

New infinite classes of 1-factorizations of complete graphs

Midori Kobayashi, Nobuaki Mutoh

School of Administration and Informatics
University of Shizuoka
Yada, Shizuoka 422-8526, Japan

Gisaku Nakamura

Tokai University
Shibuya-ku, Tokyo, 151-0063, Japan

Abstract

Some classes of 1-factorizations of complete graphs are known. They are GK_{2n} , AK_{2n} , WK_{2n} and their variations, and automorphism-free 1-factorizations. In this paper, for any positive integer t , we construct new 1-factorizations N_tK_{2n} which are defined for all $2n$ with $2n \geq 6t$. They also have some variations.

1. INTRODUCTION

Let $K_{2n} = (V_{2n}, E_{2n})$ be the complete graph on $2n$ vertices. Put $m = 2n - 1$. A 1-factor of K_{2n} is a set of n edges that partition the vertex set V_{2n} . A 1-factorization of K_{2n} is a set of m 1-factors that partition the set of edges E_{2n} .

Some 1-factorizations of K_{2n} are known [1,2,3,4]. They have been dubbed GK_{2n} , WK_{2n} and AK_{2n} . They have some variations; $G'K_{2n}$, $W'K_{2n}$, $W''K_{2n}$, $W^{(1)}K_{2n}$, $W^{(1)'}K_{2n}$, $W^{(1)''}K_{2n}$ (Figures 1 to 10). Among these, GK_{2n} is the most simple and famous 1-factorization of K_{2n} ; it is called the patterned 1-factorization.

Moreover, it is known there exists an automorphism-free 1-factorization of K_{2n} ($n \geq 5$) [1].

These 1-factorizations are defined for every $2n$ except a few small $2n$. On the other hand, not for every $2n$, various 1-factorizations have been constructed; for example, cyclic 1-factorizations when $2n \neq 2^k$, geometric 1-factorizations when $2n = 2^k$ and affine 1-factorizations when $2n = 3^k + 1$ ([2], p604).

In this paper, for any positive integer t , we construct a new 1-factorization N_tK_{2n} which is defined for all $2n$ with $2n \geq 6t$. And we show their variations; N'_tK_{2n} , N''_tK_{2n} , $N_t^{(1)}K_{2n}$, $N_t^{(1)'}K_{2n}$, $N_t^{(1)''}K_{2n}$.

