A recursive theorem on matching extension

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Abstract

A graph G having a perfect matching (or 1-factor) is called n-extendable if every matching of size n is extended to a 1-factor. Further, G is said to be $\langle r : m, n\rangle$-extendable if, for every connected subgraph S of order $2r$ for which $G \setminus V(S)$ is connected, S is m-extendable and $G \setminus V(S)$ is n-extendable. We prove the following: Let p, r, m, and n be positive integers with $p - r > n$ and $r > m$. Then every 2-connected $\langle r : m, n\rangle$-extendable graph of order $2p$ is $\langle r + 1 : m + 1, n - 1\rangle$-extendable.

1. Introduction

We consider only finite simple graphs and follow Bondy and Murty [1] for general terminology and notation. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $A \subset V(G)$, $G[A]$ denotes the subgraph of G induced by A and $G \setminus A$ is the subgraph of G induced by $V(G) \setminus A$. If $G[A]$ is connected, then a subset A is said to be *connected* (if A is a empty set, then it is considered to be connected). Further, we often identify $G[A]$ with A. If H is a subgraph and v is a vertex, we may write $G \setminus H$ or $G \setminus v$ instead of $G \setminus V(H)$ or $G \setminus \{v\}$, respectively. If A and B are disjoint subsets of $V(G)$, then $E(A, B)$ denotes the set of edges with one end in A and the other in B. For $e \in E(G)$, $V(e)$ denotes the set of endvertices of e.

Let $n \geq 0$ and $p > 0$ be integers with $n \leq p - 1$ and G a graph with $2p$ vertices having a 1-factor (a perfect matching). Then G is said to be *n-extendable* if every matching of size n in G can be extended to a 1-factor. In particular, G is 0-extendable if and only if G has a 1-factor. Further, G is said to be *(r, n)-extendable* (resp. *$[r, n]$-extendable*) if $G[S]$ (resp. $G \setminus S$) is n-extendable for every connected subset S of order $2r$. Furthermore, a connected graph G is called *$\langle r, n\rangle$-extendable* if $G[S]$ is n-extendable for every connected subset S of order $2r$ for which $G \setminus S$ is connected.

Theorem A (Nishimura and Saito [5]). Let p, r and n be integers with $p > r > n > 0$. Then every (r, n)-extendable graph of order $2p$ is $(r + 1, n + 1)$-extendable. □

Theorem B([5]). Let p, r and n be integers with $r > 0$ and $p - r > n \geq 0$. Then every connected $[r, n]$-extendable graph of order $2p$ is $[r-1, n]$-extendable. $\square$

Theorem C(Nishimura[4]). Let p, r and n be positive integers with $p > r > n$. Then every 2-connected $\langle r, n\rangle$-extendable graph of order $2p$ is $\langle r+1, n\rangle$-extendable. $\square$

In this paper, we present an extended theorem which is similar to the theorems above. A connected graph G is called $\langle r : m, n\rangle$-*extendable* if, for every connected subset S of order $2r$ for which $G \setminus S$ is connected, $G[S]$ is m-extendable and $G \setminus S$ is n-extendable. From this definition, if G is an $\langle r : m, n\rangle$-extendable graph of order $2p$, then two inequalities $p > r + n$ and $r > m$ are required.

Theorem 1. Let $p, r, m,$ and n be positive integers with $p - r > n$ and $r > m$. Then every 2-connected $\langle r : m, n\rangle$-extendable graph of order $2p$ is $\langle r+1 : m+1, n-1\rangle$-extendable.

Note that if a graph G is $\langle r : m, 0\rangle$-extendable, then G is $\langle r, m\rangle$-extendable. Furthermore, by Theorem C, if an even order graph G is 2-connected $\langle r, m\rangle$-extendable, then G is m-extendable. So, we have the following corollary immediately.

Corollary 2. If a graph G is 2-connected and $\langle r : m, n\rangle$-extendable, then G is $(m+n)$-extendable.

