

Three-line chromatic indices of Steiner triple systems

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Abstract

There are five possible structures for a set of three lines of a Steiner triple system. Each of these three-line “configurations” gives rise to a colouring problem in which a partition of all the lines of an $\text{STS}(v)$ is sought, the components of the partition each having the property of not containing any copy of the configuration in question. For a three-line configuration B , and $\text{STS}(v)$ S , the minimum number of classes required is denoted by $\chi(B, S)$ and is called the B -chromatic index of S . This generalises the ordinary chromatic index $\chi'(S)$ and the 2-parallel chromatic index $\chi''(S)$. (For the latter see [7].) In this paper we obtain results concerning $\chi(B, v) = \min\{\chi(B, S) : S \text{ is an } \text{STS}(v)\}$ for four of the five three-line configurations B . In three of the cases we give precise values for all sufficiently large v and in the fourth case we give an asymptotic result. The values of the four chromatic indices for $v \leq 13$ are also determined.

1 Introduction.

Considerable research activity has recently been seen on the topic of configurations in Steiner triple systems. Aspects which have received significant attention have been the counting of configurations in Steiner triple systems [12, 8, 15], the decomposition of Steiner triple systems into various n -line configurations [16, 13, 14], and the avoidance of certain configurations [3, 5, 17]. A general survey of all these aspects,

and others, is given in [11]. There are a variety of colouring problems related to the issues of decomposition and avoidance. In an earlier paper [7], the current authors together with P. Danziger introduced the notion of a generalised chromatic index associated with a given configuration in a Steiner triple system. That paper focused on the so-called 2-parallel chromatic index. In this paper we investigate four of the possible five 3-line chromatic indices.

A Steiner triple system of order v is an ordered pair $(V, \mathcal{B})$, where V is a set of cardinality v (the *points*) and $\mathcal{B}$ is a collection of 3-element subsets of V (the *lines* or *blocks*) which has the property that every 2-element subset of V is contained in precisely one block. It is well-known that an STS(v) exists if and only if $v \equiv 1$ or $3 \pmod{6}$; such values of v are called *admissible*. If S is an STS(v) then its *chromatic index* $\chi'(S)$ is the smallest number of colours required to colour the blocks of S , each with a single colour, so that no two intersecting blocks receive the same colour. The generalisation of this concept given in [7] relates to colouring the blocks of an STS(v) so as to avoid monochromatic copies of a configuration C . By a *configuration* C we simply mean a collection of lines of an STS(v). The resulting chromatic index is denoted by $\chi(C, S)$. The possible 2-line configurations are: (a) two lines intersecting in a point, and (b) two parallel (i.e. non-intersecting) lines. In the former case $\chi(C, S)$ is just the ordinary chromatic index $\chi'(S)$. The latter case gives rise to the 2-parallel chromatic index denoted by $\chi''(S)$.

It is generally difficult to determine the precise value of $\chi(C, S)$ for given C and S . However, it is possible to obtain upper and lower bounds in some cases. For admissible v we may define

$$\begin{aligned}\overline{\chi'}(v) &= \max\{\chi'(S) : S \text{ is an STS}(v)\} \text{ and} \\ \underline{\chi'}(v) &= \min\{\chi'(S) : S \text{ is an STS}(v)\}.\end{aligned}$$

(And we can make similar definitions for $\chi''(S)$ and $\chi(C, S)$.) It may be shown that

$$\underline{\chi'}(v) = \begin{cases} (v-1)/2 & \text{if } v \equiv 3 \pmod{6}, \\ (v+1)/2 & \text{if } v \equiv 1 \pmod{6} \text{ and } v \geq 19. \end{cases}$$

This follows from results of Ray-Chaudhuri and Wilson [20] on the existence of Kirkman triple systems for $v \equiv 3 \pmod{6}$, and Vanstone *et al* [22] on the existence of Hanani triple systems for $v \equiv 1 \pmod{6}$, $v \geq 19$. (A definition of these systems is given below.) It is also known that for $v \geq 9$, $\overline{\chi'}(v) \leq 3(v-3)/2$, [4], and that for sufficiently large v , $\chi'(S) = v/2 + o(v)$ for any STS(v), S , [19].

In [7] it is shown that for $v \geq 27$

$$\underline{\chi''}(v) = \begin{cases} (v-1)/2 & \text{if } v \equiv 3 \text{ or } 7 \pmod{12}, \\ (v+1)/2 & \text{if } v \equiv 1 \text{ or } 9 \pmod{12}. \end{cases}$$

and, using a result of Phelps and Rödl [18], that $\overline{\chi''}(v) \leq v - c\sqrt{v \log v}$ for some absolute constant c .

There are five 3-line configurations $(B_1, B_2, \dots, B_5)$ for Steiner triple systems and these are shown in Figure 1 with their traditional names.

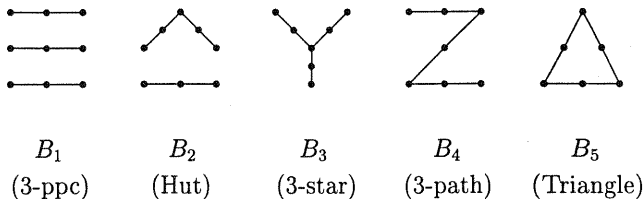


Figure 1: The five 3-line configurations.

We investigate below the generalised chromatic indices associated with the first four of these configurations. The fifth configuration (the triangle) gives rise to very different results and estimation of that chromatic index is connected with the partition of projective spaces $PG(n, 3)$ into caps; see [9] for definitions of these terms. We hope to make the triangle chromatic index the subject of a future paper.

