

Generalized Steiner triple systems with group size $g \equiv 1, 5 \pmod{6}$ *

G. Ge

Institute of Economics
Suzhou University, Suzhou 215006
People's Republic of China

Abstract

Generalized Steiner triple systems, $\text{GS}(2, 3, n, g)$ are equivalent to maximum constant weight codes over an alphabet of size $g + 1$ with distance 3 and weight 3 in which each codeword has length n . The existence of $\text{GS}(2, 3, n, g)$ has been solved by several authors for $2 \leq g \leq 10$. The necessary conditions for the existence of a $\text{GS}(2, 3, n, g)$ are $(n - 1)g \equiv 0 \pmod{2}$, $n(n - 1)g^2 \equiv 0 \pmod{6}$, and $n \geq g + 2$. Recently, D. Wu et al proved that for any given $g \geq 7$, if there exists a $\text{GS}(2, 3, n, g)$ for all n , $g + 2 \leq n \leq 9g + 158$, satisfying the above two congruences, then the necessary conditions are also sufficient. In this paper, the result is partially improved. It is shown that for any given g , $g \equiv 1, 5 \pmod{6}$ and $g \geq 11$, if there exists a $\text{GS}(2, 3, n, g)$ for all n , $n \equiv 1, 3 \pmod{6}$ and $g + 2 \leq n \leq 9g + 4$, then the necessary conditions are also sufficient. As an application, it is proved that the necessary conditions for the existence of a $\text{GS}(2, 3, n, g)$ are also sufficient for $g = 11$.

1 Introduction

A $(g + 1)$ -ary *constant weight code* (n, w, d) is a code $C \subseteq (Z_{g+1})^n$ of length n and minimum distance d , such that every $c \in C$ has Hamming weight w . To construct a constant weight code (n, w, d) with $w = 3$, a *group divisible design* (GDD) will be used. A K -GDD is an ordered triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ where $\mathcal{V}$ is a set of n elements, $\mathcal{G}$ is a collection of subsets of $\mathcal{V}$ called *groups* which partition $\mathcal{V}$, and $\mathcal{B}$ is a set of some subsets of $\mathcal{V}$ called *blocks*, such that each block intersects each group in at most one element and that each pair of elements from distinct groups occurs together in exactly one block in $\mathcal{B}$, where $|B| \in K$ for any $B \in \mathcal{B}$. The group type is the multiset $\{|G| : G \in \mathcal{G}\}$. A k -GDD(g^n) denotes a K -GDD with n groups

*Research supported in part by NSFC Grant 19231060-2.

of size g and $K = \{k\}$. If all blocks of a GDD can be partitioned into parallel classes, then the GDD is called *resolvable* GDD and denoted by RGDD, where a parallel class is a set of blocks partitioning the element set $\mathcal{V}$. In a 3-GDD(g^n), let $\mathcal{V} = (Z_{g+1} \setminus \{0\}) \times (Z_{n+1} \setminus \{0\})$ with n groups $G_i \in \mathcal{G}$, $G_i = (Z_{g+1} \setminus \{0\}) \times \{i\}$, $1 \leq i \leq n$ and blocks $\{(a, i), (b, j), (c, k)\} \in \mathcal{B}$. One can construct a constant weight code $(n, 3, d)$ as stated in [5], [7]. From each block we form a codeword of length n by putting an a , b and c in positions i , j and k respectively and zeros elsewhere. This gives a constant weight code over Z_{g+1} with minimum distance 2 or 3. If the minimum distance is 3, then the code is a $(g+1)$ -ary *maximum constant weight code* (MCWC) $(n, 3, 3)$ and the 3-GDD(g^n) is called *generalized Steiner triple system*, denoted by GS(2, 3, n, g). It is easy to see that a 3-GDD(g^n) is a GS(2, 3, n, g) iff any two intersecting blocks meet at most two common groups of the GDD. The following result is known.

Lemma 1.1 ([5], [7]) *The following are the necessary conditions for the existence of a GS(2, 3, n, g):*

- (1) $(n-1)g \equiv 0 \pmod{2}$;
- (2) $n(n-1)g^2 \equiv 0 \pmod{6}$;
- (3) $n \geq g+2$.

The necessary conditions are shown to be sufficient by several authors with one exception for $2 \leq g \leq 10$. Hence, we have the following lemma.

Lemma 1.2 ([5], [7], [8], [3], [4], [9], [6]) *The necessary conditions for the existence of a GS(2, 3, n, g) are also sufficient for $2 \leq g \leq 10$ with one exception of $(g, n) = (2, 6)$.*

Blake-Wilson and Phelps [2] proved that the necessary conditions for the existence of a GS(2, 3, n, g) are also asymptotically sufficient for any g . Recently, D. Wu et al [9] proved that for any given $g \geq 7$, if there exists a GS(2, 3, n, g) for all n , $g+2 \leq n \leq 9g+158$, satisfying $(n-1)g \equiv 0 \pmod{2}$ and $n(n-1)g^2 \equiv 0 \pmod{6}$, then the necessary conditions for the existence of a GS(2, 3, n, g) are also sufficient.

Since the existence of GS(2, 3, n, g) has been solved for $g \leq 10$, we need only to consider the case $g \geq 11$. For $g \equiv 1, 5 \pmod{6}$ and $g \geq 11$, let $T_g = \{n: \text{there exists a GS(2, 3, } n, g)\}$, $B_g = \{n: n \text{ satisfying the necessary conditions listed in Lemma 1.1}\}$, $M_g = \{n: n \in B_g, n \leq 9g+4\}$. In this paper, the results of [9] will be partially improved and the following results are obtained.

Theorem 1.3 *For any $g \equiv 1, 5 \pmod{6}$ and $g \geq 11$, if $M_g \subset T_g$, then $B_g = T_g$. That is, the necessary conditions for the existence of a GS(2, 3, n, g) are also sufficient.*

Theorem 1.4 *$B_{11} = T_{11}$, that is, the necessary conditions for the existence of a GS(2, 3, n, g) are also sufficient for $g = 11$.*

Combining Lemma 1.2 and Theorem 1.4 shows that the existence of a GS(2, 3, n, g) is completely determined for any $g \leq 11$.

2 Product Constructions

In product constructions, we will need the concept of both *holey generalized Steiner triple systems* and *disjoint incomplete Latin squares*.

A *holey group divisible design*, K -HGDD, is a fourtuple $(\mathcal{V}, \mathcal{G}, \mathcal{H}, \mathcal{B})$, where $\mathcal{V}$ is a set of points, $\mathcal{G}$ is a partition of $\mathcal{V}$ into subsets called *groups*, $\mathcal{H} \subset \mathcal{G}$, $\mathcal{B}$ is a set of *blocks* such that a group and a block contain at most one common point and every pair of points from distinct groups, not both in $\mathcal{H}$, occurs in a unique block in $\mathcal{B}$, where $|B| \in K$ for any $B \in \mathcal{B}$. A k -HGDD($g^{(n,u)}$) denotes a K -HGDD with n groups of size g in $\mathcal{G}$, u groups in $\mathcal{H}$ and $K = \{k\}$. A *holey generalized Steiner triple system*, HGS(2, 3, (n, u) , g), is a 3-HGDD($g^{(n,u)}$) with the property that any two intersecting blocks meet at most two common groups.