From this corollary, we understand that if a graph G is 2-connected and $\langle r, m\rangle$-extendable but not $(m+n)$-extendable, then there exists a connected subset T of order $2r$ such that $G \setminus T$ is connected and not n-extendable.

If $p = 2r$, then the connectedness condition of Theorem 1 cannot be weakened. For example, let K_{2n+2} and K'_{2n+2} be two disjoint complete graphs with order $2n+2$. Let $u \in V(K_{2n+2})$ and $v \in V(K'_{2n+2})$. Add an edge uv between K_{2n+2} and K'_{2n+2}. Let G be the resulting graph. Now we can easily check that S and $G \setminus S$ are n-extendable for every connected subset S of order $2n+2$ for which $G \setminus S$ is connected. It is obvious however that G is not $2n$-extendable since G cannot have a 1-factor which contains uv.

2. Preliminary Lemmas.

Our proof of Theorem 1 depends heavily on the following two theorems. We denote the number of odd components of a graph G by $o(G)$.

Lemma 1 (Tutte [7]).
(I) A graph G has a 1-factor iff $o(G \setminus S) \leq |S|$ for all $S \subset V(G)$. $\square$
(II) $o(G \setminus S) - |S| \equiv 0 \pmod 2$ if G has even order.

Lemma 2 (Plummer [6]).

(I) If G is n-extendable, then G is $(n-1)$-extendable.

(II) If G is connected and n-extendable, then G is $(n+1)$-connected. □

Next, the following two lemmas are easily deduced from the definitions of variations of extendability.

Lemma 3 . Let m, n, and r be positive integers with $r > m$. If G is 2-connected and $\langle r : m, n\rangle$-extendable, then G is $\langle r+1, m\rangle$-extendable.

Proof. Let G be a graph satisfying the hypothesis. By the definitions and Lemma 2 (I), if G is $\langle r : m, n\rangle$-extendable, then G is $\langle r : m, n-1\rangle$-extendable. So, we have G is $\langle r, m, 0\rangle$-extendable, inductively. Then G is also $\langle r, m\rangle$-extendable. Since G is 2-connected, G becomes $\langle r+1, m\rangle$-extendable by Theorem C. □

Lemma 4 . Let m, n, and r be positive integers with $r > m$. If G is 2-connected $\langle r : m, n\rangle$-extendable, then, for every connected subset T of order $2(r+1)$ for which $G \setminus T$ is connected, $G \setminus T$ is $(n-1)$-extendable.

Proof. Let G be a graph satisfying the hypothesis, T a connected subset of order $2(r+1)$ for which $G \setminus T$ is connected. Then we may assume that T is m-extendable by Lemma 3 and that T is $(m+1)(\geq 2)$-connected by Lemma 2 (II). So, since G is connected, there exists an edge uv in $E(T)$ such that $T \setminus \{u, v\}$ is connected and $E(u, G \setminus T) \neq \emptyset$. Let M be an arbitrary matching of $G \setminus T$ with size $n-1$. Set $S = T \setminus \{u, v\}$. Clearly, $G \setminus S$ is connected. Therefore, S is m-extendable and $G \setminus S$ is n-extendable by hypothesis. Then $M \cup \{uv\}$ can be extended to a 1-factor F of $G \setminus S$. Thus $G \setminus T$ has a 1-factor $F \setminus \{uv\}$ which contains M, or $G \setminus T$ is $(n-1)$-extendable. □

3. Proof of Theorem 1.

Let p, r, m, n, and G be as in the theorem. Suppose, to the contrary of the conclusion, G is not $\langle r+1 : m+1, n-1\rangle$-extendable. So, there exists a connected subset T of order $2(r+1)$ for which $G \setminus T$ is connected, and which satisfies the following:

(i) T is not $(m+1)$-extendable or (ii) $G \setminus T$ is not $(n-1)$-extendable.