In the course of our investigations we need to mention certain specific configurations and designs of various types. A set of n lines of an $STS(v)$ intersecting in a common point is called an (n) -star; it is sometimes convenient to include the cases $n = 0$ and $n = 1$. A set of n parallel (i.e. mutually disjoint) lines of an $STS(v)$ is called an (n) -partial parallel class, abbreviated to (n) -ppc. Again, it is sometimes convenient to include the values $n = 0$ and $n = 1$. If $n = v/3$, the maximum possible value, then an n -ppc is called a (full) parallel class. An $STS(v)$ whose lines may be partitioned into full parallel classes is said to be *resolvable*. Such a design together with its partition is called a Kirkman triple system, $KTS(v)$. These exist if and only if $v \equiv 3 \pmod{6}$ [20]. For $v \equiv 1 \pmod{6}$ and $v \geq 19$ there exists an $STS(v)$ whose lines may be partitioned into $(v-1)/2$ $((v-1)/3)$ -ppcs together with one $((v-1)/6)$ -ppc. Such a design together with its partition is called a Hanani triple system, $HATS(v)$ [22].

The $STS(7)$ is unique up to isomorphism and is called a *Fano plane*. We will refer to a Fano plane itself and those configurations which are not stars and are obtained from it by deleting lines as *Fano derivatives*. The isomorphism classes for these are illustrated in Figure 2 with their traditional names.

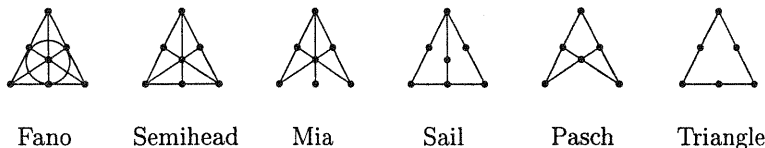


Figure 2: The six Fano derivatives.

The $STS(9)$ is also unique up to isomorphism and, in fact, forms a $KTS(9)$ re-

solvable into four parallel classes each containing three lines.

If in the definition of an STS(v) we replace the 2-element and 3-element subsets respectively by t -element and k -element subsets ($t < k$) then we obtain the definition of a Steiner system $S(t, k, v)$. An $S(2, 3, v)$ is just an STS(v). Concepts such as parallel classes and resolvability are easily extended from STS(v) to $S(t, k, v)$.

An m -GDD (*Group Divisible Design*) of type g^u is an ordered triple $(V, \mathcal{G}, \mathcal{B})$ with the following properties.

- (i) $\mathcal{G}$ is a partition of the set V of points into u subsets each of cardinality g (so that $|V| = gu$). These subsets are called the *groups*.
- (ii) $\mathcal{B}$ is a collection of m -element subsets of V (the *blocks*) with the property that each group intersects each block in at most one point.
- (iii) Every pair of points from distinct groups occurs in a unique block.

A partial parallel class of blocks of an m -GDD of type g^u is simply a collection of non-intersecting blocks. Such a partial parallel class is said to be a (full) parallel class if the union of all its blocks contains all the points of the design. The GDD is said to be resolvable if its blocks may be partitioned into full parallel classes.

A *transversal design* TD(k, v) is a k -GDD of type v^k . It is well-known that the existence of transversal designs and resolvable transversal designs is related to that of mutually orthogonal latin squares (MOLS) (see, e.g., [1]).

2 The 3-ppc (B_1).

Our first result gives an upper bound for $\underline{\chi}(B_1, v)$.

Theorem 2.1

$$\underline{\chi}(B_1, v) \leq \begin{cases} (v+1)/4 & \text{if } v \equiv 3 \text{ or } 7 \pmod{12}, \\ (v+3)/4 & \text{if } v \equiv 1 \text{ or } 9 \pmod{12}. \end{cases}$$

Proof From [7] we have

$$\underline{\chi}''(v) \leq \begin{cases} (v-1)/2 & \text{if } v \equiv 3 \text{ or } 7 \pmod{12}, \\ (v+1)/2 & \text{if } v \equiv 1 \text{ or } 9 \pmod{12}. \end{cases}$$

By combining these 2-parallel-free classes together in pairs (with one class unpaired) we obtain the desired result. $\square$

We shall now prove that, for sufficiently large v , the bound given by the Theorem above cannot be improved. As a first step we prove the following Lemma. In essence, and for large v , this enables us to deal only with colourings whose classes are stars or unions of two stars.

Lemma 2.1 *Suppose that S is a set of lines of an $STS(v)$, no three of these lines being parallel. Then, if $|S| \geq 23$, S may be partitioned into two disjoint subsets, S_1 and S_2 , neither of which contains two parallel lines and one of which is a star.*

Proof If S itself does not contain a pair of parallel lines then we can take $S_1 = S$ and $S_2 = \emptyset$. Note that S_1 is then either a Fano derivative or a star; however it cannot be the former since $|S| > 7$. We may therefore assume that S has a pair of parallel lines, l_1 and l_2 say. Put $A = \{l \in S : l \cap l_2 = \emptyset\}$ and $B = \{l \in S : l \cap l_1 = \emptyset\}$. If A contained two parallel lines, say l_3 and l_4 then $\{l_2, l_3, l_4\}$ would form a set of three parallel lines in S . Hence neither A nor, similarly, B can contain two parallel lines. Thus A and B are either stars or Fano derivatives.

Next put $C = \{l \in S : l \cap l_1 \neq \emptyset \text{ and } l \cap l_2 \neq \emptyset\}$. Then, clearly, $|C| \leq 9$. But actually, $|C| \leq 8$. To see this, assume $|C| = 9$, and let $l_1 = \{a, b, c\}$ and $l_2 = \{d, e, f\}$. Then C contains three lines through a , say $\{a, d, x\}$, $\{a, e, y\}$ and $\{a, f, z\}$, and three lines through b which, without loss of generality may be taken as one of the following four alternatives.