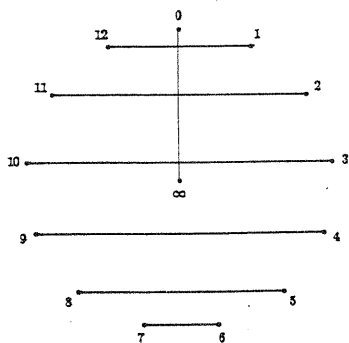


Figure 1: GK_{14}

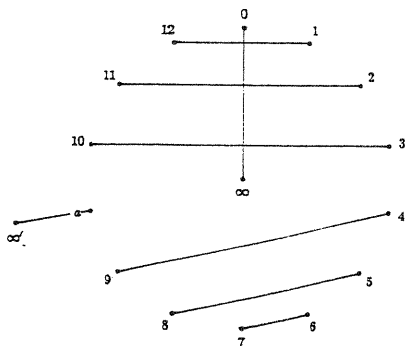


Figure 2: $G'K_{16}$

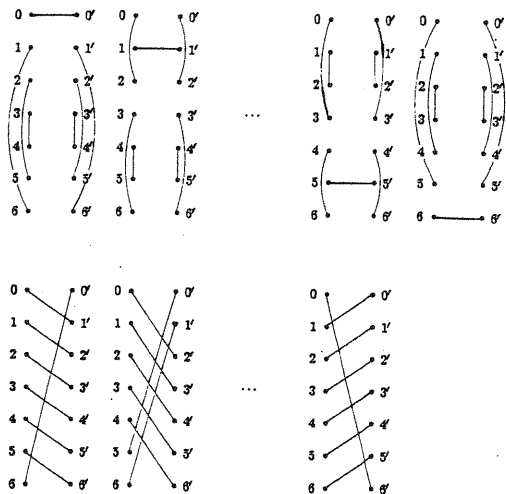


Figure 3: AK_{14}

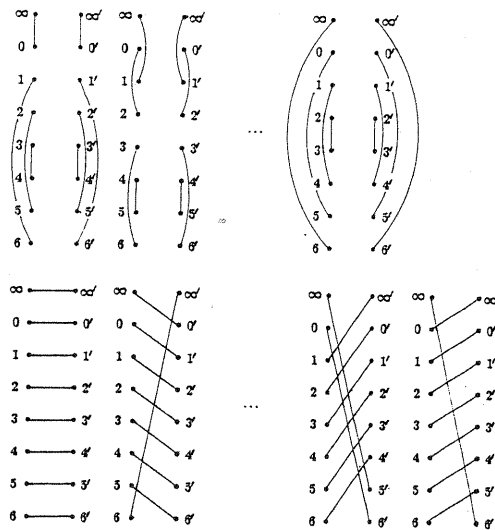


Figure 4: AK_{16}

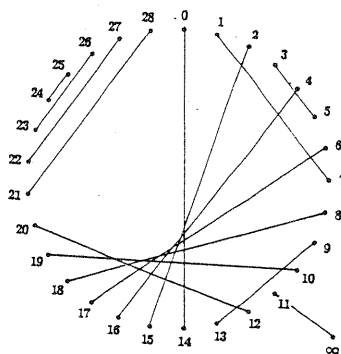


Figure 5: WK_{30}

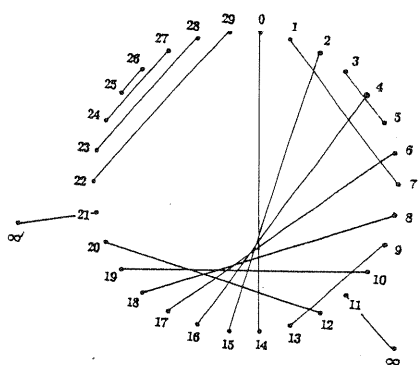


Figure 6: $W'K_{32}$

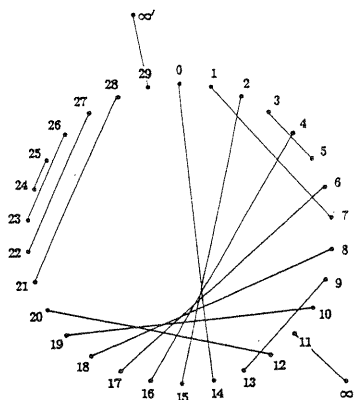


Figure 7: $W''K_{32}$

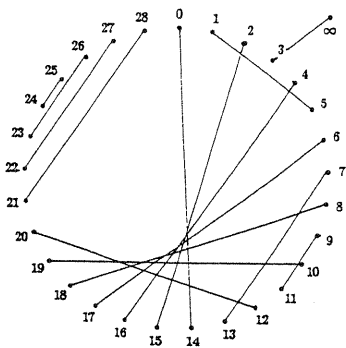


Figure 8: $W^{(1)}K_{30}$

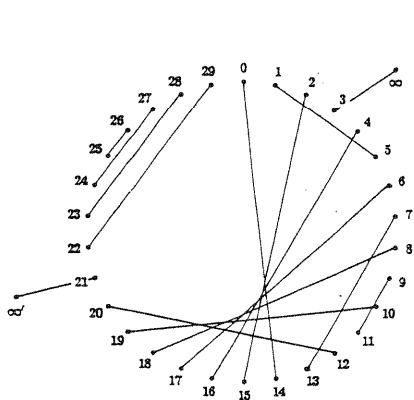


Figure 9: $W^{(1)'}K_{32}$

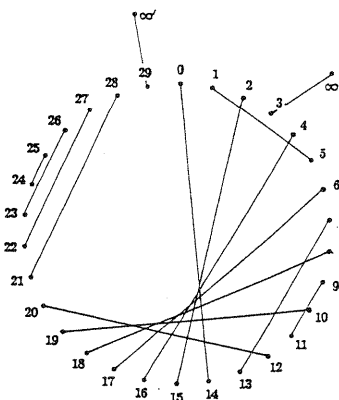


Figure 10: $W^{(1)''}K_{32}$

2. PRELIMINARIES

Put $r = n - 1$. Assume the vertices are labeled as $V_{2n} = \{\infty\} \cup \{0, 1, 2, \dots, m-1\} = \{\infty\} \cup Z_m$. Let σ be the vertex permutation $\sigma = (\infty)(0 \ 1 \ 2 \ \dots \ m-1)$ and put $\Sigma = \langle \sigma \rangle$. Then for any edge $\{a, b\}$ of K_{2n} , we define the length of the edge with respect to Σ :

