b_2 + a_{n+1}, b_2 + a_{n+2} + n, b_3 + a_{n+2}, \ldots, b_{n-1} + a_{2n-1} + n, b_n + a_{2n-1}, b_{2n} + n$ *are distinct* mod $2n$.

**Proposition 5** *Let $A$ and $B$ be two $n \times n$ matrices defined by*

$$A(i,j) = \begin{cases} 3n/2 + i + j - 1 & \textit{mod } 2n \textit{ if } i \le n/2 \\ 3n/2 + i + j & \textit{mod } 2n \textit{ if } i > n/2 \end{cases}$$

$$B(i,j) = \begin{cases} n/2 + i + j - 1 & \textit{mod } 2n \textit{ if } i \le n/2 \\ n/2 + i + j & \textit{mod } 2n \textit{ if } i > n/2. \end{cases}$$

*Then the dicyclic group $Q_{2n}$ is R-sequenceable if there exist two permutations $\pi$ and $\theta$ of degree $n$ such that $\pi \circ \theta^{-1}$ is a cycle of length $n$ with $\theta(1) = n$, and $\pi(A) \cup \theta(B)$ is a complete set of residues modulo $2n$.*

**Theorem 2** *$Q_{2n}$ is R-sequenceable if and only if $n$ is an even integer greater than 2.*

**Proof.** It was shown in [8] that for $n = 2$, and for $n$ odd, $Q_{2n}$ is not $R$-sequenceable, and for $n \equiv 0 \bmod 4$, $Q_{2n}$ is $R$-sequenceable. Thus we assume that $n = 4k - 2$ where $k > 1$ is an integer. We modify the proof of the case of $n = 4k$ in [8] by defining the permutations of $\pi$ and $\theta$ by

$$\pi(x) = \begin{cases} x + 2k - 1 & \text{if } 1 \le x \le 2k - 1, \\ x - 2k + 1 & \text{if } 2k \le x \le 4k - 2, \end{cases}$$

$$\theta(y) = \begin{cases} y + 2k - 2 & \text{if } 2 \le y \le 2k - 1, y \ne k + 1, \\ y - 2k + 2 & \text{if } 2k \le y \le 4k - 3, y \ne 3k - 2, \end{cases} \text{ together with}$$

$$\theta(1) = 4k - 2, \theta(4k - 2) = 1, \theta(k + 1) = k, \text{ and } \theta(3k - 2) = 3k - 1.$$

By the definitions of $A$ and $B$ in Proposition 5, we have $\pi(A) = \{1, 2, 3, \ldots, 4k - 2\}$ and $\theta(B) = \{4k - 1, 4k, 4k + 1, \ldots, 8k - 4\}$. Hence $\pi(A) \cup \theta(B)$ is a complete set of residues modulo $2n$.

Notice $k > 1$. By direct calculation, we have $\pi \circ \theta^{-1} = (2k-1\ 2k-2\ \ldots\ k+1\ k\ 3k\ 3k+1\ 3k+2\ \ldots\ 4k-2\ 2k\ 2k+1\ 2k+2\ \ldots\ 3k-1\ k-1\ k-2\ \ldots\ 2\ 1)$ which is a cycle of length $n$. Hence by Proposition 5, $Q_{2n}$ is $R$-sequenceable. ∎

By using Proposition 4 we can give a distinct $R$-sequencing of $Q_{2n}$ from the construction indicated in the proof of Theorem 2. This construction can serve as an alternative proof to Theorem 2 for the case $n = 8k + 2$.

We define the sequences (1) and (2) in Proposition 4 as follows:

Sequence (1), of $2n - 1$ elements, can be given in seven segments, with numbers of elements and rule of construction given by

(i) $8k + 2$ elements: $0, 8k + 1, 1, 8k, 2, 8k - 1, \ldots, 4k, 4k + 1$;

(ii) $2k$ elements: $16k + 3, 16k + 2, 16k + 5, \ldots, 14k + 5, 14k + 4$;

(iii) $2k - 2$ elements: $14k + 1, 14k + 2, 14k - 1, 14k, 14k - 3, 14k - 2, \ldots, 12k + 5, 12k + 6$;

(iv) $2k$ elements: $12k + 3, 12k + 2, 12k + 1, 12k, \ldots, 10k + 4$;

(v) The single element $14k + 3$;

(vi) $2k + 1$ elements: $10k + 3, 10k + 2, \ldots, 8k + 3$;

(vii) The final element $12k + 4$.