It is easy to see that if $u = 0$ or $u = 1$, then a HGS(2, 3, $(n + u, u)$, g) is just a GS(2, 3, n, g) or a GS(2, 3, $n + 1, g$) respectively.

A *Latin square* of side n , LS(n), is an $n \times n$ array based on some set S of n symbols with the property that every row and every column contains every symbol exactly once. An *incomplete Latin square*, ILS($n + a, a$), denotes a LS($n + a$) "missing" a sub LS(a). Without loss of generality, we may assume that the missing subsquare, or *hole*, is at the lower right corner. We say $(i, j, s) \in \text{ILS}(n + a, a)$ if the entry in the cell (i, j) is s . Let A_1, A_2 be two ILS($n + a, a$)s on the same symbol set. If $(i, j, s_1) \neq (i, j, s_2)$ for any $(i, j, s_1) \in A_1, (i, j, s_2) \in A_2$, then we say that A_1 and A_2 are *disjoint*. We use r DILS($n + a, a$) to denote r pairwise disjoint ILS($n + a, a$)s.

For the existence of r DILS($n + a, a$), we have the following two lemmas.

Lemma 2.1 ([3]) *There exist $\delta(a)$ DILS($n + a, a$), where $\delta(0) = n$ and $\delta(a) = a$ for $1 \leq a \leq n$.*

Lemma 2.2 ([9]) *There exist n DILS($n + a, a$) for all $n \equiv 0 \pmod{4}$ and $0 \leq a \leq n$.*

The following singular indirect product construction for GS(2, 3, n, g)s is first stated in [3].

Lemma 2.3 (*Singular Indirect Product (SIP)*) *Let m, n, t, u and a be integers such that $0 \leq a \leq u < n$. Suppose the following designs exist:*

- (1) t DILS($n + a, a$);
 - (2) a 3-GDD(g^m) with the property that all blocks of the design can be partitioned into t sets $S_0, S_1, \dots, S_{t-1}$, such that the minimum distance in $S_r, 0 \leq r \leq t - 1$, is 3;
 - (3) a HGS(2, 3, $(n + u, u)$, g).
- Then there exists a HGS(2, 3, (c, d) , g), where $c = m(n + a) + u - a, d = ma + u - a$. Further, if there exists
- (4) a GS(2, 3, $ma + u - a, g$),
- then there exists a GS(2, 3, $m(n + a) + u - a, g$).

Taking $a = 0$ in Lemma 2.3, we get the singular direct product construction, which first appeared in [8].

Lemma 2.4 (*Singular Direct Product (SDP)*) Let m, n, t , and u be integers such that $0 \leq u < n$. Suppose $t \leq n$ and the following designs exist:

(1) a $3-GDD(g^m)$ with the property that all blocks of the design can be partitioned into t sets $S_0, S_1, \dots, S_{t-1}$, such that the minimum distance in $S_r, 0 \leq r \leq t-1$, is 3;

(2) a $HGS(2, 3, (n+u, u), g)$.

Then there exists a $HGS(2, 3, (mn+u, u), g)$. Further, if there exists a $GS(2, 3, u, g)$, then there exists a $GS(2, 3, mn+u, g)$.

Taking $u=0$ or 1 in Lemma 2.4, we get the Construction C or D of Etzion in [5] respectively.

Lemma 2.5 (*Direct Product (DP)*) Let $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ be a $3-GDD(g^m)$, and suppose there exists a $GS(2, 3, n, g)$. Then there exists a $GS(2, 3, mn, g)$ if $\mathcal{B}$ can be partitioned into t sets $S_0, \dots, S_{t-1}$, such that the minimum distance in $S_r, 0 \leq r \leq t-1$, is 3 and $t \leq n$.

Lemma 2.6 ([5]) Let $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ be a $3-GDD(g^m)$, and suppose there exists a $GS(2, 3, n, g)$. Then there exists a $GS(2, 3, m(n-1)+1, g)$ if $\mathcal{B}$ can be partitioned into t sets $S_0, \dots, S_{t-1}$, such that the minimum distance in $S_r, 0 \leq r \leq t-1$, is 3 and $t \leq n-1$.

It is easy to notice that the derived generalized Steiner triple system in Lemma 2.5 and Lemma 2.6 has a sub $GS(2, 3, n, g)$. Hence, we have the following.

Lemma 2.7 Let $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ be a $3-GDD(g^m)$. Suppose there exists a $GS(2, 3, n, g)$. Then there exists a $HGS(2, 3, (mn, n), g)$ or a $HGS(2, 3, (m(n-1)+1, n), g)$ if $\mathcal{B}$ can be partitioned into t sets $S_0, \dots, S_{t-1}$, such that the minimum distance in $S_r, 0 \leq r \leq t-1$, is 3 and $t \leq n$ or $t \leq n-1$ respectively.

If one uses a $3-RGDD(g^m)$ in the constructions, then each parallel class becomes an S_r and there are $t = \frac{g(m-1)}{2}$ such classes. The following is stated in [3].

Lemma 2.8 If there exists a $GS(2, 3, n, g)$ and a $3-RGDD(g^m)$ with $t = \frac{g(m-1)}{2} \leq n$ or $n-1$, then there exists a $GS(2, 3, mn, g)$ or a $GS(2, 3, m(n-1)+1, g)$ respectively.

For the existence of a $3-RGDD(g^m)$, we have the following.

Lemma 2.9 ([1]) A $3-RGDD(g^m)$ exists iff $(m-1)g \equiv 0 \pmod{2}$, $mg \equiv 0 \pmod{3}$ and $g^m \neq 2^3, 2^6$ and 6^3 .

By combining Lemmas 2.7-2.9, we have the following.

Lemma 2.10 For any $g \geq 11$, if there exists a $GS(2, 3, n, g)$, then there exists a $GS(2, 3, 3n, g)$ and a $GS(2, 3, 3(n-1)+1, g)$. Consequently, there exists a $HGS(2, 3, (3n, n), g)$ and a $HGS(2, 3, (3(n-1)+1, n), g)$.

3 Proof of Theorem 1.3

In this section, we will show the proof of Theorem 1.3. First, we need the following lemmas.