$$d(\{a, b\}) = \begin{cases} \min\{|b - a|, m - |b - a|\} & (a, b \neq \infty) \\ \infty & (\text{otherwise}) \end{cases}$$

We note that $1 \leq d(\{a, b\}) \leq r$ when $a, b \neq \infty$.

A starter of Z_m is a set of edges $S = \{\{v_i, w_i\} \in E_{2n} \mid 1 \leq i \leq r\}$ such that the set of vertices of S is $V_{2n} \setminus \{\infty, 0\}$ and $d(S) = \{1, 2, \dots, r\}$. If S is a starter of Z_m , $S' = S \cup \{\{\infty, 0\}\}$ is called a starter 1-factor of K_{2n} , and we obtain a 1-factorization of K_{2n} by rotating S' according to σ , that is, $\Sigma S' = \{\sigma^i S' \mid 0 \leq i \leq m-1\}$.

For any positive integer t , we will construct starters of K_{2n} in sections 3 and 4.

3. $N_t K_{2n}$ WHEN t IS ODD

Assume t is odd ≥ 1 and $2n \geq 6t$. Put $s = (t-1)/2$ and put

- (1) $V_I = \{a_1, a_2, \dots, a_t, a'_1, a'_2, \dots, a'_t\},$
 $E_I = \{\{a_1, a'_1\}, \{a_2, a'_2\}, \dots, \{a_t, a'_t\}\},$
- (2) $V_{II} = \{b_1, b_2, \dots, b_t, b'_1, b'_2, \dots, b'_t\},$
 $E_{II} = \{\{b_1, b'_{s+1}\}, \{b_2, b'_{s+2}\}, \dots, \{b_{s+1}, b'_t\}\} \cup \{\{b_{s+2}, b'_1\}, \{b_{s+3}, b'_2\}, \dots, \{b_t, b'_s\}\},$
- (3) $V_{III} = \{\infty, 0, c_1, c_2, \dots, c_s, c'_1, c'_2, \dots, c'_s\},$
 $E_{III} = \{\{\infty, 0\}, \{c_1, c'_1\}, \{c_2, c'_2\}, \dots, \{c_s, c'_s\}\},$
- (4) $V_{IV} = \{d_1, d_2, \dots, d_s, d'_1, d'_2, \dots, d'_s\},$
 $E_{IV} = \{\{d_1, d'_1\}, \{d_2, d'_2\}, \dots, \{d_s, d'_s\}\}.$
- (5) When $r(=n-1)$ is odd, put $u = (r-3t+2)/2$, $v = (r-3t)/2$. When r is even, put $u = v = (r-3t+1)/2$. And put
 $V_{V_1} = \{e_1, e_2, \dots, e_u, e'_1, e'_2, \dots, e'_u\}, E_{V_1} = \{\{e_1, e'_1\}, \{e_2, e'_2\}, \dots, \{e_u, e'_u\}\},$
 $V_{V_2} = \{f_1, f_2, \dots, f_v, f'_1, f'_2, \dots, f'_v\}, E_{V_2} = \{\{f_1, f'_1\}, \{f_2, f'_2\}, \dots, \{f_v, f'_v\}\}.$

We define V_{2n} and σ as follows;

$$V_{2n} = \{\infty\} \cup V_I \cup V_{II} \cup V_{III} \cup V_{IV} \cup V_{V_1} \cup V_{V_2},$$

$$\begin{aligned}
\sigma = & (\infty)(f_v f_{v-1} \cdots f_1 \\
& c_s c_{s-1} \cdots c_1 0 c'_1 c'_2 \cdots c'_s \\
& b_1 b_2 \cdots b_t \\
& f'_1 f'_2 \cdots f'_v \\
& a_t a_{t-1} \cdots a_1 a'_1 a'_2 \cdots a'_t \\
& e'_u e'_{u-1} \cdots e'_1 \\
& d'_s d'_{s-1} \cdots d'_1 \\
& b'_t b'_{t-1} \cdots b'_1 \\
& d_1 d_2 \cdots d_s \\
& e_1 e_2 \cdots e_u)
\end{aligned}$$

and put $\Sigma = \langle \sigma \rangle$ (see Figure 11).