Case 1: $\{b, d, y\}, \{b, e, z\}, \{b, f, x\}$,

Case 2: $\{b, d, y\}, \{b, e, z\}, \{b, f, u\}$,

Case 3: $\{b, d, v\}, \{b, e, z\}, \{b, f, u\}$,

Case 4: $\{b, d, v\}, \{b, e, w\}, \{b, f, u\}$,

(where x, y, z, u, v, w are pairwise distinct points). Consider the third point α of the line containing c and e . In Cases 1 and 2, α cannot be y or z . But then $\{a, f, z\}$, $\{b, d, y\}$ and $\{c, e, \alpha\}$ are three parallel lines, a contradiction. In Cases 3 and 4, because $\{a, d, x\}$ and $\{b, f, u\}$ are parallel, we must have $\alpha = x$ or u . At the same time, because $\{a, f, z\}$ and $\{b, d, v\}$ are parallel, we must also have $\alpha = z$ or v , again a contradiction. Thus, indeed, $|C| \leq 8$. (This cannot be improved as the following set of eight lines shows: $\{\{a, d, x\}, \{a, e, y\}, \{a, f, z\}, \{b, d, y\}, \{b, e, z\}, \{b, f, u\}, \{c, d, u\}, \{c, f, x\}\}$.)

Since $A \cup B \cup C = S$ and $(A \cup B) \cap C = \emptyset$, we have $|A \cup B| \geq 15$. Thus either $|A| \geq 8$ or $|B| \geq 8$. Without loss of generality we can assume that $|A| \geq 8$, so that A is necessarily a star with a star-centre of degree at least 8. Let the star-centre be a .

Put $S_1 = \{l \in S : a \in l\}$ and $S_2 = \{l \in S : a \notin l\}$. Then S_1 is a star centred on a and therefore has no pair of parallel lines. Also, $A \subseteq S_1$ and so $|S_1| \geq 8$. Now suppose that S_2 has pair of parallel lines, λ_1 and λ_2 say. Neither λ_1 nor λ_2 contain a . Since there are more than six lines through a which lie in S , there must be at least one line of S through a which does not meet λ_1 or λ_2 . But this is a contradiction and consequently we see that S_2 cannot contain a pair of parallel lines. $\square$

Theorem 2.2 *Suppose that S is an $STS(v)$. Then*

(i) *if $v \geq 63$ and $v \equiv 3$ or $7 \pmod{12}$, $\chi(B_1, S) \geq (v+1)/4$,*

(ii) if $v \geq 133$ and $v \equiv 1$ or $9 \pmod{12}$, $\chi(B_1, S) \geq (v + 3)/4$.

Proof Suppose that S is an STS(v) with $\chi(B_1, S) = k$ where

$$k < \begin{cases} (v + 1)/4 & \text{if } v \equiv 3 \text{ or } 7 \pmod{12}, \\ (v + 3)/4 & \text{if } v \equiv 1 \text{ or } 9 \pmod{12}, \end{cases}$$

and

$$v \geq \begin{cases} 63 & \text{if } v \equiv 3 \text{ or } 7 \pmod{12}, \\ 133 & \text{if } v \equiv 1 \text{ or } 9 \pmod{12}. \end{cases}$$

We will prove that there is a B_1 -free colouring of S in at most k colours in which every colour class is either a star or the union of two stars. From this we will deduce a contradiction.

Firstly, take the original colouring and consider those colour classes C_i with $|C_i| \geq 23$ which are neither a single star nor the union of two stars. By the previous Lemma we may partition such a class C_i into C_i^* and C_i' where C_i^* is a star and C_i' is a Fano derivative. The stars C_i^* may then be combined in pairs (including with them any of the original colour classes which were single stars) and the Fano derivatives C_i' may also be combined in pairs. If there is an even number of the configurations C_i as described then we obtain a colouring of S in at most k colours. If there is an odd number then we may need one additional colour class and, if we do need $k + 1$ colours, one of the colour classes will be a single star and one will be a Fano derivative. In either circumstance each colour class now comprises one of the following:

- (a) A union of two stars (not being a single star), or
- (b) A single star (at most one of these), or
- (c) A class of cardinality less than or equal to 22 and not of the forms described in (a) or (b).

We may further assume that none of the blocks lying in any of the classes of type (c) contains a star centre from any of the classes of types (a) or (b). (If it did then it could be transferred to an appropriate star. This might create a new class of type (a) or (b) and the process might have to be repeated; however it will eventually terminate.)

In the revised colouring obtained by the process described above, let t denote the number of colour classes of type (a) and let s be the number of star centres from classes of types (a) and (b), so that either $s = 2t$ or $s = 2t + 1$. Each such star centre is incident with $(v - 1)/2$ blocks, none of which lies in the category (c) classes. If a blocks contain two star centres and b blocks contain three star centres then

$$a + 3b = \frac{s(s - 1)}{2}.$$

Hence the total number of blocks in the classes of types (a) and (b) is

$$\frac{s(v - 1)}{2} - a - 2b = \frac{s(v - 1)}{2} - \frac{s(s - 1)}{3} - \frac{a}{3}.$$

The total number of blocks, say l , in the classes of type (c) is therefore

$$\begin{aligned}
 l &= \frac{v(v-1)}{6} - \frac{s(v-1)}{2} + \frac{s(s-1)}{3} + \frac{a}{3} \\
 &= \frac{(v-s)(v-1-2s)}{6} + \frac{a}{3} \\
 &\geq \frac{(v-s)(v-1-2s)}{6}.
 \end{aligned} \tag{1}$$

The argument now splits into two cases.

Case 1. If the colouring has $k+1$ classes then, necessarily, $s = 2t+1$ and the number of type (c) colour classes is $k+1 - (t+1) = k-t$. Also, there is at least one such class and so $k-t \geq 1$. The total number of blocks contained in the type (c) classes is at most $22(k-t)$. Therefore

$$22(k-t) \geq \frac{(v-1-2t)(v-3-4t)}{6}$$

It follows that

$$k \geq \frac{(v-1)(v-3) + (142-6v)t + 8t^2}{132} = \frac{g(t)}{132}, \text{ say.}$$

But $g'(x) = 142 - 6v + 16x \leq 122 - 2v$ if $x \leq (v-5)/4$. Thus, for $v \geq 61$, the function g is strictly decreasing on the interval $(-\infty, (v-5)/4]$.