Now, for such a subset T, $G \setminus T$ is $(n-1)$-extendable by Lemma 4. Therefore we may assume that T is not $(m+1)$-extendable. Let $M = \{e_1, e_2, ..., e_{m+1}\}$ be a matching of T which is not extended to a 1-factor of T. And we set $B = \bigcup_{i=1}^{m+1} V(e_i)$. Then, by Lemma 1 (I), there exists a set $A \subset T \setminus B$ such that $o((T \setminus B) \setminus A) > |A|$. Clearly since G is even order, for this set A there exists a positive integer k such that

$$o(T \setminus B \setminus A) = o((T \setminus B) \setminus A) = |A| + 2k$$

by Lemma 1 (II). Throughout our proof of Theorem 1, we consider that such a set A is fixed. By the way, we may assume that T is m-extendable by Lemma 3. So, for every edge $e_i \in M$, T must have a 1-factor which contains $M \setminus \{e_i\}$. Again, by Lemma 1 (I), we have

$$o((T \setminus (B \setminus V(e_i)) \setminus A) \leq |A|.$$

Thus every $V(e_i)$ must join at least $2k$ odd components in $T \setminus B \setminus A$.

Since T is connected m-extendable and $m > 0$, T is $(m+1)(\geq 2)$-connected by Lemma 2 (II). Therefore, we can decompose T into $V(O_1) \cup V(P_2) \cup ... \cup V(P_l)$ satisfying the following:

(i) O_1 is a longest cycle of T and

(ii) P_i $(2 \leq i \leq l)$ is a longest path of $T \setminus (V(O_1) \cup (\bigcup_{j=2}^{i-1} V(P_j)))$ with end vertices a_i, b_i such that $a_i x_i, b_i y_i \in E(T)$, where $x_i, y_i \in V(O_1) \cup (\bigcup_{j=2}^{i-1} V(P_j))$ and $x_i \neq y_i$.

If $P_i = w_1 w_2 \dots w_c$ ($w_1 = a_i$ and $w_c = b_i$), then $x_i P_i y_i$ denotes the path $x_i w_1 w_2 \dots w_c y_i$. For O_1 and $x_i P_i y_i$ $(2 \leq i \leq k)$, we define an orientation, respectively. And we denote by x^+, x^- the succesor and the predecessor of a vertex x on O_1 (or P_i) according to the orientation, respectively. Since G is connected, there exists a vertex v of $T = V(O_1) \cup V(\bigcup_{j=2}^{l} P_j)$ which is adjacent to a vertex of $G \setminus T$. Then, by the property of P_i, $T \setminus \{v, v^+\}$ is connected. Obviously $G \setminus (T \setminus \{v, v^+\})$ is also connected. Hence $T \setminus \{v, v^+\}$ and $G \setminus (T \setminus \{v, v^+\})$ are m-extendable and n-extendable, respectively. Now since $G \setminus (T \setminus \{v, v^+\})$ is $(n+1)(\geq 2)$-connected by Lemma 2 (II), $E(v^+, G \setminus T) \neq \emptyset$. Applying the same argument but replacing v^+ to v, we have $E(v^{++}, G \setminus T) \neq \emptyset$. Similarly, we have $E(v^-, G \setminus T)), E(v^{--}, G \setminus T) \neq \emptyset$, etc. Consequently we can prove that each vertex of T is adjacent to a vertex of $G \setminus T$. In particular, we have the following:

$T \setminus \{u, v\}$ and $G \setminus (T \setminus \{u, v\})$ are connected for each edge uv on $O_1 \cup (\bigcup_{i=2}^{l} x_i P_i y_i)$.

Let $\{u, v\}$ be a set of distinct two vertices of T such that $T \setminus \{u, v\}$ is connected. Here notice that u might be non-adjacent to v and that $G \setminus (T \setminus \{u, v\})$ is connected. Let $\mathcal{C} = \{C_1, C_2, ..., C_\alpha\}$ (resp. $\mathcal{D} = \{D_1, D_2, ..., D_\beta\}$) be the set of odd components (resp. even components) of $(T \setminus B) \setminus A$. Then $T = A \cup B \cup (\bigcup_{i=1}^{\alpha} V(C_i)) \cup (\bigcup_{j=1}^{\beta} V(D_j))$. We consider nine cases.

