

A q -analogue of a formula of Hernandez obtained by inverting a result of Dilcher

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Abstract

We prove a q -analogue of the formula

$$\sum_{1 \leq k \leq n} \binom{n}{k} (-1)^{k-1} \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_m = k} \frac{1}{i_1 i_2 \dots i_m} = \sum_{1 \leq k \leq n} \frac{1}{k^m}$$

by inverting a formula due to Dilcher.

1 The identities

Hernández in [6] proved the following identity:

$$\sum_{1 \leq k \leq n} \binom{n}{k} (-1)^{k-1} \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_m = k} \frac{1}{i_1 i_2 \dots i_m} = \sum_{1 \leq k \leq n} \frac{1}{k^m}. \quad (1)$$

However this identity does not really require a proof, since we will show that it is just an inverted form of an identity of Dilcher [2];

$$\sum_{1 \leq k \leq n} \binom{n}{k} (-1)^{k-1} \frac{1}{k^m} = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq n} \frac{1}{i_1 i_2 \dots i_m}. \quad (2)$$

For $k \geq 1$, define

$$a_k := - \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_m = k} \frac{1}{i_1 i_2 \dots i_m} \quad \text{and} \quad b_k := \frac{1}{k^m},$$

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then the identities are

$$\begin{aligned}\sum_{1 \leq k \leq n} \binom{n}{k} (-1)^k a_k &= \sum_{1 \leq k \leq n} b_k, \\ \sum_{1 \leq k \leq n} \binom{n}{k} (-1)^k b_k &= \sum_{1 \leq k \leq n} a_k.\end{aligned}\tag{3}$$

These are *inverse relations*, as can be seen by introducing ordinary generating functions $A(z) = \sum a_n z^n$ and $B(z) = \sum b_n z^n$. Then (3) gives immediately

$$\begin{aligned}A\left(\frac{z}{z-1}\right) &= B(z), \\ B\left(\frac{z}{z-1}\right) &= A(z).\end{aligned}$$

However

$$w = \frac{z}{z-1} \longleftrightarrow z = \frac{w}{w-1},$$

and the proof is finished. An alternative argument that will be useful in the sequel when we do the q -analogue, is as follows. We take differences in (3) of the lines indexed with n resp. $n-1$; then we have to prove that

$$b_n = \sum_{1 \leq k \leq n} \binom{n-1}{k-1} (-1)^k a_k \iff a_n = \sum_{1 \leq k \leq n} \binom{n-1}{k-1} (-1)^k b_k.$$

Now in this form this is a traditional inverse relation; see e. g. [7]. An explicit argument will follow in the next section for the q -instance.

We note that Dilcher's sum appears also in disguised form in [3].

2 A q -analogue

Dilcher's formula (2) is a corollary of his elegant q -version;

$$\sum_{1 \leq k \leq n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{k-1} \frac{q^{\binom{k+1}{2} + (m-1)k}}{(1-q^k)^m} = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq n} \frac{q^{i_1}}{1-q^{i_1}} \cdots \frac{q^{i_m}}{1-q^{i_m}}.$$

Here, $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the Gaussian polynomial

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$$

with

$$(x; q)_n := (1 - x)(1 - xq) \dots (1 - xq^{n-1}).$$

Apart from Dilcher's paper [2], the article [1] is also of some relevance in this context. Therefore it is a natural question to find a q -analogue of Hernández' formula, or, what amounts to the same, to find the appropriate inverse relations for the q -analogues. We state them in the following lemma.

Lemma 1.

$$\begin{aligned} \sum_{1 \leq k \leq n} b_k &= \sum_{1 \leq k \leq n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} a_k, \\ \sum_{1 \leq k \leq n} q^{-k} a_k &= \sum_{1 \leq k \leq n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-kn + \binom{k}{2}} b_k. \end{aligned} \quad (4)$$

Proof. Again, taking differences in (4), we have to prove that

$$b_n = \sum_{1 \leq k \leq n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} a_k \iff a_n = \sum_{1 \leq k \leq n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^k q^{-(n-1)k + \binom{k}{2}} b_k. \quad (5)$$

However, after trivial modifications, this is the inverse pair reported in [5], exercise (2.6.6 (b)). Credits for it are given to Carlitz, Szegő, and Rogers; compare the references in [5].

After a first version of this note was circulated, O. Warnaar kindly informed me that this lemma would also follow from results in [4]. $\square$

We would like to remark that an alternative formulation can be given in terms of matrices of *connection coefficients*.

This can be done in terms of the original formulæ (4), but looks much nicer when referring to (5):

Define matrices

$$T := \left[\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} \right]_{n,k}, \quad U := \left[\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^k q^{-(n-1)k + \binom{k}{2}} \right]_{n,k},$$

then

$$TU = I.$$

Theorem 2. [q -analogue of Hernández' formula]

$$\sum_{1 \leq k \leq n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{k-1} q^{-kn + \binom{k}{2}} \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_m = k} \frac{q^{i_1}}{1 - q^{i_1}} \dots \frac{q^{i_m}}{1 - q^{i_m}} = \sum_{1 \leq k \leq n} \frac{q^{k(m-1)}}{(1 - q^k)^m}.$$

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