However, if $v \equiv 3$ or $7 \pmod{12}$, then $t \leq k-1 \leq (v-7)/4$. Therefore $g(t) \geq g((v-7)/4)$. This reduces to $g(t) \geq 35v - 221$. Hence $k \geq (35v - 221)/132$. But $k \leq (v-3)/4$, and so

$$\frac{v-3}{4} \geq \frac{35v-221}{132}$$

This gives $v \leq 61$, which contradicts the initial assumption that $v \geq 63$.

Similarly, if $v \equiv 1$ or $9 \pmod{12}$, then $t \leq k-1 \leq (v-5)/4$. Therefore $g(t) \geq g((v-5)/4)$. This reduces to $g(t) \geq 34v - 162$. Hence $k \geq (34v - 162)/132$. But $k \leq (v-1)/4$, and so

$$\frac{v-1}{4} \geq \frac{34v-162}{132}$$

This gives $v \leq 129$, which contradicts the initial assumption that $v \geq 133$.

It follows that, for all possible residue classes for v , Case 1 cannot apply.

Case 2. If the colouring has k classes then either there are no classes of type (c) (which is what we are trying to prove) or there is at least one. So suppose that there is at least one class of type (c) and then either

(i) $s = 2t$ and $k \geq t+1$, or

(ii) $s = 2t+1$ and $k \geq t+2$.

In either case $s \leq 2t + 1$ and $t \leq k - 1 \leq (v - 5)/4$, and so $s \leq (v - 3)/2$. It follows that $v - 3 - 2s \geq 0$ and hence that $v - 3 - 4t > 0$. Consequently

$$(v - s)(v - 1 - 2s) \geq (v - 1 - 2t)(v - 3 - 4t).$$

The total number of blocks contained in the classes of type (c) is at most $22(k - t)$ and so

$$22(k - t) \geq \frac{(v - 1 - 2t)(v - 3 - 4t)}{6}.$$

As before, this gives a contradiction. Hence our colouring in k colours can contain no colour classes of type (c).

From the conclusions of Cases 1 and 2 it follows that our revised colouring has at most k colour classes and that every colour class is either a star or the union of two stars. By splitting those classes which are the union of two stars into the two constituent stars, we obtain a partition of the blocks of S into at most $2k$ stars and so $\chi''(S) \leq 2k$. Hence

$$\chi''(S) \leq \begin{cases} (v - 3)/2 & \text{if } v \equiv 3 \text{ or } 7 \pmod{12}, \\ (v - 1)/2 & \text{if } v \equiv 1 \text{ or } 9 \pmod{12}. \end{cases}$$

However, it is shown in [7] that this is impossible for $v \geq 39$. □

Corollary 2.1

$$\chi(B_1, v) = \begin{cases} (v + 1)/4 & \text{if } v \geq 63 \text{ and } v \equiv 3 \text{ or } 7 \pmod{12}, \\ (v + 3)/4 & \text{if } v \geq 133 \text{ and } v \equiv 1 \text{ or } 9 \pmod{12}. \end{cases}$$

□

3 The hut (B_2).

We start with an upper bound for $\chi(B_2, v)$.

Theorem 3.1 *If $v \neq 13$ then $\chi(B_2, v) \leq (v - 1)/2$.*

Proof A colouring of lines which avoids monochromatic intersecting line pairs will provide a B_2 -free colouring of an STS(v). Thus $\chi(B_2, v) \leq \chi'(v)$. This deals with the cases $v \equiv 3$ and $9 \pmod{12}$. A colouring of lines which avoids monochromatic parallel line pairs also provides a B_2 -free colouring of an STS(v). Thus $\chi(B_2, v) \leq \chi''(v)$. This deals with the case $v \equiv 7 \pmod{12}$. If $v \equiv 1 \pmod{12}$ and $v \neq 13$ then there exists a nearly-Kirkman triple system of order $v - 1$, i.e. a resolvable 3-GDD of type $2^{(v-1)/2}$ [21]. By adjoining a single additional point, say ∞ , to each group and taking these extended groups together with the existing blocks we obtain an STS(v) whose blocks may be partitioned into a single star (through ∞) and $(v - 3)/2$ partial

parallel classes. This partition provides a B_2 -free colouring of the system in $(v-1)/2$ colours. $\square$

We shall now prove that, for sufficiently large v , the bound given by the Theorem above cannot be improved. As a first step we prove the following Lemma. In essence, and for large v , this enables us to deal only with colourings whose classes are stars or partial parallel classes.

Lemma 3.1 *Suppose that S is a set of lines of an STS(v), not containing a B_2 (hut) configuration. Then either S is a partial parallel class, or a star, or S contains at most 12 lines.*

Proof We start by assuming that S is not a partial parallel class or a star. Consider the points lying at the intersections of the lines of S . From amongst these points, select one of maximum degree, δ ($\delta \geq 2$).

If $\delta \geq 5$ then take an intersection point, say a , of maximum degree. At least five lines of S pass through a . Take a line of S not passing through a ; this line can intersect at most three of these five lines and we therefore have a B_2 -configuration in S . This is a contradiction and we therefore conclude that $\delta \leq 4$.

If $\delta = 2$ then consider two of the intersecting lines of S , say $\{a, b, c\}$ and $\{a, d, e\}$. Each additional line of S must intersect one of these two lines. However, since $\delta = 2$, we can have at most one further line of S through each of the points b, c, d, e . Thus $|S| \leq 6$.

If $\delta = 3$ then consider three of the lines of S through a common point of intersection, say $\{a, b, c\}$, $\{a, d, e\}$ and $\{a, f, g\}$. Each additional line of S must intersect two of these three lines. Since $\delta = 3$, we can have at most six further lines in S , giving $|S| \leq 9$.