Set $S = T \setminus \{u, v\}$. Note that S is m-extendable and k is positive.

Case 1. $u, v \in A$.

Let $e \in M$ and set $A' = (A \setminus \{u, v\}) \cup V(e)$. Since $S \setminus (B \setminus V(e))$ has a 1-factor, we have

$$|A| = |A'| \geq o(S \setminus (B \setminus V(e)) \setminus A') = o(T \setminus B \setminus A) = |A| + 2k,$$

or $k \leq 0$, which contradicts that k is positive.

Case 2. $u \in A$ and $v \in B$.
Let $vy \in M$ and set $A' = (A \setminus \{u\}) \cup \{y\}$. Then we have

$$|A| = |A'| \geq o(S \setminus (B \setminus \{v, y\}) \setminus A') = o(T \setminus B \setminus A) = |A| + 2k,$$

which is a contradiction.

Case 3. $u \in A$ and $v \in D_i$.
Let $e \in M$ and set $A' = (A \setminus \{u\}) \cup V(e)$. Then we have

$$|A| + 1 = |A'| \geq o(S \setminus (B \setminus V(e)) \setminus A') \geq o(T \setminus B \setminus A) + 1 = |A| + 2k + 1,$$

which is a contradiction.

Case 4. $u, v \in B$ and $uv \in M$.
We have

$$|A| \geq o(S \setminus (B \setminus \{u, v\}) \setminus A) = o(T \setminus B \setminus A) = |A| + 2k,$$

which is a contradiction.

Case 5. $u \in B$ and $v \in D_i$.
Let $ux \in M$ and set $A' = A \cup \{x\}$. Then we have

$$|A| + 1 = |A'| \geq o(S \setminus (B \setminus \{u, x\}) \setminus A') \geq o(T \setminus B \setminus A) + 1 = |A| + 2k + 1,$$

which is a contradiction.

Case 6. $u \in A$ and $v \in C_i$.
Let $e \in M$ and set $A' = (A \setminus \{u\}) \cup V(e)$. Then we have

$$|A| + 1 = |A'| \geq o(S \setminus (B \setminus V(e)) \setminus A') \geq o(T \setminus B \setminus A) - 1 = |A| + 2k - 1,$$

or $k \leq 1$. Then we have $k = 1$ since k is positive.

Case 7. $u, v \in B$ and $uv \notin M$.
Note that S has a 1-factor even if S is 0-extendable. Let $ux, vy \in M$ and set $A' = A \cup \{x, y\}$. Since S is $(m-1)$-extendable by Lemma 2 (I), we have

$$|A| + 2 \geq |A'| \geq o(S \setminus (B \setminus \{x, y\}) \setminus A') = o(T \setminus B \setminus A) = |A| + 2k,$$

which implies $k = 1$.

Case 8. $u \in B$ and $v \in C_i$.

Let $ux \in M$ and set $A' = A \cup \{x\}$. Then we have

$$|A| + 1 = |A'| \geq o(S \setminus (B \setminus \{x\}) \setminus A') \geq o(T \setminus B \setminus A) - 1 = |A| + 2k - 1.$$

We have $k = 1$.

Case 9. $u, v \in C_i$ or $u, v \in D_j$.

Let $e \in M$ and set $A' = A \cup V(e)$. Then

$$|A| + 2 = |A'| \geq o(S \setminus (B \setminus V(e)) \setminus A') \geq o(T \setminus B \setminus A) = |A| + 2k.$$

We have $k = 1$.