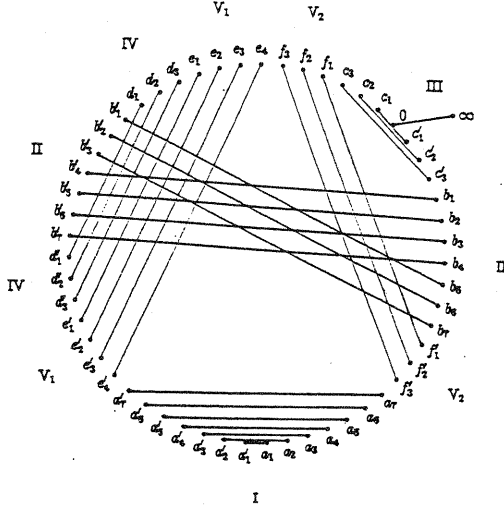


Figure 11: N_7K_{56}

Proposition 3.1 Assume t is odd ≥ 1 and $2n \geq 6t$. Then

$$S_t = E_I \cup E_{II} \cup E_{III} \cup E_{IV} \cup E_{V_1} \cup E_{V_2}$$

is a starter 1-factor of K_{2n} with respect to $\Sigma = \langle \sigma \rangle$.

Proof. We need only show that the lengths of the edges of S_t are all different.

- (1) $|E_I| = t$. $d(E_I) = \{1, 3, 5, \dots, 2t-1\}$.
- (2) $|E_{II}| = t$. Since $d(\{b_1, b'_{s+1}\}) = |V_{III}| - 1 + (|V_{V_2}| + |V_{V_1}| + |V_{IV}|)/2 + s + 1 = r - t + 1$, we have $d(E_{II}) = \{r - t + 1, r - t + 2, \dots, r\}$.
- (3) $|E_{III}| = s + 1$. $d(E_{III}) = \{\infty, 2, 4, 6, \dots, t-1\}$.

$$(4) |E_{IV}| = s. d(E_{IV}) = \{t+1, t+3, t+5, \dots, 2t-2\}.$$

We have $|E_I \cup E_{II} \cup E_{III} \cup E_{IV}| = 3t$, so it doesn't depend on $2n$. (Note that we are assuming $2n \geq 6t$.) Only E_{V_1} and E_{V_2} depend on $2n$.

$$(5) |E_{V_1}| = u \text{ and } |E_{V_2}| = v. \text{ When } r \text{ is odd, } d(E_{V_1}) = \{2t, 2t+2, 2t+4, \dots, r-t\} \text{ and } d(E_{V_2}) = \{2t+1, 2t+3, 2t+5, \dots, r-t-1\}$$

When r is even, $d(E_{V_1}) = \{2t, 2t+2, 2t+4, \dots, r-t-1\}$ and $d(E_{V_2}) = \{2t+1, 2t+3, 2t+5, \dots, r-t\}$.

Therefore $d(S_t) = \{\infty, 1, 2, \dots, r\}$, from which it follows that S_t is a starter 1-factor of K_{2n} .

4. $N_t K_{2n}$ WHEN t IS EVEN

Assume t is even ≥ 2 and $2n \geq 6t$. Put $s = t/2$ and put

$$(1) V_I = \{a_1, a_2, \dots, a_t, a'_1, a'_2, \dots, a'_t\}, \\ E_I = \{\{a_1, a'_1\}, \{a_2, a'_2\}, \dots, \{a_t, a'_t\}\},$$