Similarly sequence (2), of $2n$ elements, is given by

(i) $2k + 1$ elements: $8k + 1, 8k, 8k - 1, \ldots, 6k + 1$;

(ii) $2k - 2$ elements: $6k - 2, 6k - 1, 6k - 4, 6k - 3, 6k - 6, 6k - 5, \ldots, 4k + 2, 4k + 3$;

(iii) $2k$ elements: $4k, 4k - 1, 4k - 2, \ldots, 2k + 1$;

(iv) the single element $6k$;

(v) $2k + 1$ elements: $2k, 2k - 1, 2k - 2, \ldots, 1, 0$;

(vi) the single element $4k + 1$;

(vii) $8k + 2$ elements: $12k + 3, 12k + 2, 12k + 4, 12k + 1, 12k + 5, 12k, 12k + 6, 12k - 1, \ldots, 16k + 3, 8k + 2$.

In the following examples, semicolons separate the segments in the listing of sequence elements. $k = 1$, one segment of each sequence is vacuous (noted by a repeated semicolon).

When $k = 1$, so that $n = 8k + 2 = 10$, the sequences are

($1_1$): 0, 9, 1, 8, 2, 7, 3, 6, 4, 5; 19, 18;; 15, 14; 17; 13, 12, 11; 16, and

($2_1$): 9, 8, 7;; 4, 3; 6; 2, 1, 0; 5; 15, 14, 16, 13, 17, 12, 18, 11, 19, 10.

When $k = 2$, so that $n = 8k + 2 = 18$, the sequences are

($1_2$): 0, 17, 1, 16, 2, 15, 3, 14, 4, 13, 5, 12, 6, 11, 7, 10; 8, 9; 35, 34, 33, 32; 29, 30; 27, 26, 25, 24; 31; 23, 22, 21, 20, 19; 28, and

($2_2$): 17, 16, 15, 14, 13; 10, 9; 8, 7, 6, 5; 12; 4, 3, 2, 1, 0; 9; 27, 26, 28, 25, 29, 24, 30, 23, 31, 22, 32, 21, 33, 20, 34, 19, 35, 18.

## 3 $R$-sequenceability of groups of order $pq$

First we state an alternative definition of $R$-sequenceability. It is easily seen that this definition is equivalent to Definition 3.

**Definition 4** (a) A group $G$ of order $m$ is called **$R$-sequenceable** if its nonidentity elements can be listed in a sequence

$$g_1, g_2, \ldots, g_{m-1}$$

such that

$$g_1^{-1}g_2, g_2^{-1}g_3, \ldots, g_{m-2}^{-1}g_{m-1}, g_{m-1}^{-1}g_1$$

are all distinct.

(b) If, further, $g_1 = g_2 g_{m-1} = g_{m-1} g_2$, $G$ is called **$R^*$-sequenceable.**

The nonabelian group $G$ of order $pq$ where $p, q$ are primes with $q \equiv 1 \pmod p$ is defined, using $r$ such that $r^p \equiv 1 \pmod q$, $r \not\equiv 1 \pmod q$, as follows:

$$G = \{(u, v) \mid u \in \mathbf{Z}/p\mathbf{Z}, v \in \mathbf{Z}/q\mathbf{Z}\}$$

with multiplication defined by

$$(u, v)(x, y) = (u + x, vr^x + y).$$

The following result was proved by Keedwell [3].

**Proposition 6** *The nonabelian group of order $pq$ is $R$-sequenceable if $p$ has 2 as a primitive root.*

This result can be extended, essentially by means of a modification of the direct product theorem in [2].

**Theorem 3** *The nonabelian group of order $pq$ is $R$-sequenceable.*

**Proof.** In view of Proposition 6 we assume $p > 5$. The (additive) cyclic group $\mathbf{Z}/p\mathbf{Z}$ is $R^*$-sequenceable [2]; let $g_1, \ldots, g_{p-1}$ be an $R^*$-sequencing with $g_1 = g_2 + g_{p-1}$.

In order to list the nonidentity elements of $G$, we introduce $p \times q$ matrices $A$ and $B$, defined over $\mathbf{Z}/p\mathbf{Z}$ and $\mathbf{Z}/q\mathbf{Z}$, respectively. Entries $a_{11}$ and $b_{11}$ are left blank; group elements are given as $(a_{ij}, b_{ij})$, and are sequenced by reading column-by-column from $A$ and $B$.