If $\delta = 4$ then consider four lines of S through a common point of intersection, say $\{a, b, c\}$, $\{a, d, e\}$, $\{a, f, g\}$ and $\{a, h, i\}$. In order to avoid a B_2 -configuration, each additional line of S must intersect three of these four lines. Hence the additional lines of S induce a partial Steiner triple system of order 8 on $\{b, c, d, e, f, g, h, i\}$. It is well-known that such a system can contain at most eight lines [6]. It follows that $|S| \leq 12$. $\square$

We note here that an STS(9) is a 12-line B_2 -free configuration which is neither a star nor a partial parallel class (and so $\chi(B_2, 9) = \bar{\chi}(B_2, 9) = 1$).

Theorem 3.2 *If S is any STS(v) with $v \geq 49$ then $\chi(B_2, S) \geq (v-1)/2$.*

Proof Suppose that we have a B_2 -free colouring of S in k colours in which there are s colour classes which are stars, p colour classes which are partial parallel classes, and q other colour classes having neither of the preceding forms and (in consequence of the previous Lemma) having at most 12 lines. We may assume that none of the classes of the latter two types contains a block incident with any of the star-centres. (If there were such a block then it could be transferred to an appropriate star. This might create a new star or partial parallel class and the process might have to be repeated; however it will eventually terminate.)

We have $s + p + q = k$ and, as before, the number of blocks in the non-star colour classes is

$$\frac{(v-s)(v-1-2s)}{6} + \frac{a}{3},$$

where a is the number of blocks of S containing two star-centres. The maximum possible number of lines in one of the ppc colour classes is $\lfloor (v-s)/3 \rfloor$. It follows that

$$p\left(\frac{v-s}{3}\right) + 12q \geq \frac{(v-s)(v-1-2s)}{6} + \frac{a}{3}.$$

Hence

$$\begin{aligned} 72q &\geq (v-s)(v-1-2s) - 2p(v-s) + 2a \\ &= (v-s)(v-1-2k+2q) + 2a \end{aligned} \quad (2)$$

If $k < (v-1)/2$ then $(v-1-2k) \geq 2$. Noting $s \leq k - q$, equation (2) gives $72q \geq (v - (v-3)/2 + q)(2 + 2q) = (v + 3 + 2q)(1 + q)$. If $v > 47$ this gives $72q > (50 + 2q)(1 + q)$, which reduces to $(q-5)^2 < 0$. Plainly this is impossible, and so for $v \geq 49$ we must have $k \geq (v-1)/2$. $\square$

Corollary 3.1 *If $v \geq 49$ then $\underline{\chi}(B_2, v) = (v-1)/2$.* $\square$

Let us remark that if $k = (v-1)/2$ then for $v \geq 73$, the inequalities obtained in the proof of the Theorem give $q = 0$, $a = 0$, and either $3|(v-s)$ or $p = 0$. Also if $s > 0$ then $a = 0$ implies that the star centres form an STS(s). Observe also that it is possible to have a B_2 -free colouring of an STS(v) with $(v-1)/2$ colour classes in which some colour classes are stars and some are partial parallel classes: take three copies of an STS($2u+1$) intersecting in a common STS(u), to form an STS($4u+3$) with the cross-system blocks from a resolvable TD($3, u$) (such a TD exists for suitably large u). We get $u+1$ classes of parallel blocks from the TD, plus u stars through the points of the STS(u). If $v = 4u+3$ then the number of classes is $2u+1 = (v-1)/2$.

4 The 3-star (B_3).

We start with an upper bound for $\underline{\chi}(B_3, v)$.

Theorem 4.1 *If $v \neq 7$ then*

$$\underline{\chi}(B_3, v) \leq \begin{cases} (v-1)/4 & \text{if } v \equiv 9 \pmod{12}, \\ (v+1)/4 & \text{if } v \equiv 3 \text{ or } 7 \pmod{12}, \\ (v+3)/4 & \text{if } v \equiv 1 \pmod{12}. \end{cases}$$

Proof For $v \equiv 3 \pmod{6}$ take a Kirkman triple system of order v . This is resolvable into $(v-1)/2$ parallel classes. By combining these classes in pairs (with one class left unpaired if $v \equiv 3 \pmod{12}$) we obtain a B_3 -free colouring of the system in $(v-1)/4$ colours for $v \equiv 9 \pmod{12}$ and $(v+1)/4$ colours for $v \equiv 3 \pmod{12}$.

For $v \equiv 1 \pmod{6}$ take a Hanani triple system of order v ; such a system exists for $v \geq 19$ and is resolvable into $(v+1)/2$ partial parallel classes. By combining these classes in pairs (with one class left unpaired if $v \equiv 1 \pmod{12}$) we obtain a B_3 -free colouring of the system in $(v+1)/4$ colours for $v \equiv 7 \pmod{12}$ and $(v+3)/4$ colours for $v \equiv 1 \pmod{12}$, $v \neq 7$ or 13 . Finally, we observe that the chromatic index of both STS(13)s is 8 [6], and thus in this case as well we can combine partial parallel classes in pairs to obtain $\underline{\chi}(B_3, 13) \leq 4$. $\square$

Our next result shows that the upper bounds given by the previous Theorem cannot be improved.