Lemma 3.1 *For $g \equiv 1, 5 \pmod{6}$ and $g \geq 11$, suppose $v = 18p + j$, $j \in \{1, 3, 7, 9\}$. If $6p + \lfloor \frac{j}{3} \rfloor + \delta(j) \in T_g$, where $\delta(j) = 0$ or 1 , and $\delta(j) \equiv j \pmod{3}$, then $v \in T_g$.*

Proof. Apply Lemma 2.10 with $n = 6p + \lfloor \frac{j}{3} \rfloor + \delta(j)$, the conclusion then follows. $\square$

Lemma 3.2 *For $g \equiv 1, 5 \pmod{6}$ and $g \geq 11$, suppose $v = 54p + j$, $j = 13$ or 15 . If $6p + 3 \in T_g$, $18p + j - 12 \in T_g$, and $p \geq \lceil \frac{15-j}{12} \rceil$, then $v \in T_g$.*

Proof. Apply Lemma 2.3 with $m = 3, n = 12p + 4, t = g, u = 6p + 3$ and $a = 6p - \frac{15-j}{2}$. It is easy to check that $a \leq u < n$. Since $p \geq \lceil \frac{15-j}{12} \rceil$, it is easy to see that $a \geq 0$. From Lemma 2.2, there exist n DILS($n+a, a$) for $0 \leq a \leq n$. We further have t DILS($n+a, a$) since $t \leq u-2 < n$. Thus the condition (1) of Lemma 2.3 is satisfied. For $g \geq 11$, a 3-RGDD(g^3) always exists by Lemma 2.9, which has g parallel classes. So, condition (2) is also satisfied. From $u \in T_g$, we apply Lemma 2.10 to obtain a HGS($2, 3, (n+u, u), g$), providing the design in condition (3). Finally, we have $ma + u - a = 18p - 12 + j \in T_g$, the condition (4) is satisfied. Therefore, we have $v \in T_g$. This completes the proof. $\square$

Lemma 3.3 *For $g \equiv 1, 5 \pmod{6}$ and $g \geq 11$, suppose $v = 54p + j$, $j \in \{31, 33, 49, 51\}$. If $6p + 7 \in T_g$, $18p + j - 36 \in T_g$, and $p \geq \lceil \frac{43-j}{12} \rceil$, then $v \in T_g$.*

Proof. Apply Lemma 2.3 with $m = 3, n = 12p + 12, t = g, u = 6p + 7$ and $a = 6p - \frac{43-j}{2}$. Then the proof is completed analogously to that of Lemma 3.2. $\square$

Now, we are in a position to prove Theorem 1.3.

Proof of Theorem 1.3. We need to show that $M_g \subset T_g$ implies that $B_g \subset T_g$. The proof is by induction on n . Suppose $n \in B_g$. If $n \in M_g$, then $n \in T_g$. Otherwise, we have $n \geq 9g + 6$ and distinguish between the following cases:

Case 1: $n = 18p + j \geq 9g + 6$, $j \in \{1, 3, 7, 9\}$. It is easy to see that $n \geq 9g + 6$ implies $\alpha = 6p + \lfloor \frac{j}{3} \rfloor + \delta(j) \geq 3g + 2 > g + 2$. It is also easy to verify that $\alpha \in B_g$. If $\alpha \in M_g$, then Lemma 3.1 guarantees that $n \in T_g$ and the proof is complete. Otherwise, we can repeat the induction process taking α as n' .

Case 2: $n = 54p + j \geq 9g + 6$, $j = 13$ or 15 . We first claim that $p \geq \lceil \frac{15-j}{12} \rceil$. If not so, then $p < \lceil \frac{15-j}{12} \rceil \leq 1$. Thus $n < 54 + j$. Since $g \geq 11$ and $j = 13$ or 15 , we have $n < 69 < 9g + 6$, a contradiction.

Next, it is easy to see that $n \geq 9g + 6$ implies $6p \geq g + \frac{6-j}{9}$. Then it is easily checked that $\alpha = 6p + 3 \geq g + 2$ and $\beta = 18p + j - 12 \geq g + 2$ for $g \geq 11$. Since $\beta \equiv 1$ or $3 \pmod{6}$, we see that $\alpha \in B_g$ and $\beta \in B_g$. If we have both $\alpha \in M_g$ and $\beta \in M_g$, then Lemma 3.2 guarantees that $n \in T_g$ and the proof is complete. If at

least one of α and β is not in M_g , then we can repeat the induction process taking the number α, β not in M_g as n' .

Case 3: $n = 54p + j \geq 9g + 6, j \in \{31, 33, 49, 51\}$. Apply Lemma 3.3, the proof of this case is similar to that of Case 2. We need only to check that $p \geq \lceil \frac{43-j}{12} \rceil$, $6p + 7 \geq g + 2$ and $18p + j - 36 \geq g + 2$. We first claim that $p \geq \lceil \frac{43-j}{12} \rceil$. If not so, then $p < \lceil \frac{43-j}{12} \rceil \leq 1$. Thus $n < 54 + j$. Since $g \geq 11$ and $j \in \{31, 33, 49, 51\}$, we have $n < 54 + 51 \leq 9g + 6$, a contradiction.

Next, it is easy to check that $n \geq 9g + 6$ implies $6p + 7 \geq g + 2$ and $18p + j - 36 \geq g + 2$ for $g \geq 11$.

After certain steps of induction on n, n' will be small enough so that n' is in M_g , consequently, $n \in T_g$. Case 1 implies the solution for $n \equiv 1, 3, 7$ or $9 \pmod{18}$; Cases 2 and 3 imply the solution for $n \equiv 13$ or $15 \pmod{18}$. This completes the proof. $\square$

4 Proof of Theorem 1.4

In this section, we will show that the necessary conditions for the existence of a $GS(2, 3, n, 11)$ are also sufficient. From Theorem 1.3, we need only to consider the case $n \in M_{11} = \{n : n \equiv 1, 3 \pmod{6} \text{ and } 13 \leq n \leq 103\}$.

For $n \equiv 3 \pmod{6}$, to construct a $GS(2, 3, n, 11)$ in Z_{11n} , it suffices to find a set of generalized base blocks, $\mathcal{A} = \{B_1, \dots, B_s\}$, $s = \frac{11(n-1)}{2}$, such that $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ forms a $GS(2, 3, n, 11)$, where $\mathcal{V} = Z_{11n}$, $G = \{G_1, G_2, \dots, G_n\}$, $G_i = \{i + nj : 0 \leq j \leq 10\}$, $1 \leq i \leq n$, and $\mathcal{B} = \{B + 3j : B \in \mathcal{A}, 0 \leq j \leq \frac{11n}{3} - 1\}$. For convenience, we write $\mathcal{A} = \bigcup_{i=1}^3 \{\{i, x, y\} : \{x, y\} \in S_i\}$. So, for each $\mathcal{A}$ we need only display the corresponding $S_i, 1 \leq i \leq 3$.