Suppose that u and v are vertices satisfying one of the situations of Cases 1–5. Then $T \setminus \{u, v\}$ is disconnected. In particular, $T \setminus V(e_i)$ is disconnected for every $e_i \in M$. Furthermore, if $uv \in E(T)$, then uv is not an edge on $O_1 \cup (\bigcup_{i=2}^{l} x_i P_i y_i)$. Conversely, since uv on $O_1 \cup (\bigcup_{i=2}^{l} x_i P_i y_i)$ does not join two distinct components of $(T \setminus A) \setminus B$, every edge uv on $O_1 \cup (\bigcup_{i=2}^{l} x_i P_i y_i)$ satisfies the one of Cases 6–9. Now since M is not empty, we have an edge $e = w_1 w_2 \in M$. Notice that w_1 is in B and that $T \setminus V(e)$ is disconnected. By observation of the various cases, w_1^+ is in $B \cup (\bigcup_{i=1}^{\alpha} C_i)$, and w_1^+ is not w_2. Similarly, $w_1^- \in B \cup (\bigcup_{i=1}^{\alpha} C_i)$ and $w_1^- \neq w_2$. Let Q be a component of $T \setminus V(e)$ containing w_1^+. Since $T \setminus \{w_1, w_1^-\}$ is connected, it is m-extendable. Hence, it is also 2-connected by Lemma 2 (II). Then there exists a vertex z of Q (or $Q \setminus \{w_1^-\}$ if w_1^- is in Q) which is adjacent to a vertex of $(T \setminus \{w_1, w_2\}) \setminus Q$. Therefore, Q is not a component of $T \setminus \{w_1, w_2\} = T \setminus V(e)$, which is a contradiction. This contradiction completes the proof of Theorem 1. □

The following property can be considerd as an extension of factor-criticality. A graph G is said to be $2n$-*factor-critical* if the graph remaining after deletion of any $2n$ vertices from G has a 1-factor (a perfect matching). Clearly, this property is stronger than that of extendability, that is, if a graph G is $2n$-factor-critical, then G is n-extendable.

Now let $r, m,$ and n be nonnegative integers. A connected graph G is called $\langle r : m, n\rangle$-*factor-critical* if, for every connected subset S of order r for which $G \setminus S$ is connected, $G[S]$ is m-factor-critical and $G \setminus S$ is n-factor-critical. Similarly, we can also define that a graph becomes $\langle r, n\rangle$-*factor-critical* (or (r, n)-*factor-critical* or $[r, n]$-*factor-critical*). Then, by the argument quite similar to that in the proof of Theorem 1, we have the following results.

Theorem 3. Let $p, r, m,$ and n be positive integers with $p - r > n$ and $r > m$. Then every 2-connected $\langle 2r : 2m, 2n\rangle$-factor-critical graph of order $2p$ is $\langle 2(r+1) : 2(m+1), 2(n-1)\rangle$-factor-critical.

Corollary 4. If a graph G is 2-connected and $\langle 2r : 2m, 2n\rangle$-factor-critical, then G is $2(m+n)$-factor-critical.

Finally, we conjecture the following:

Conjecture. Let n, p, and r be integers such that $1 \leq n < r$ and $p - r > n$, and let G be an $(n+1)$-connected graph of order $2p$. If for every connected subset $S \subset V(G)$ with $|S| = 2r$ (for which $G \setminus S$ is connected), S or $G \setminus S$ is n-extendable, then G is also n-extendable.

In [4], we proved that for 2-connected graphs, Theorem C contains the following theorems:

Theorem D (Nishimura [2]). Let G be a connected graph of order $2p$ ($p \geq 3$), and let r and n be integers such that $1 \leq n < r < p$. If for some integer r, every induced connected subgraph of order $2r$ is n-extendable, then G is n-extendable.

Theorem E (Nishimura [3]). Let G be a connected graph of order $2p$. Let r and n be positive integers such that $p - r \geq n + 1$. If $G \setminus S$ is n-extendable for every connected subset S of order $2r$, then G is n-extendable.

If the conjecture above is correct, then this will be 'another' extension of these theorems.

Acknowledgements. We would like to acknowledge the referees for their comments.

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(Received 12/11/98)