Theorem 4.2

$$\underline{\chi}(B_3, v) \geq \begin{cases} (v-1)/4 & \text{if } v \equiv 9 \pmod{12}, \\ (v+1)/4 & \text{if } v \equiv 3 \text{ or } 7 \pmod{12}, \\ (v+3)/4 & \text{if } v \equiv 1 \pmod{12}. \end{cases}$$

Proof Any B_3 -free configuration of lines of an STS(v) cannot have a point of degree greater than 2. Thus the largest possible configuration of this type has at most $\lfloor 2v/3 \rfloor$ lines. It follows that

$$\underline{\chi}(B_3, v) \geq \left\lceil \frac{v(v-1)}{6} \right\rceil \left\lfloor \frac{2v}{3} \right\rfloor.$$

Examining the residue classes for v modulo 12 we obtain the desired result. $\square$

Corollary 4.1 *If $v \neq 7$ then*

$$\underline{\chi}(B_3, v) = \begin{cases} (v-1)/4 & \text{if } v \equiv 9 \pmod{12}, \\ (v+1)/4 & \text{if } v \equiv 3 \text{ or } 7 \pmod{12}, \\ (v+3)/4 & \text{if } v \equiv 1 \pmod{12}. \end{cases}$$

$\square$

The value omitted in the Corollary above is easily dealt with. If F denotes the unique STS(7) then $\chi(B_3, F) > 2$ because the complement of a Pasch configuration in F is a 3-star. On the other hand it is trivial to obtain a B_3 -free colouring of F in three colours. Therefore $\chi(B_3, F) = 3$.

5 The 3-path (B_4).

Our results concerning the 3-path rely on the following Lemma.

Lemma 5.1 *If C is a set of lines of an STS(v) and $|C| \geq v+1$ then C contains a B_4 configuration.*

Proof Suppose that C contains at least $v+1$ lines of an STS(v) but does not contain a B_4 configuration. If the lines of C span p points then the average degree (in C) of

these points is at least $(3v+3)/p \geq (3v+3)/v > 3$. Thus there is a point of degree at least four. Now take a point of maximum degree $m \geq 4$, and consider all the lines of C which are incident with it. If any other point on these lines had degree (in C) greater than one then C would contain a B_4 configuration. Hence all the other points on these m lines have degree one and the m lines form an m -star component of C . Now delete this m -star from C to form C' .

We have $|C'| \geq v+1-m$ and C' spans $p-(2m+1)$ points. Note that $p-(2m+1) \neq 0$ because if $m = (p-1)/2$ then $|C'| > 0$ and clearly C' cannot have lines without spanning points. The average degree (in C') of these points is at least $3(v+1-m)/(p-2m-1) > 3$. Thus there is a point of degree at least four. We may now iterate the earlier process; at each stage we remove more points from C than lines, eventually arriving at the contradiction of a non-empty set of lines C^* spanning no points. The result follows. $\square$

Theorem 5.1 $\chi(B_4, v) \geq (v-1)/6$.

Proof From the previous lemma we have that the maximum cardinality of a colour class in a B_4 -free colouring of an STS(v) is v . $\square$

We shall now prove that the bound given in the Theorem above is attained for an infinite number of values v .

Theorem 5.2 *If there exists a resolvable Steiner system $S(2, 7, v)$ then $\chi(B_4, v) = (v-1)/6$. In particular, $\chi(B_4, 7^n) = (7^n-1)/6$ for all $n \geq 1$.*

Proof Replace each 7-block of the resolvable $S(2, 7, v)$ by an STS(7) to form an STS(v). Each original parallel class now gives $v/7$ disjoint STS(7)s which we may take as a B_4 -free colour class. Therefore there are $(v-1)/6$ colour classes. This, together with Theorem 5.1 and the fact that an $S(2, 7, 7^n)$ (an affine geometry $AG(n, 7)$) exists for all $n \geq 1$, (see, e.g., [2]) implies the result. $\square$

Although we are unable to prove that the bound discussed in the preceding Theorems is attained for all (or almost all) admissible v , we can establish an asymptotic result (Theorem 5.3). Note also that the method of Theorem 3.1 gives an upper bound for $\chi(B_4, v)$ of $(v-1)/2$.

Theorem 5.3 *As $v \rightarrow \infty$, $\chi(B_4, v) = (v-1)/6 + o(v)$.*

Proof There exists v_0 such that for all $v > v_0$, the number of MOLS of side v , say $N(v)$, satisfies $N(v) \geq v^{\frac{1}{14.8}}$ [1]. Hence, for $v > v_0$ and $m \leq v^{\frac{1}{14.8}}$, there exists a resolvable transversal design TD(m, v) [1]. We will assume that v_0 is so large that

$$14.8 \frac{\log(v_0 + 4)}{\log v_0} < 15.$$

Take $u \geq \max\{(v_0+4)^{\frac{16}{15}}, 7^{16}\}$ and admissible. Define $k = \lfloor \log_7 u/16 \rfloor$ so that $k \geq 1$, $k \leq \log_7 u/16 < k+1$, and $7^{16k} \leq u < 7^{16(k+1)}$. We may write $u = \sum_{i=0}^n u_i 7^i$, where $0 \leq u_i < 7$ and $u_n \neq 0$.

Next choose $\alpha \in \{0, 1, 2, 3\}$ so that $w = \alpha 7^k + u_{k-1} 7^{k-1} + u_{k-2} 7^{k-2} + \dots + u_0$ is admissible. Put $v = u_n 7^{n-k} + u_{n-1} 7^{n-k-1} + \dots + u_k - \alpha$ so that $u = 7^k v + w$. Note that

$$v + w \equiv (u_n + u_{n-1} + \dots + u_k - \alpha) + (\alpha + u_{k-1} + u_{k-2} + \dots + u_0) \equiv u \pmod{6},$$

so that $v + w$ is admissible.

By our choice of α , we have $0 \leq w < 4 \cdot 7^k$. Hence $(v + 4) 7^k > 7^k v + w = u$ and so $v + 4 > u 7^{-k} \geq u^{\frac{15}{16}} \geq v_0 + 4$, giving $v > v_0$. Also, $7^k \leq u^{\frac{1}{16}} < (7^k(v + 4))^{\frac{1}{16}}$ and so $7^k < (v + 4)^{\frac{1}{15}}$. But $v > v_0$ and so $(v + 4)^{\frac{1}{15}} < v^{\frac{1}{14.8}}$, giving $7^k < v^{\frac{1}{14.8}}$. It follows that there is a resolvable $\text{TD}(7^k, v)$.