Lemma 4.1 *There exists a $GS(2, 3, n, 11)$ for all $n \in F_1$, where $F_1 = \{15, 21, 27, 33, 51, 69\}$.*

Proof. For the values $n \in F_1$, with the aid of a computer, we have found a set of generalized base blocks of a $GS(2, 3, n, 11)$. Here, we only list the $S_i, 1 \leq i \leq 3$ for $n = 15$. For the remaining values n , the corresponding $S_i, 1 \leq i \leq 3$ are listed in the Appendix.

$$n = 15, \mathcal{A} = \bigcup_{i=1}^3 \{\{i, x, y\} : \{x, y\} \in S_i\},$$

$$S_1 = \{ \{24, 95\}, \{111, 154\}, \{6, 162\}, \{90, 148\}, \{11, 53\}, \{15, 20\}, \{139, 159\}, \{72, 134\}, \\ \{71, 129\}, \{2, 109\}, \{66, 122\}, \{65, 128\}, \{14, 67\}, \{47, 103\}, \{114, 160\}, \{39, 138\}, \\ \{142, 158\}, \{130, 141\}, \{22, 63\}, \{99, 105\}, \{41, 44\}, \{55, 133\}, \{74, 124\}, \{81, 140\}, \\ \{80, 85\}, \{137, 165\}, \{115, 119\}, \{9, 157\}, \{118, 143\} \};$$

$$S_2 = \{ \{38, 106\}, \{130, 158\}, \{144, 165\}, \{116, 143\}, \{23, 102\}, \{19, 78\}, \{15, 83\}, \{123, \\ 145\}, \{36, 129\}, \{114, 155\}, \{8, 43\}, \{79, 128\}, \{90, 110\}, \{6, 98\}, \{24, 64\}, \{12, \\ 63\}, \{22, 54\}, \{35, 117\}, \{101, 149\}, \{87, 160\}, \{13, 85\}, \{74, 76\}, \{67, 153\}, \{16, \\ 112\}, \{3, 91\}, \{42, 61\}, \{21, 57\}, \{66, 150\} \};$$

$$S_3 = \{ \{38, 81\}, \{6, 118\}, \{37, 120\}, \{36, 144\}, \{139, 142\}, \{100, 131\}, \{11, 42\}, \{35, \\ 122\}, \{32, 143\}, \{126, 152\}, \{15, 133\}, \{34, 73\}, \{16, 72\}, \{66, 161\}, \{30, 77\}, \{19, \\ 21\}, \{52, 119\}, \{14, 94\}, \{114, 124\}, \{67, 101\} \}. \quad \square$$

Lemma 4.2 *There exists a $GS(2, 3, n, 11)$ for all $n \in F_2 = \{13, 19, 25, 31\}$.*

Proof. With the aid of a computer, we have found a set of base blocks $\mathcal{A}$ of a $GS(2, 3, n, 11)$ for $n \in F_2$.

For convenience, we write $\mathcal{A} = \{\{1, x, y\} : \{x, y\} \in S\}$. So, for each $\mathcal{A}$ we need only display the corresponding S .

$n = 13$, $S = \{\{62, 65\}, \{5, 88\}, \{130, 136\}, \{127, 134\}, \{117, 128\}, \{114, 126\}, \{24, 58\}, \{21, 123\}, \{32, 129\}, \{25, 95\}, \{29, 54\}, \{34, 63\}, \{36, 68\}, \{56, 125\}, \{44, 81\}, \{99, 104\}, \{55, 102\}, \{52, 100\}, \{59, 60\}, \{67, 135\}, \{3, 39\}, \{23, 73\}\}$.

$n = 19$, $S = \{\{28, 108\}, \{86, 93\}, \{76, 124\}, \{42, 133\}, \{30, 100\}, \{89, 193\}, \{43, 152\}, \{31, 80\}, \{64, 175\}, \{82, 138\}, \{15, 204\}, \{69, 95\}, \{54, 166\}, \{2, 189\}, \{107, 178\}, \{126, 144\}, \{17, 79\}, \{88, 91\}, \{29, 75\}, \{35, 199\}, \{3, 173\}, \{55, 163\}, \{117, 141\}, \{90, 145\}, \{151, 174\}, \{13, 74\}, \{83, 114\}, \{143, 195\}, \{146, 197\}, \{6, 10\}, \{44, 84\}, \{11, 61\}, \{177, 202\}\}$.

$n = 25$, $S = \{\{91, 162\}, \{20, 153\}, \{99, 113\}, \{50, 220\}, \{71, 148\}, \{65, 118\}, \{64, 229\}, \{240, 260\}, \{44, 234\}, \{42, 169\}, \{181, 255\}, \{140, 194\}, \{39, 248\}, \{97, 109\}, \{173, 174\}, \{70, 74\}, \{152, 261\}, \{6, 9\}, \{79, 155\}, \{132, 266\}, \{136, 138\}, \{62, 102\}, \{196, 230\}, \{63, 252\}, \{38, 156\}, \{133, 160\}, \{192, 225\}, \{147, 195\}, \{84, 116\}, \{31, 66\}, \{60, 73\}, \{127, 182\}, \{8, 177\}, \{23, 120\}, \{105, 184\}, \{46, 69\}, \{40, 92\}, \{61, 90\}, \{10, 68\}, \{7, 94\}, \{154, 165\}, \{27, 45\}, \{114, 131\}, \{32, 89\}\}$.

$n = 31$, $S = \{\{113, 139\}, \{83, 245\}, \{183, 272\}, \{111, 149\}, \{234, 290\}, \{267, 270\}, \{198, 248\}, \{24, 128\}, \{80, 231\}, \{200, 307\}, \{106, 334\}, \{188, 227\}, \{124, 181\}, \{102, 283\}, \{103, 222\}, \{115, 323\}, \{132, 186\}, \{14, 177\}, \{189, 242\}, \{82, 328\}, \{77, 273\}, \{35, 119\}, \{184, 208\}, \{254, 332\}, \{133, 142\}, \{33, 123\}, \{137, 338\}, \{61, 153\}, \{37, 136\}, \{165, 195\}, \{87, 271\}, \{140, 217\}, \{75, 97\}, \{44, 317\}, \{28, 174\}, \{12, 305\}, \{298, 326\}, \{284, 325\}, \{175, 255\}, \{170, 221\}, \{86, 236\}, \{321, 336\}, \{176, 302\}, \{13, 225\}, \{278, 341\}, \{19, 287\}, \{193, 276\}, \{3, 214\}, \{43, 48\}, \{68, 171\}, \{138, 199\}, \{46, 92\}, \{244, 309\}, \{226, 335\}, \{293, 313\}\}$. $\square$

The following lemma is a combination of Theorem 1 and Lemma 7 in [2]. Here, we need a new concept. A *maximum packing with triangles*, $MPT(n)$, is an ordered triple $(\mathcal{P}, \mathcal{T}, \mathcal{L})$, where $\mathcal{P}$ is the vertex set of K_n , $\mathcal{T}$ is a collection of edge disjoint triangles from the edge set of K_n with $|\mathcal{T}|$ as large as possible, and $\mathcal{L}$ is the collection of edges in K_n not belonging to one of the triangles of $\mathcal{T}$. The collection of edges $\mathcal{L}$ is called the *leave*.