Let $a_{ij} = g_1$ for $i = 1$, $2 \le j \le (q+1)/2$, and for $i = 2$, $1 \le j \le (q+1)/2$. For $i = 1, 2$ and $j > (q+1)/2$, let $a_{ij} = 0$. For $i > 2$ and $1 \le j \le q$, let $a_{ij} = g_{i-1}$. The definition of $B$ uses the nonzero element $c = -r^{g_1 - g_{p-2} - g_2}$ of $\mathbf{Z}/q\mathbf{Z}$; all arithmetic is in $\mathbf{Z}/q\mathbf{Z}$. For $j \ge 2$, let $b_{1j} = b_{3j} = (j-1) \cdot c$. Complete row 3 by letting $b_{31} = 0$, and define row 2 by letting $b_{2j} = -b_{3j}$, $1 \le j \le q$. For $4 \le i \le p-1$ (except as noted below), row $i$ is the $q$-tuple $(0, q-1, q-2, \ldots, 2, 1)$. The final row is $(r^k, 2r^k, \ldots, (q-1)r^k, 0)$, where $k = g_{p-1} - g_{p-2} \bmod p$. (A modification is needed when $cr^{g_3 - g_2} \equiv -1$, or equivalently, when $g_3 - g_2 + g_{p-1} - g_{p-2} \equiv 0 \bmod p$. In this case, row 4 of $B$ is defined by the $q$-tuple $(0, 1, 2, \ldots, q-1)$, affecting rows 4 and 5 of $B'$ below.)

To verify that $(a_{ij}, b_{ij})$ form the desired $R$-sequencing of $G$, one must study the matrices $A'$ and $B'$ with $a'_{11}$ and $b'_{11}$ blank and other entries defined by

$$(a'_{ij}, b'_{ij}) = \begin{cases} (a_{i-1,j}, b_{i-1,j})^{-1}(a_{ij}, b_{ij}) & \text{if } i > 1, \\ (a_{p,j-1}, b_{p,j-1})^{-1}(a_{ij}, b_{ij}) & \text{if } i = 1 \text{ and } j > 1, \end{cases}$$

together with $(a'_{21} b'_{21}) = (a_{pq} b_{pq})^{-1}(a_{21} b_{21})$. Thus the $(a'_{ij}, b'_{ij})$ list the products required by Definition 3.

The verification that the listed products are distinct is aided by the following observations:

(i) The matrix $A'$ has a particularly simple form, namely,

$$\begin{bmatrix} - & g_1 - g_{p-1} & \cdots & g_1 - g_{p-1} & g_2 - g_1 & \cdots & g_2 - g_1 \\ g_1 - g_{p-1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ g_2 - g_1 & g_2 - g_1 & \cdots & g_2 - g_1 & g_1 - g_{p-1} & \cdots & g_1 - g_{p-1} \\ g_3 - g_2 & g_3 - g_2 & \cdots & & & & g_3 - g_2 \\ \cdots & & & & & & \\ g_{p-1} - g_{p-2} & g_{p-1} - g_{p-2} & \cdots & & & & g_{p-1} - g_{p-2} \end{bmatrix},$$

so that, from row 4 onwards, it suffices to verify that the entries in each row of $B'$ are distinct.

(ii) $q = 2hp + 1$ and $r^p = 1$, so that none of $r^{g_2}$ or $r^{g_2 - g_1}, r^{g_4 - g_3}, \ldots, r^{g_{p-2} - g_{p-3}}$ can be $-1$.

(iii) The analysis of row 1 and row 3 is made easy by observing the two equalities

$$c - r^{g_1 - g_{p-2}} = cr^{g_2} + c, \quad c - r^{-g_{p-2}} = cr^{g_2 - g_1} + c.$$

(iv) The last row of $B'$ is given by $(r^k, 3r^k, \ldots, (q-2)r^k, 0, 2r^k, 4r^k \ldots, (q-1)r^k)$.

(v) If $cr^{g_3 - g_2} = -1$, then row 4 of $B'$ as "generally" defined consists entirely of zeros. The modification noted above corrects this difficulty so that the affected rows of $B'$ have distinct entries. ∎

**Remark:** It should be pointed out that the $R$-sequencing of the previous result is, in fact, an $R^*$-sequencing because $(g_1, 0) = (g_2, 0)(g_{p-1}, 0) = (g_{p-1}, 0)(g_2, 0)$.

It is useful to have an example for the construction of the $R$-sequencing in Theorem 3. We include the smallest example for which the prime $p$ does not have 2 as a primitive root.