Since $v + 4 > u^{\frac{15}{16}}$, $w < 4 \cdot 7^k \leq 4 \cdot u^{\frac{1}{16}}$ and $u \geq 7^{16}$, we have $v > u^{\frac{15}{16}} - 4 > 4u^{\frac{1}{16}} + 1 > w + 1$. Consequently by [10] there exists an $\text{STS}(v + w)$ containing an $\text{STS}(w)$ subsystem. We now take 7^k copies of this $\text{STS}(v + w)$ intersecting in a common $\text{STS}(w)$ subsystem; we may take the points of the i^{th} copy to be

$$1, 2, \dots, w, 1_i, 2_i, \dots, v_i.$$

Altogether there are $7^k v + w = u$ points and we may form an $\text{STS}(u)$ on these points by taking as blocks all the blocks of all the $\text{STS}(v + w)$ s (the *horizontal* blocks) together with certain other blocks which we describe below (the *vertical* blocks). The horizontal blocks cover all pairs of the forms $\{a, b\}, \{a, c_i\}, \{c_i, d_i\}$ for $a, b = 1, 2, \dots, w$, $c, d = 1, 2, \dots, v$ and $i = 1, 2, \dots, 7^k$. The vertical blocks must cover every pair of the form $\{c_i, d_j\}$ for $c, d = 1, 2, \dots, v$, $i, j = 1, 2, \dots, 7^k$ and $i \neq j$. To form the vertical blocks we take a resolvable $\text{TD}(7^k, v)$ with groups $\{1_i, 2_i, \dots, v_i\}$ for $i = 1, 2, \dots, 7^k$. We then replace each block of size 7^k with an $\text{STS}(7^k)$ (on the same points) having $(7^k - 1)/6$ B_4 -free colour classes (see the previous Theorem).

An original parallel class of the $\text{TD}(7^k, v)$ will contain v 7^k -blocks and will give rise to v copies of each of the $(7^k - 1)/6$ colour classes. The v copies of each such class are pointwise disjoint and may therefore be combined to form a single class of 3-blocks not containing any B_4 configurations. Since there are v parallel classes in the $\text{TD}(7^k, v)$ we obtain a partition of the vertical blocks into $(7^k - 1)v/6$ B_4 -free colour classes.

It remains to deal with the horizontal blocks. We do this using stars, some of which may be empty and, therefore, redundant. We firstly take w stars on the points $1, 2, \dots, w$; these give w B_4 -free colour classes and cover all blocks of the form $\{a, b, c\}$ for $a, b, c = 1, 2, \dots, w$, as well as certain other horizontal blocks. (A block $\{a, b, c\}$ may be assigned to any one of the three stars centred on a, b or c .) However blocks of the form $\{d_i, e_i, f_i\}$ (if any) remain uncovered. To deal with these we take each point d_i for $i = 1, 2, \dots, 7^k$ and $d = 1, 2, \dots, v$, and we form a star on each of these points using the blocks $\{d_i, e_i, f_i\}$. Such a block may be assigned to any one of the three stars centred on d_i, e_i or f_i . For each $d = 1, 2, \dots, v$, we may combine the stars centred on the points d_i for $i = 1, 2, \dots, 7^k$ into a single B_4 -free colour class. A total of v classes therefore suffices to deal with the remaining horizontal blocks.

The total number (say X) of B_4 -free colour classes in our partition of the blocks

of the STS(u) satisfies

$$X \leq \left(\frac{7^k - 1}{6} \right) v + w + v = \frac{u - 1}{6} + \frac{5v + 5w + 1}{6}.$$

But $v > w$ and $v < u \cdot 7^{-k} = 7u \cdot 7^{-(k+1)} \leq 7u^{\frac{15}{16}}$. Hence $v + w = o(u)$ as $u \rightarrow \infty$. It follows that

$$\underline{\chi}(B_4, u) \leq \frac{u - 1}{6} + o(u) \text{ as } u \rightarrow \infty$$

□

6 The chromatic indices for $v = 7, 9$ and 13 .

We give below the values of $\chi(B_i, S)$ for $i = 1, 2, 3, 4$ when S is, in turn, the STS(7) (S_7), the STS(9) (S_9), and each of the two non-isomorphic STS(13)s. Here we denote by C the STS(13) which has a cyclic automorphism group of order 13. We use the realisation of C which is generated by the blocks $\{0, 1, 4\}$ and $\{0, 2, 7\}$ under the action of the mapping $i \rightarrow i + 1 \pmod{13}$. We denote by N the STS(13) which is non-cyclic. A realisation of this system may be obtained by taking C and replacing any Pasch configuration with the opposite Pasch configuration, i.e. replacing four blocks of C which have the structure $\{a, b, c\}, \{a, y, z\}, \{x, b, z\}, \{x, y, c\}$ with the four blocks $\{x, y, z\}, \{x, b, c\}, \{a, y, c\}, \{a, b, z\}$.

Case 1, the 3-ppc.

The STS(7) has no parallel line pairs, so $\chi(B_1, S_7) = 1$. The STS(9) does have B_1 -configurations, so $\chi(B_1, S_9) > 1$. On the other hand, S_9 has a 2-colouring with one class comprising any 4-star and the other class comprising the remaining blocks. Therefore $\chi(B_1, S_9) = 2$. As shown in [7], $\chi''(C) = \chi''(N) = 6$. By combining the corresponding colour classes in pairs we have $\chi(B_1, C) \leq 3$ and $\chi(B_1, N) \leq 3$. A computer search has shown that the largest 3-ppc-free subsets of lines from C and from N have cardinality 12. Thus neither C nor N has a 2-colouring. Consequently $\chi(B_1, C) = \chi(B_1, N) = 3$.

Case 2, the hut.