Lemma 4.3 *There exists a $GS(2, 3, n, 11)$ for any prime power $n \equiv 1 \pmod{6}$ and $n \geq 43$.*

Proof. Apply Theorem 1 and Lemma 7 in [2], it suffices to show that there exists a $MPT(11) = (\mathcal{P}, \mathcal{T}, \mathcal{L})$ with 6 partial parallel classes, which is listed below.

$$\begin{aligned} \mathcal{P} &= \{1, 2, \dots, 11\}, \mathcal{T} = \bigcup_{i=1}^6 P_i, \mathcal{L} = \{\{2, 4\}, \{4, 3\}, \{3, 10\}, \{10, 2\}\}. \\ P_1 &= \{\{4, 8, 11\}, \{1, 5, 9\}, \{2, 6, 7\}\}; P_2 = \{\{1, 7, 8\}, \{2, 5, 11\}, \{6, 9, 10\}\}; \\ P_3 &= \{\{7, 10, 11\}, \{1, 2, 3\}, \{4, 5, 6\}\}; P_4 = \{\{2, 8, 9\}, \{3, 5, 7\}, \{1, 4, 10\}\}; \\ P_5 &= \{\{5, 8, 10\}, \{1, 6, 11\}, \{3, 9, 11\}\}; P_6 = \{\{3, 6, 8\}, \{4, 7, 9\}\}. \end{aligned} \quad \square$$

Lemma 4.4 *There exists a $GS(2, 3, v, 11)$ for all $v \in F_3 = \{37, 39, 45, 55, 57, 63, 75, 81, 91, 93, 99\}$.*

Proof. From Lemmas 4.1 and 4.2, we have a $GS(2, 3, n, 11)$ for all $n \in F = \{13, 15, 19, 21, 25, 27, 31, 33\}$. Apply Lemma 2.10 with $n \in F$, we get a $GS(2, 3, v, 11)$ for all $v \in F_3$. $\square$

Lemma 4.5 *There exists a $GS(2, 3, v, 11)$ for all $v \in F_4 = \{85, 87\}$.*

Proof. There exist 24 DILS($24+a, a$) for $a = 0$ or 1 by Lemma 2.2. There exist also a $GS(2, 3, 13, 11)$ and a $GS(2, 3, 15, 11)$ by Lemma 4.2 and Lemma 4.1. From Lemma 2.10, we have a $HGS(2, 3, (37, 13), 11)$. Apply Lemma 2.3 with $m = 3, n = 24, t = 11, u = 13, a = 0$ or 1 , we get a $GS(2, 3, 85, 11)$ or a $GS(2, 3, 87, 11)$ respectively. $\square$

Now, we are in a position to prove Theorem 1.4.

Proof of Theorem 1.4: From Theorem 1.3, we need only to consider the values v , such that $v \in M_{11}$. Lemma 4.3 provides a $GS(2, 3, v, 11)$ for all $v \in F_5 = \{43, 49, 61, 67, 73, 79, 97, 103\}$. It is readily checked that the union of F_i , for $1 \leq i \leq 5$, is the same as M_{11} . The conclusion then follows. $\square$

Appendix

$$n = 21, \mathcal{A} = \bigcup_{i=1}^3 \{\{i, x, y\} : \{x, y\} \in S_i\},$$

$$S_1 = \{ \{135, 202\}, \{96, 205\}, \{74, 159\}, \{61, 171\}, \{162, 230\}, \{121, 187\}, \{79, 214\}, \{217, 227\}, \{38, 181\}, \{99, 213\}, \{118, 200\}, \{75, 229\}, \{176, 184\}, \{107, 158\}, \{179, 208\}, \{36, 53\}, \{129, 220\}, \{109, 116\}, \{41, 128\}, \{150, 206\}, \{104, 139\}, \{120, 157\}, \{48, 82\}, \{12, 14\}, \{113, 222\}, \{18, 138\}, \{71, 173\}, \{23, 223\}, \{15, 218\}, \{17, 130\}, \{21, 110\}, \{51, 226\}, \{93, 178\}, \{133, 212\}, \{163, 228\}, \{90, 183\}, \{153, 199\}, \{42, 72\} \};$$

$$S_2 = \{ \{70, 142\}, \{199, 200\}, \{56, 204\}, \{166, 168\}, \{67, 227\}, \{45, 88\}, \{57, 175\}, \{143, 219\}, \{14, 164\}, \{72, 180\}, \{66, 190\}, \{77, 90\}, \{41, 172\}, \{25, 119\}, \{135, 157\}, \{17, 84\}, \{36, 100\}, \{74, 122\}, \{140, 187\}, \{62, 63\}, \{68, 207\}, \{27, 40\}, \{54, 82\}, \{52, 109\}, \{38, 79\}, \{47, 105\}, \{93, 108\}, \{5, 94\}, \{112, 228\}, \{85, 215\}, \{18, 159\}, \{19, 155\}, \{42, 129\} \};$$

$$S_3 = \{ \{19, 99\}, \{26, 227\}, \{135, 216\}, \{41, 103\}, \{35, 44\}, \{133, 137\}, \{172, 211\}, \{28, 83\}, \{51, 229\}, \{34, 119\}, \{23, 226\}, \{6, 75\}, \{38, 62\}, \{116, 215\}, \{91, 110\}, \{155, 166\}, \{151, 177\}, \{53, 183\}, \{47, 78\}, \{140, 197\}, \{9, 10\}, \{58, 122\}, \{55, 80\}, \{198, 207\}, \{175, 224\}, \{168, 222\}, \{161, 205\}, \{42, 89\}, \{43, 79\}, \{185, 212\}, \{134, 230\}, \{109, 196\}, \{61, 202\}, \{14, 148\}, \{27, 98\}, \{48, 81\}, \{63, 190\}, \{65, 188\}, \{11, 132\} \}.$$

$$n = 27, \mathcal{A} = \bigcup_{i=1}^3 \{\{i, x, y\} : \{x, y\} \in S_i\},$$

$$S_1 = \{ \{64, 266\}, \{147, 208\}, \{29, 116\}, \{192, 195\}, \{97, 230\}, \{186, 222\}, \{187, 233\}, \{139, 268\}, \{117, 131\}, \{61, 227\}, \{21, 149\}, \{76, 115\}, \{209, 228\}, \{94, 98\}, \{69, 154\}, \{13, 191\}, \{25, 264\}, \{124, 216\}, \{66, 198\}, \{239, 265\}, \{23, 26\}, \{138, 193\}, \{133, 182\}, \{46, 119\}, \{35, 58\}, \{104, 287\}, \{86, 256\}, \{143, 196\}, \{232, 282\}, \{174, 194\}, \{110, 247\}, \{19, 236\}, \{85, 100\}, \{140, 288\}, \{170, 262\}, \{9, 44\}, \{157, 278\}, \{132, 151\}, \{141, 295\}, \{102, 220\}, \{59, 210\}, \{99, 108\}, \{225, 289\}, \{155, 215\}, \{114, 137\}, \{63, 211\}, \{60, 176\}, \{181, 188\}, \{33, 277\}, \{105, 126\} \};$$

$$S_2 = \{ \{240, 280\}, \{79, 149\}, \{37, 66\}, \{75, 93\}, \{27, 158\}, \{26, 244\}, \{154, 192\}, \{6, 289\}, \{142, 179\}, \{254, 259\}, \{215, 232\}, \{15, 92\}, \{40, 81\}, \{140, 216\}, \{16, 138\}, \{103, 147\}, \{127, 144\}, \{234, 249\}, \{228, 252\}, \{65, 237\}, \{36, 38\}, \{80, 176\}, \{187, 193\}, \{156, 257\}, \{10, 87\}, \{53, 123\}, \{188, 260\}, \{167, 284\}, \{50, 286\}, \{4, 95\}, \{59, 205\}, \{32, 219\}, \{111, 247\}, \{76, 107\}, \{115, 197\}, \{11, 270\}, \{3, 20\}, \{204, 211\}, \{102, 266\}, \{33, 146\}, \{52, 54\}, \{100, 224\}, \{131, 200\} \};$$