**Example.** The nonabelian group $G$ of order 203 has $p = 7$ and $q = 29$. We use the $R^*$-sequencing 3, 2, 5, 6, 4, 1 of $\mathbf{Z}/7\mathbf{Z}$, so that $A$ is as follows:

$$\left[\begin{array}{ccccccccccccccccccccccccccccc} - & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

Use $r = 7$ (which implies $c = 6$). Since $g_3 - g_2 + g_{p-1} - g_{p-2} \equiv 0 \bmod 7$, we modify row 4 and obtain the following matrix as $B$:

$$\begin{bmatrix}
- & 06 & 12 & 18 & 24 & 01 & 07 & 13 & 19 & 25 & 02 & 08 & 14 & 20 & 26 & 03 & 09 & 15 & 21 & 27 & 04 & 10 & 16 & 22 & 28 & 05 & 11 & 17 & 23 \\
0 & 23 & 17 & 11 & 05 & 28 & 22 & 16 & 10 & 04 & 27 & 21 & 15 & 09 & 03 & 26 & 20 & 14 & 08 & 02 & 25 & 19 & 13 & 07 & 01 & 24 & 18 & 12 & 06 \\
0 & 06 & 12 & 18 & 24 & 01 & 07 & 13 & 19 & 25 & 02 & 08 & 14 & 20 & 26 & 03 & 09 & 15 & 21 & 27 & 04 & 10 & 16 & 22 & 28 & 05 & 11 & 17 & 23 \\
0 & 01 & 02 & 03 & 04 & 05 & 06 & 07 & 08 & 09 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 \\
0 & 28 & 27 & 26 & 25 & 24 & 23 & 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 09 & 08 & 07 & 06 & 05 & 04 & 03 & 02 & 01 \\
0 & 28 & 27 & 26 & 25 & 24 & 23 & 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 09 & 08 & 07 & 06 & 05 & 04 & 03 & 02 & 01 \\
23 & 17 & 11 & 05 & 28 & 22 & 16 & 10 & 04 & 27 & 21 & 15 & 09 & 03 & 26 & 20 & 14 & 08 & 02 & 25 & 19 & 13 & 07 & 01 & 24 & 18 & 12 & 06 & 0
\end{bmatrix}$$

By the definitions of the elements $(a'_{ij}, b'_{ij})$ we have $A'$ as follows:

$$\begin{bmatrix}
- & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4
\end{bmatrix}$$

and, finally, $B'$ is as follows:

$$\begin{bmatrix}
- & 10 & 20 & 01 & 11 & 21 & 02 & 12 & 22 & 03 & 13 & 23 & 04 & 14 & 24 & 20 & 02 & 13 & 24 & 06 & 17 & 28 & 10 & 21 & 03 & 14 & 25 & 07 & 18 \\
0 & 17 & 05 & 22 & 10 & 27 & 15 & 03 & 20 & 08 & 25 & 13 & 01 & 18 & 06 & 23 & 11 & 28 & 16 & 04 & 21 & 09 & 26 & 14 & 02 & 19 & 07 & 24 & 12 \\
0 & 11 & 22 & 04 & 15 & 26 & 08 & 19 & 01 & 12 & 23 & 05 & 16 & 27 & 09 & 05 & 15 & 25 & 06 & 16 & 26 & 07 & 17 & 27 & 08 & 18 & 28 & 09 & 19 \\
0 & 02 & 04 & 06 & 08 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 01 & 03 & 05 & 07 & 09 & 11 & 13 & 15 & 17 & 19 & 21 & 23 & 25 & 27 \\
0 & 21 & 13 & 05 & 26 & 18 & 10 & 02 & 23 & 15 & 07 & 28 & 20 & 12 & 04 & 25 & 17 & 09 & 01 & 22 & 14 & 06 & 27 & 19 & 11 & 03 & 24 & 16 & 08 \\
0 & 15 & 01 & 16 & 02 & 17 & 03 & 18 & 04 & 19 & 05 & 20 & 06 & 21 & 07 & 22 & 08 & 23 & 09 & 24 & 10 & 25 & 11 & 26 & 12 & 27 & 13 & 28 & 16 \\
23 & 11 & 28 & 16 & 04 & 21 & 09 & 26 & 14 & 02 & 19 & 07 & 24 & 12 & 0 & 17 & 05 & 22 & 10 & 27 & 15 & 03 & 20 & 08 & 25 & 13 & 01 & 18 & 06
\end{bmatrix}$$

The listing of the elements $(a_{ij}, b_{ij})$ by columns gives the indicated $R^*$-sequencing of $G$, as can be checked by examining the $(a'_{ij}, b'_{ij})$.

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