The STS(7) has no parallel line pairs, so $\chi(B_2, S_7) = 1$. For the STS(9), given any two parallel lines, any other line is either parallel to both or intersects both. Thus S_9 contains no B_2 -configurations and so $\chi(B_2, S_9) = 1$. A 4-colouring of C is provided by the partition:

Class 1	Class 2	Class 3	Class 4
{0, 1, 4}	{2, 3, 6}	{3, 4, 7}	{4, 5, 8}
{1, 2, 5}	{6, 7, 10}*	{8, 9, 12}	{12, 0, 3}
{5, 6, 9}	{7, 8, 11}	{10, 11, 1}	{3, 5, 10}
{11, 12, 2}	{9, 10, 0}	{1, 3, 8}	{5, 7, 12}
{2, 4, 9}	{0, 2, 7}*	{7, 9, 1}	{10, 12, 4}
{4, 6, 11}	{6, 8, 0}*	{9, 11, 3}	
{11, 0, 5}	{8, 10, 2}*		
{12, 1, 6}			

If the Pasch configuration indicated by the asterisks above is replaced by the opposite Pasch configuration then we obtain a 4-colouring of N . A computer search has shown that the largest hut-free subsets of lines from C and from N have cardinality 8. Thus neither C nor N has a 3-colouring. Consequently $\chi(B_2, C) = \chi(B_2, N) = 4$.

Case 3, the 3-star.

As remarked earlier, $\chi(B_3, S_7) = 3$. From Corollary 4.1, $\chi(B_3, S_9) = 2$. From Theorem 4.2, $\chi(B_3, C) \geq 4$ and $\chi(B_3, N) \geq 4$, whilst from the comment in the proof of Theorem 4.1, $\chi(B_3, C) \leq 4$ and $\chi(B_3, N) \leq 4$. Thus $\chi(B_3, C) = \chi(B_3, N) = 4$.

Case 4, the 3-path.

The STS(7) has no parallel line pairs, so $\chi(B_4, S_7) = 1$. For the STS(9), a colouring in three classes is obtained by taking one class to be any 4-star and then partitioning the remaining blocks into two sails. Thus $\chi(B_4, S_9) \leq 3$. On the other hand no colour class for S_9 can contain more than four lines as this would require at least two lines from one parallel class together with at least one line from a different parallel class, thereby forming a B_4 -configuration. Consequently S_9 has no 2-colouring and so $\chi(B_4, S_9) = 3$. From the comment following Theorem 5.2 we have $\chi(B_4, C) \leq \chi''(C) = 6$ and $\chi(B_4, N) \leq \chi''(N) = 6$. A computer analysis has shown that the largest 3-path-free subsets of lines from C and from N are 6-stars. Since any pair of 6-stars from C (or from N) intersect in a common line, at most one 6-star can be employed in any colouring of C (or N). Thus, to provide a 5-colouring of C or N , the colour class sizes must be 6, 5, 5, 5, 5. We consider the two cases of C and N separately.

(i) $\chi(B_4, C)$.

Further computer analysis shows that there are 572 possible colour classes of cardinality 5 or 6, including the thirteen 6-stars. A computer search shows that no five of these partition the blocks of C . Thus $\chi(B_4, C) = 6$.

(ii) $\chi(B_4, N)$.

Here the computer analysis shows that there are 607 possible colour classes of cardinality 5 or 6, including the thirteen 6-stars. A computer search gives precisely six collections of five classes which partition the blocks of N . An example of such a partition is given below. Here the system N is realised by switching the Pasch configuration from C which has the blocks {10, 11, 1}, {12, 1, 6}, {4, 6, 11} and {10, 12, 4}.

Class 1	Class 2	Class 3	Class 4	Class 5
{1, 2, 5}	{0, 1, 4}	{4, 5, 8}	{6, 7, 10}	{7, 8, 11}
{2, 3, 6}	{3, 4, 7}	{6, 8, 0}	{1, 3, 8}	{9, 10, 0}
{11, 12, 2}	{5, 6, 9}	{7, 9, 1}	{4, 6, 12}	{12, 0, 3}
{0, 2, 7}	{8, 9, 12}	{10, 12, 1}	{5, 7, 12}	{3, 5, 10}
{2, 4, 9}	{10, 11, 4}	{11, 0, 5}	{9, 11, 3}	{11, 1, 6}
{8, 10, 2}				

It follows that $\chi(B_4, N) \leq 5$. Since there are no 3-path-free subsets of N having cardinality greater than 6, we must have $\chi(B_4, N) = 5$.

The results of this section are summarised in the table below.

$B \setminus S$	S_7	S_9	C	N
B_1	1	2	3	3
B_2	1	1	4	4
B_3	3	2	4	4
B_4	1	3	6	5

Table 1: Three-line chromatic indices for $v = 7, 9$ and 13.

7 Concluding remarks.

Some of our results concerning the 3-ppc and 3-star may be extended to the n -ppc (P_n) and n -star (S_n). Taking the result of [7] given in the proof of Theorem 2.1 and combining 2-ppc-free classes together in groups of up to $(n-1)$ gives an upper bound for $\chi(P_n, v)$ of order $v/2n$. Similarly, combining up to $(n-1)$ (partial) parallel classes of a Kirkman triple system (or Hanani triple system) in the manner described for $n = 3$ in Theorem 4.1 gives an upper bound for $\chi(S_n, v)$, also of order $v/2n$.

Obtaining good estimates for the values of $\bar{\chi}(B_i, v)$ seems much more difficult than the corresponding problem for $\chi(B_i, v)$. The results of Phelps and Rödl [18], and of Pippenger and Spencer [19] cited in our Introduction may be used, but the bounds obtained in this way do not seem at all tight.

Finally, we note that for each of the four three-line configurations B_i considered in this paper, the lower bound $\chi(B_i, v)$ is of order $c_i v$ for an appropriate constant c_i . This is not the case for the fifth configuration B_5 (the triangle), where a much lower growth rate pertains. We hope to deal with this remaining case in a future paper.

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