

On the normality of Cayley digraphs of valency 2 on nonabelian groups of odd square free order*

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Abstract

In this paper, we prove that all Cayley digraphs of valency 2 on non-abelian groups of odd square-free order are normal.

For a given subset S of a finite group G without the identity element 1, the Cayley *digraph* on G with respect to S is denoted by $\Gamma = \text{Cay}(G, S)$ where $V(\Gamma) = G$, $E(\Gamma) = \{(g, sg) \mid g \in G, s \in S\}$. It is clear that $\text{Aut}(\Gamma)$, the automorphism group of Γ , contains the right regular representation G_R of G as a subgroup. Moreover Γ is connected if and only if $G = \langle S \rangle$, and Γ is undirected if and only if $S^{-1} = S$.

Γ is called normal if G_R is a normal subgroup of $\text{Aut}(\Gamma)$. The concept of normality for Cayley digraphs is known to be important in the study of arc-transitive digraphs and half-transitive graphs. A natural problem is, for a given finite group G , to determine all normal or nonnormal Cayley digraphs of G . However this is a very difficult problem. The groups for which complete information about the normality of Cayley digraphs is available are cyclic groups of prime order (see [1]) and groups of order $2p$ (see [3]). Wang, Wang and Xu [9] determined all disconnected normal Cayley digraphs. Therefore we always suppose, in this paper, that the Cayley digraph $\text{Cay}(G, S)$ is connected, that is, S is a generating subset of G . Xu [11, Problem 6] asked the following question: when S is a minimal generating set of G , are the corresponding Cayley digraph and graph normal? For abelian groups, Feng and Gao [5] proved that if the Sylow 2-subgroups of G are cyclic then the answers to the question are positive, and otherwise negative in general.

About nonabelian groups, Feng and Xu [6] proved that there are only two non-normal connected Cayley digraphs of valency 2 on nonabelian groups of order p^3 and p^4 . This also implies that there are few nonnormal connected Cayley digraphs. Feng [4] determined all nonnormal Cayley digraphs of valency 2 on nonabelian groups of order $2p^2$. Wang and Li [10] also proved that the Cayley graphs of nonabelian groups

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of order $2pq$ and of degree 2 are normal. In this paper we discuss the normality of connected Cayley digraphs of valency 2 on nonabelian groups of odd square-free order. Our result is the following:

Main Theorem *Let G be a nonabelian group of odd square-free order and let $|S| = 2$. Then $\Gamma = \text{Cay}(G, S)$ is normal.*

To prove our result, we need the following lemmas:

Lemma 1 ([11, Prop. 1.5]) *Let $A = \text{Aut}(\Gamma)$ be the automorphism group of the Cayley digraph Γ of a group G with respect to its generating subset S and let A_1 be the stabilizer subgroup of A fixing the identity element 1 of G . Then Γ is normal if and only if A_1 is contained in the automorphism group $\text{Aut}(G)$ of G .*

Lemma 2 ([4]) *Let $S = \{e, f\}$ be a two-generating subset of G without the identity 1 and let A_1^* be the subgroup of A which fixes the elements 1, e and f of G . Then Γ is normal if and only if $A_1^* = 1$.*

In this paper, we mainly discuss a normal subgroup A of the automorphism group of the Cayley digraph $\Gamma = \text{Cay}(G, S)$ of valency 2 to determine whether Γ is normal. It is clear that $|A : G|$ is a power of 2. To prove our theorem, we can assume that $\text{Cay}(G, S)$ is not normal, where G is the smallest counterexample of odd square-free order. Let N be a smallest normal subgroup of A . Then $N = T_1 \times T_2 \times \cdots \times T_k$ where T_i is isomorphic to Z_p or a simple group. Since G is of odd square-free order, $k = 1$. When N is simple, since G is a Hall odd-subgroup of A , $N \cap G$ is also a Hall odd-subgroup of N . Hence, by Corollary 5.6 of [2], $N \cong \text{PSL}(2, p)$ where p is a Mersenne prime. Moreover, by Theorem II.8.27 of [7], G is the semidirect product of Z_p by $Z_{(p-1)/2}$.

Now, we deal with the case when N is transitive on the set $V(\Gamma)$ of the digraph Γ .

Let (u, v) be a directed arc of Γ (the direction is from u to v). Then u and v are the tail and head of (u, v) respectively. If Γ has a circuit such that for every vertex u on this circuit, u is the tail of two incident arcs of the circuit or the head of two incident arcs, then the circuit is called an alternating circuit of Γ . Furthermore, if u is the tail of two incident arcs, then there exists at most one alternating circuit containing these two incident arcs; in which case we denote the circuit by $O(u)$. Similarly if u is the head of two incident arcs of an alternating circuit we denote the circuit by $I(u)$.

Claim 3 In Γ , an alternating circuit must be an alternating cycle.