$S_3 = \{ \{120, 260\}, \{47, 102\}, \{72, 205\}, \{34, 82\}, \{105, 228\}, \{44, 170\}, \{148, 217\}, \{89, 145\}, \{28, 253\}, \{122, 166\}, \{257, 263\}, \{25, 101\}, \{31, 42\}, \{203, 244\}, \{4, 171\}, \{70, 71\}, \{94, 214\}, \{52, 180\}, \{141, 173\}, \{77, 124\}, \{161, 190\}, \{96, 249\}, \{69, 277\}, \{226, 290\}, \{63, 159\}, \{9, 243\}, \{14, 189\}, \{16, 216\}, \{33, 295\}, \{8, 29\}, \{140, 274\}, \{206, 272\}, \{129, 233\}, \{112, 212\}, \{92, 150\}, \{118, 288\}, \{108, 293\}, \{7, 210\}, \{182, 194\}, \{239, 255\}, \{55, 213\}, \{40, 251\}, \{56, 78\}, \{46, 117\}, \{185, 267\}, \{51, 59\}, \{176, 222\}, \{49, 220\}, \{181, 197\}, \{95, 258\} \}.$

$n = 33, \mathcal{A} = \bigcup_{i=1}^3 \{ \{i, x, y\} : \{x, y\} \in S_i \},$

$S_1 = \{ \{168, 306\}, \{3, 7\}, \{147, 225\}, \{299, 324\}, \{140, 343\}, \{136, 339\}, \{142, 354\}, \{18, 123\}, \{48, 222\}, \{235, 246\}, \{164, 336\}, \{282, 333\}, \{46, 138\}, \{237, 326\}, \{61, 323\}, \{58, 281\}, \{148, 268\}, \{33, 40\}, \{11, 137\}, \{45, 139\}, \{179, 301\}, \{120, 187\}, \{8, 15\}, \{106, 214\}, \{162, 262\}, \{207, 340\}, \{212, 249\}, \{111, 113\}, \{50, 125\}, \{16, 362\}, \{74, 248\}, \{351, 356\}, \{37, 193\}, \{77, 80\}, \{124, 144\}, \{175, 194\}, \{51, 122\}, \{70, 134\}, \{26, 273\}, \{202, 286\}, \{54, 114\}, \{105, 185\}, \{53, 353\}, \{110, 303\}, \{43, 55\}, \{30, 95\}, \{66, 311\}, \{126, 183\}, \{261, 283\}, \{24, 318\}, \{36, 314\}, \{38, 211\}, \{96, 289\}, \{19, 145\}, \{206, 253\}, \{49, 348\}, \{63, 200\}, \{128, 271\}, \{2, 272\}, \{47, 215\}, \{198, 327\}, \{59, 135\}, \{56, 169\} \};$

$S_2 = \{ \{163, 224\}, \{64, 268\}, \{112, 347\}, \{212, 217\}, \{162, 303\}, \{57, 72\}, \{109, 189\}, \{45, 359\}, \{6, 80\}, \{308, 317\}, \{29, 305\}, \{82, 254\}, \{21, 339\}, \{58, 114\}, \{116, 337\}, \{56, 215\}, \{69, 309\}, \{73, 156\}, \{84, 137\}, \{123, 263\}, \{105, 348\}, \{31, 210\}, \{141, 218\}, \{131, 182\}, \{127, 296\}, \{331, 361\}, \{28, 274\}, \{32, 198\}, \{257, 321\}, \{98, 295\}, \{119, 164\}, \{76, 165\}, \{63, 148\}, \{208, 298\}, \{16, 293\}, \{33, 221\}, \{213, 343\}, \{61, 279\}, \{12, 42\}, \{179, 330\}, \{97, 280\}, \{46, 322\}, \{150, 325\}, \{139, 227\}, \{178, 183\}, \{83, 275\}, \{144, 345\}, \{125, 159\}, \{158, 235\}, \{70, 86\}, \{333, 352\}, \{44, 262\}, \{192, 326\}, \{34, 283\}, \{25, 234\}, \{23, 157\} \};$

$S_3 = \{ \{292, 330\}, \{104, 105\}, \{217, 289\}, \{52, 138\}, \{159, 363\}, \{239, 259\}, \{112, 276\}, \{84, 190\}, \{11, 345\}, \{46, 240\}, \{90, 211\}, \{95, 358\}, \{235, 353\}, \{14, 257\}, \{62, 181\}, \{53, 265\}, \{164, 295\}, \{131, 146\}, \{87, 128\}, \{115, 166\}, \{55, 314\}, \{327, 350\}, \{125, 249\}, \{12, 110\}, \{116, 152\}, \{30, 287\}, \{233, 268\}, \{167, 272\}, \{71, 99\}, \{298, 307\}, \{189, 293\}, \{96, 150\}, \{193, 308\}, \{111, 255\}, \{98, 122\}, \{73, 76\}, \{269, 360\}, \{50, 238\}, \{86, 175\}, \{27, 348\}, \{291, 325\}, \{66, 251\}, \{75, 221\}, \{82, 236\}, \{91, 197\}, \{15, 183\}, \{117, 340\}, \{58, 174\}, \{216, 230\}, \{79, 344\}, \{4, 156\}, \{139, 320\}, \{151, 182\}, \{41, 208\}, \{100, 127\}, \{266, 278\}, \{51, 161\} \}.$

$n = 51, \mathcal{A} = \bigcup_{i=1}^3 \{ \{i, x, y\} : \{x, y\} \in S_i \},$

$S_1 = \{ \{24, 74\}, \{71, 429\}, \{21, 58\}, \{87, 106\}, \{149, 283\}, \{116, 314\}, \{14, 327\}, \{376, 485\}, \{138, 548\}, \{371, 374\}, \{12, 514\}, \{216, 284\}, \{67, 435\}, \{405, 560\}, \{96, 320\}, \{217, 367\}, \{16, 310\}, \{418, 478\}, \{259, 416\}, \{195, 360\}, \{294, 510\}, \{47, 372\}, \{258, 488\}, \{408, 440\}, \{221, 248\}, \{134, 220\}, \{44, 319\}, \{160, 501\}, \{184, 353\}, \{541, 550\}, \{28, 97\}, \{114, 389\}, \{378, 387\}, \{299, 441\}, \{55, 227\}, \{247, 339\}, \{40, 424\}, \{214, 296\}, \{168, 226\}, \{130, 442\}, \{177, 555\}, \{62, 251\}, \{190, 479\}, \{4, 69\}, \{218, 430\}, \{497, 498\}, \{37, 270\}, \{370, 426\}, \{191, 223\}, \{132, 471\}, \{292, 344\}, \{203, 233\}, \{125, 382\}, \{275, 540\}, \{189, 558\}, \{72, 172\}, \{439, 445\}, \{76, 108\}, \{274, 401\}, \{255, 444\}, \{325, 423\}, \{228, 359\}, \{236, 365\}, \{31, 375\}, \{467, 484\}, \{153, 352\}, \{94, 518\}, \{415, 468\}, \{286, 489\}, \{20, 477\}, \{84, 140\}, \{420, 475\}, \{222, 263\}, \{208, 213\}, \{300, 538\}, \{2, 323\}, \{51, 179\}, \{9, 123\}, \{393, 427\}, \{36, 264\}, \{30, 481\}, \{65, 432\}, \{361, 521\}, \{231, 463\}, \{117, 174\} \};$