Proof. When an alternating circuit A' of Γ is not an alternating cycle, there exist vertices which appear at least two times in A' . Since Γ is vertex-transitive and of valency 2, each vertex of A' must appear two times in A' . Hence, vertices not in A' are not adjacent to the vertices of A' . However, Γ is connected. Thus, all vertices appear in A' . Hence, the subgroup A_1^* , fixing A' pointwise, must fix all vertices of Γ . In other words, $A_1^* = 1$. By Lemma 2, Γ is normal. This is impossible.

Now, we consider the alternating cycle construction of Γ . Since A is transitive, the length of the alternating cycles is a constant $2m$ where m is the number of vertices of valency 2 in an alternating cycle. Since A_1^* fixes the alternating cycle $O(1)$ pointwise, it must fix the set $I((e^{-1}f)^i)$ for $0 \leq i < m$ (see Figure 1 for m

odd). If $|O(1) \cap I((e^{-1}f)^i)| > 2$ for some i , A_1^* fixes all vertices in $I((e^{-1}f)^i)$. Since Γ is transitive and connected, A_1^* fixes all alternating cycles and all vertices. Hence, $A_1^* = 1$, which is impossible. Similarly, $|O(1) \cap O(f(e^{-1}f)^i)| \leq 2$. Since Γ is transitive, $|O(g) \cap O(h)| \leq 2$, where $O(g)$ and $O(h)$ are distinct alternating cycles. Let k be the number of alternating cycles. Then, $km = |G|$ by calculating the number of vertices of valency 2 in the alternating circuits. If $m \geq k$, then there are i, j with $i \neq j$ such that $I((e^{-1}f)^i) = I((e^{-1}f)^j)$. Moreover, there is a vertex $f((e^{-1}f)^l)$ or $(e^{-1}f)^l(l \neq i, j)$ contained in $O(1) \cap I((e^{-1}f)^i)$, which is impossible. Hence, $m < k$.

We define a new digraph $A(\Gamma)$ as follows (see Figure 1 for m odd): $V(A(\Gamma))$ is the set of different alternating cycles; for $O(g), O(h) \in V(A(\Gamma))$, $(O(g), O(h)) \in E(A(\Gamma))$ if and only if $O(g) \cap O(h)$ contains vertices which are of valency 2 in $O(h)$.

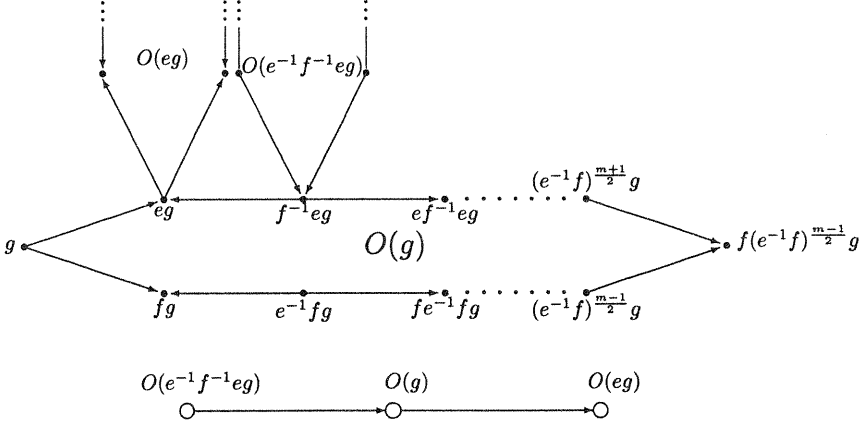


Figure 1

It is clear that there are no loops in $A(\Gamma)$, and that $A(\Gamma)$ is of order k and of out-degree m or $m/2$. Further, we have the following:

Lemma 4 Two alternating cycles $O(g)$ and $O(h)$ of Γ have at most two common vertices. If $O(g)$ and $O(h)$ have a common vertex, or have two common vertices that have different valencies in the same alternating cycle, then $A \leq \text{Aut}(A(\Gamma))$ and $\text{Aut}(A(\Gamma))$ has a regular arc-transitive subgroup isomorphic to G .

Proof. The first conclusion comes from the previous discussion. Since $A = \text{Aut}(\Gamma)$ preserves the alternating cycle construction of Γ , there is a homomorphism from A to $\text{Aut}(A(\Gamma))$ such that the image of A permutes the vertices of $A(\Gamma)$ (that are the alternating cycles of Γ). Let K be the kernel of this homomorphism. When two alternating cycles have only one common vertex or have two common vertices that have different valencies in the same alternating cycle, since K fixes all alternating cycles, K must fix all vertices in Γ . Hence, $K = 1$. So, $A \leq \text{Aut}(A(\Gamma))$. Moreover, as a subgroup of A , G permutes transitively the arcs of the digraph $A(\Gamma)$. It is clear that the action of G on $A(\Gamma)$ is regular arc-transitive.