$S_2 = \{ \{61, 514\}, \{93, 196\}, \{68, 295\}, \{14, 322\}, \{121, 166\}, \{134, 286\}, \{285, 445\}, \{143, 534\}, \{96, 112\}, \{63, 78\}, \{418, 508\}, \{235, 433\}, \{480, 488\}, \{276, 453\}, \{375, 482\}, \{299, 425\}, \{231, 412\}, \{416, 503\}, \{287, 528\}, \{188, 237\}, \{177, 537\}, \{123, 176\}, \{25, 264\}, \{9, 420\}, \{157, 173\}, \{27, 371\}, \{120, 165\}, \{195, 215\}, \{81, 325\}, \{10, 458\}, \{309, 314\}, \{15, 160\}, \{26, 540\}, \{50, 400\}, \{113, 158\}, \{91, 290\}, \{66, 427\}, \{111, 209\}, \{48, 248\}, \{118, 468\}, \{116, 496\}, \{232, 557\}, \{442, 464\}, \{167, 404\}, \{186, 318\}, \{58, 358\}, \{102, 397\}, \{94, 421\}, \{92, 292\}, \{38, 254\}, \{483, 484\}, \{335, 355\}, \{133, 553\}, \{293, 362\}, \{52, 399\}, \{55, 169\}, \{184, 348\}, \{114, 478\}, \{253, 414\}, \{529, 554\}, \{183, 463\}, \{179, 349\}, \{42, 151\}, \{31, 271\},$

{7, 198}, {117, 429}, {30, 341}, {54, 455}, {35, 282}, {303, 387}, {127, 320}, {40, 208}, {126, 561}, {95, 152}, {181, 440}, {72, 338}, {345, 383}, {305, 376}, {37, 302}, {460, 462}, {12, 446}, {187, 234}, {18, 451}, {178, 401}, {17, 261}, {283, 479}, {137, 192}, {44, 542}, {70, 87}, {221, 319}, {103, 443}, {526, 559}, {28, 256}, {69, 109}, {226, 477}, {145, 409}, {185, 247} };

$S_3 = \{ \{125, 356\}, \{418, 458\}, \{445, 521\}, \{419, 491\}, \{243, 409\}, \{249, 262\}, \{200, 527\}, \{211, 253\}, \{77, 287\}, \{280, 420\}, \{268, 394\}, \{213, 359\}, \{28, 415\}, \{85, 92\}, \{20, 164\}, \{24, 456\}, \{27, 291\}, \{73, 378\}, \{159, 275\}, \{166, 395\}, \{319, 416\}, \{140, 526\}, \{126, 285\}, \{176, 344\}, \{69, 81\}, \{82, 428\}, \{255, 303\}, \{144, 209\}, \{365, 522\}, \{242, 444\}, \{46, 109\}, \{506, 545\}, \{232, 304\}, \{13, 463\}, \{182, 487\}, \{308, 396\}, \{116, 244\}, \{502, 520\}, \{49, 155\}, \{6, 289\}, \{237, 296\}, \{528, 542\}, \{47, 530\}, \{95, 560\}, \{199, 230\}, \{410, 550\}, \{151, 316\}, \{181, 482\}, \{115, 143\}, \{175, 393\}, \{120, 523\}, \{228, 277\}, \{154, 338\}, \{377, 474\}, \{75, 322\}, \{439, 533\}, \{183, 459\}, \{99, 273\}, \{72, 100\}, \{236, 431\}, \{357, 504\}, \{261, 292\}, \{229, 460\}, \{345, 558\}, \{270, 385\}, \{245, 256\}, \{80, 246\}, \{65, 538\}, \{366, 477\}, \{78, 461\}, \{70, 161\}, \{212, 266\}, \{311, 358\}, \{14, 293\}, \{91, 490\}, \{30, 382\}, \{483, 501\}, \{340, 525\}, \{328, 484\}, \{215, 429\}, \{127, 185\}, \{170, 188\}, \{165, 434\}, \{96, 333\}, \{74, 341\}, \{122, 426\}, \{326, 430\}, \{128, 208\}, \{254, 465\}, \{198, 274\}, \{33, 224\}, \{392, 531\}, \{57, 475\} \}.$

$n = 69, \mathcal{A} = \bigcup_{i=1}^3 \{ \{i, x, y\} : \{x, y\} \in S_i \},$

$S_1 = \{ \{239, 535\}, \{13, 63\}, \{249, 563\}, \{469, 524\}, \{127, 369\}, \{316, 715\}, \{212, 513\}, \{371, 485\}, \{110, 459\}, \{103, 231\}, \{179, 645\}, \{60, 759\}, \{267, 630\}, \{402, 561\}, \{589, 704\}, \{284, 305\}, \{25, 322\}, \{490, 504\}, \{169, 384\}, \{258, 664\}, \{279, 565\}, \{621, 754\}, \{108, 257\}, \{532, 716\}, \{156, 302\}, \{26, 638\}, \{349, 562\}, \{290, 541\}, \{512, 613\}, \{531, 633\}, \{383, 597\}, \{368, 430\}, \{235, 625\}, \{618, 699\}, \{649, 757\}, \{648, 733\}, \{523, 560\}, \{217, 718\}, \{323, 673\}, \{45, 171\}, \{176, 423\}, \{247, 707\}, \{37, 388\}, \{117, 221\}, \{372, 694\}, \{101, 334\}, \{2, 377\}, \{416, 519\}, \{79, 270\}, \{324, 345\}, \{612, 671\}, \{262, 709\}, \{441, 538\}, \{536, 577\}, \{162, 701\}, \{339, 741\}, \{71, 745\}, \{242, 453\}, \{81, 351\}, \{264, 362\}, \{123, 392\}, \{380, 444\}, \{57, 672\}, \{269, 274\}, \{40, 584\}, \{131, 616\}, \{23, 357\}, \{272, 424\}, \{330, 404\}, \{367, 725\}, \{336, 706\}, \{174, 688\}, \{332, 462\}, \{155, 161\}, \{69, 381\}, \{75, 550\}, \{206, 253\}, \{154, 268\}, \{501, 696\}, \{32, 712\}, \{118, 546\}, \{198, 703\}, \{160, 213\}, \{12, 266\}, \{163, 303\}, \{20, 692\}, \{95, 355\}, \{5, 147\}, \{263, 636\}, \{293, 667\}, \{178, 428\}, \{164, 735\}, \{340, 350\}, \{245, 256\}, \{729, 731\}, \{413, 600\}, \{233, 436\}, \{374, 714\}, \{210, 750\}, \{447, 702\}, \{228, 488\}, \{182, 288\}, \{177, 574\}, \{168, 250\}, \{254, 385\}, \{397, 529\}, \{422, 742\}, \{50, 497\}, \{496, 637\}, \{314, 674\}, \{170, 593\}, \{173, 717\}, \{586, 595\}, \{6, 36\}, \{128, 465\}, \{151, 158\}, \{100, 457\}, \{472, 596\}, \{165, 639\}, \{722, 758\}, \{414, 480\}, \{21, 552\}, \{260, 575\}, \{150, 687\}, \{483, 528\}, \{149, 329\}, \{39, 521\}, \{744, 753\}, \{34, 551\}, \{82, 685\}, \{328, 752\}, \{3, 343\} \};$