By the above lemma, we know that N is isomorphic to a subgroup of $\text{Aut}(A(\Gamma))$. When N is transitive on $V(\Gamma)$, it is also transitive on $V(A(\Gamma))$. Hence, the order of

its stabilizer subgroup is $(p+1)p(p-1)/(2k)$. However, by Theorem II.8.27 of [7], $PSL(2, p)$ has no subgroup of order $(p+1)p(p-1)/(2k)$. Hence N is not transitive on $V(\Gamma)$. We consider the graph Γ_N , where $V(\Gamma_N)$ is the set of all N -orbits on $V(\Gamma)$, and two vertices $U, V \in V\Gamma_N$ are adjacent in Γ_N if and only if there exist $\beta \in U$ and $\alpha \in V$ which are adjacent in Γ . In our case, Γ_N is also a Cayley digraph $\text{Cay}(\overline{G}, \overline{S})$ where $\overline{G} = GN/N$, $\overline{S} = SN/N$. Hence, by Lemma 2.5 of [8], Γ_N is a dicycle. We denote the orbits of N by $\{V_0, V_1, \dots, V_l\}$ where the out-neighbors of vertices in V_i are in V_{i+1} and $l = |G|/|G \cap N|$. Assume that N is isomorphic to $PSL(2, p)$. Let N_α be the stabilizer subgroup of N fixing the vertex $\alpha \in V\Gamma$. Then, by Theorem II.8.27 of [7], N_α is the dihedral group of order $p+1$. Let M be its cyclic subgroup of order $(p+1)/2$. Since M is cyclic, it has an orbit C of order $(p+1)/2$ in some set V_i . Hence, the out-neighbors and in-neighbors of C are of order $p+1$, $(p+1)/2$ or $(p+1)/4$. If its out-neighbors or in-neighbors are of order $(p+1)/4$ or $(p+1)/2$, the length $2|H|$ of the alternating cycle is a divisor of $(p+1)$, where $H = \langle e^{-1}f \rangle$ is a subgroup of G . This is impossible. Hence, the out-neighbors and in-neighbors of C are of order $p+1$ and consist of two orbits of M of order $(p+1)/2$. Thus, $\text{Cay}(G, S)$ is not connected. Hence, $G > N = N \cap G \cong Z_p$. Then, by Lemma 2.5 of [8], Γ_N is a dicycle or G/N -arc transitive of valency 2. If Γ_N is a dicycle, there are $2p$ arcs between V_i and V_{i+1} and the number of arcs also is $2|H|$. Since Γ is connected, A^* fixes all vertices. Thus $\text{Cay}(G, S)$ is normal. So, we assume that Γ_N is G/N -arc transitive of valency 2. Let K be the kernel of A acting on the set $\{V_1, V_2, \dots, V_l\}$ and h an element in K fixing a vertex $\alpha \in V_1$. Then, since K fixes all V_i , h fixes the in-neighbors and out-neighbors of α and so fixes all vertices. Hence, K must be regular and be N . Then, since G is the smallest counterexample, $G/N \triangleleft A/N$. Hence, $G \triangleleft A$ and Γ is normal. The proof of our main Theorem is completed.

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References

- [1] B. Alspach, Point-symmetric graphs of prime order and transitive permutation groups of prime degree, *J. Combin. Theory*, **15**(1973), 12-17.
- [2] Z. Arad, E. Fisman, On finite factorization groups *J. of Algebra* **86**(1984), 522-548.
- [3] S. F. Du, R. J. Wang, M. X. Xu, On the normality Cayley digraphs of groups of twice a prime, Preprint. Peking University, 1997.
- [4] Y. Q. Feng, A family of nonnormal Cayley digraphs. in preparation.

- [5] Y. Q. Feng, T. P. Gao, Automorphism groups and isomorphisms of Cayley digraphs of abelian groups, *Australasian Journal of Combinatorics*, **16**(1997), 183-187.
- [6] Y. Q. Feng, M. Y. Xu, The Normality on a Family of Cayley Digraphs, Preprint. Peking University, 1997.
- [7] B. Huppert, Finite Groups (I), Springer-Verlag, Berlin, 1967.
- [8] C. H. Li, On isomorphism of connected Cayley graphs, III, *Bull of Austral. Math. Soc.*, **58**, (1998), 137-145.
- [9] C. Q. Wang, D. J. Wang, M. Y. Xu, On normal Cayley graphs of finite groups, *Science in China, Ser. A*, Vol. **28(2)**(1998), 131-139.
- [10] P. Wang, J. S. Li, On the normality of Cayley digraphs of degree 2 on nonabelian group of order $2pq$, submitted.
- [11] M. Y. Xu, Automorphism groups and isomorphisms of Cayley digraphs, *Discrete Math.* **182**(1998), 309-319.

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