$S_2 = \{ \{504, 716\}, \{74, 164\}, \{32, 572\}, \{635, 660\}, \{119, 655\}, \{44, 254\}, \{184, 703\}, \{50, 223\}, \{53, 670\}, \{181, 481\}, \{41, 200\}, \{77, 142\}, \{183, 480\}, \{113, 131\}, \{136, 415\}, \{534, 596\}, \{147, 316\}, \{348, 681\}, \{237, 284\}, \{272, 745\}, \{110, 570\}, \{138, 578\}, \{148, 443\}, \{96, 574\}, \{488, 651\}, \{177, 482\}, \{22, 453\}, \{287, 658\}, \{193, 239\}, \{590, 727\}, \{544, 748\}, \{283, 738\}, \{365, 397\}, \{427, 617\}, \{48, 539\}, \{86, 160\}, \{80, 548\}, \{410, 522\}, \{292, 389\}, \{333, 591\}, \{150, 567\}, \{36, 218\}, \{174, 426\}, \{24, 649\}, \{305, 616\}, \{315, 633\}, \{95, 109\}, \{331, 417\}, \{107, 204\}, \{461, 559\}, \{145, 229\}, \{562, 741\}, \{558, 694\}, \{155, 420\}, \{122, 543\}, \{104, 290\}, \{251, 662\}, \{87, 622\}, \{127, 225\}, \{230, 231\}, \{450, 629\}, \{286, 611\}, \{438, 560\}, \{236, 328\}, \{234, 484\}, \{59, 364\}, \{159, 728\}, \{214, 756\}, \{332, 390\}, \{106, 547\}, \{121, 310\}, \{383, 392\}, \{102, 157\}, \{78, 378\}, \{156, 361\}, \{363, 494\}, \{33, 530\}, \{68, 219\}, \{143, 158\}, \{515, 685\}, \{532, 665\}, \{69, 247\}, \{421, 516\}, \{73, 163\}, \{198, 356\}, \{5, 243\}, \{151, 179\}, \{319, 697\}, \{263, 675\}, \{597, 729\}, \{304, 325\}, \{56, 83\}, \{271, 553\}, \{642, 679\}, \{258, 721\}, \{587, 747\}, \{552, 723\}, \{648, 701\}, \{266, 521\}, \{429, 610\}, \{39, 93\}, \{329, 369\}, \{28, 208\}, \{45, 245\}, \{295, 638\}, \{54, 569\}, \{296, 424\}, \{514, 557\}, \{366, 726\}, \{440, 464\}, \{341, 404\}, \{536, 669\}, \{171, 535\}, \{259, 311\}, \{555, 678\} \};$

$S_3 = \{ \{415, 456\}, \{320, 713\}, \{286, 407\}, \{276, 355\}, \{511, 746\}, \{558, 738\}, \{36, 356\}, \{269, 463\}, \{53, 377\}, \{362, 715\}, \{73, 294\}, \{554, 689\}, \{437, 625\}, \{94, 635\}, \{268, 469\}, \{148,$

720}, {59, 313}, {246, 623}, {419, 752}, {87, 758}, {178, 596}, {117, 663}, {229, 584}, {75, 529}, {281, 576}, {68, 661}, {568, 600}, {436, 509}, {236, 697}, {47, 490}, {186, 641}, {329, 528}, {234, 297}, {31, 227}, {39, 79}, {108, 251}, {188, 238}, {143, 734}, {29, 482}, {267, 755}, {305, 612}, {432, 445}, {144, 247}, {123, 679}, {691, 754}, {440, 453}, {204, 434}, {37, 195}, {74, 638}, {226, 287}, {52, 112}, {99, 442}, {285, 522}, {54, 290}, {271, 743}, {140, 370}, {594, 683}, {403, 426}, {278, 536}, {154, 670}, {556, 685}, {193, 643}, {70, 376}, {155, 513}, {32, 673}, {98, 291}, {327, 578}, {452, 464}, {371, 487}, {128, 205}, {332, 684}, {557, 637}, {365, 448}, {299, 354}, {90, 266}, {44, 157}, {160, 479}, {295, 383}, {21, 705}, {11, 121}, {337, 631}, {14, 109}, {292, 644}, {170, 619}, {454, 647}, {114, 330}, {264, 526}, {15, 393}, {358, 756}, {580, 610}, {249, 470}, {55, 175}, {397, 502}, {177, 736}, {132, 342}, {228, 245}, {388, 606}, {93, 270}, {158, 523}, {328, 669}, {187, 597}, {110, 733}, {25, 138}, {156, 446}, {306, 472}, {196, 503}, {71, 488}, {97, 408}, {359, 645}, {201, 473}, {103, 577}, {200, 223}, {6, 277}, {318, 745}, {241, 747}, {80, 150}, {199, 680}, {387, 414}, {368, 385}, {282, 654}, {119, 369}, {441, 652}, {192, 353}, {392, 658}, {137, 723}, {653, 714}, {7, 574}}.

References

- [1] F. E. Bennett, R. Wei and L. Zhu, *Resolvable Mendelsohn triple systems with equal sized holes*, J. Combin. Designs **5** (1997), 329-340.
- [2] S. Blake-Wilson and K. Phelps, *Constant weight codes and group divisible design*, Designs, Codes and Cryptography **16** (1999), 11-27.
- [3] K. Chen, G. Ge and L. Zhu, *Generalized Steiner triple systems with group size five*, preprint.
- [4] K. Chen, G. Ge and L. Zhu, *Starters and related codes*, preprint.
- [5] T. Etzion, *Optimal constant weight codes over Z_k and generalized designs*, Discrete Math. **169** (1997), 55-82.
- [6] G. Ge and D. Wu, *Generalized Steiner triple systems with group size ten*, preprint.
- [7] K. Phelps and C. Yin, *Generalized Steiner systems with block three and group size $g \equiv 3 \pmod{6}$* , J. Combin. Designs **5** (1997), 417-432.
- [8] K. Phelps and C. Yin, *Generalized Steiner systems with block three and group size four*, Ars Combin. to appear.
- [9] D. Wu, L. Zhu and G. Ge, *Generalized Steiner triple systems with group size $g = 7, 8$* , preprint.

(Received 16/10/98)