$$(2) V_{II} = \{b_0, b_1, b_2, \dots, b_{t-1}, b'_0, b'_1, b'_2, \dots, b'_{t-1}\}, \\ E_{II} = \{\{b_0, b'_0\}\} \cup \{\{b_1, b'_s\}, \{b_2, b'_{s+1}\}, \dots, \{b_s, b'_{t-1}\}\} \\ \cup \{\{b_{s+1}, b'_1\}, \{b_{s+2}, b'_2\}, \dots, \{b_{t-1}, b'_{s-1}\}\},$$

$$(3) V_{III} = \{\infty, 0, c_1, c_2, \dots, c_{s-1}, c'_1, c'_2, \dots, c'_{s-1}\}, \\ E_{III} = \{\{\infty, 0\}, \{c_1, c'_1\}, \{c_2, c'_2\}, \dots, \{c_{s-1}, c'_{s-1}\}\},$$

$$(4) V_{IV} = \{d_1, d_2, \dots, d_s, d'_1, d'_2, \dots, d'_s\}, \\ E_{IV} = \{\{d_1, d'_1\}, \{d_2, d'_2\}, \dots, \{d_s, d'_s\}\}.$$

$$(5) \text{ When } r \text{ is odd, put } u = v = (r-3t+1)/2. \text{ When } r \text{ is even, put } u = (r-3t)/2, \\ v = (r-3t+2)/2. \text{ Put}$$

$$V_{V_1} = \{e_1, e_2, \dots, e_u, e'_1, e'_2, \dots, e'_u\}, E_{V_1} = \{\{e_1, e'_1\}, \{e_2, e'_2\}, \dots, \{e_u, e'_u\}\}, \\ V_{V_2} = \{f_1, f_2, \dots, f_v, f'_1, f'_2, \dots, f'_v\}, E_{V_2} = \{\{f_1, f'_1\}, \{f_2, f'_2\}, \dots, \{f_v, f'_v\}\}.$$

We define V_{2n} and σ as follows;

$$V_{2n} = \{\infty\} \cup V_I \cup V_{II} \cup V_{III} \cup V_{IV} \cup V_{V_1} \cup V_{V_2}$$

$$\sigma = (\infty)(f_v \ f_{v-1} \ \dots \ f_1 \\ c_{s-1} \ c_{s-2} \ \dots \ c_1 \ 0 \ c'_1 \ c'_2 \ \dots \ c'_{s-1} \\ b_0 \ b_1 \ b_2 \ \dots \ b_{t-1} \\ f'_1 \ f'_2 \ \dots \ f'_v \\ a_t \ a_{t-1} \ \dots \ a_1 \ a'_1 \ a'_2 \ \dots \ a'_t \\ e'_u \ e'_{u-1} \ \dots \ e'_1 \\ b'_0 \\ d'_s \ d'_{s-1} \ \dots \ d'_1 \\ b'_{t-1} \ b'_{t-2} \ \dots \ b'_1 \\ d_1 \ d_2 \ \dots \ d_s \\ e_1 \ e_2 \ \dots \ e_u)$$

and put $\Sigma = \langle \sigma \rangle$ (see Figure 12).

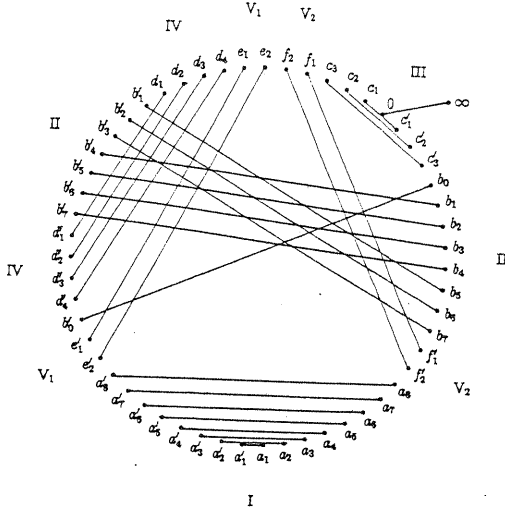


Figure 12: N_8K_{56}

Proposition 4.1 Assume t is even ≥ 2 and $2n \geq 6t$. Then

$$S_t = E_I \cup E_{II} \cup E_{III} \cup E_{IV} \cup E_{V_1} \cup E_{V_2}$$

is a starter 1-factor of K_{2n} with respect to $\Sigma = \langle \sigma \rangle$.

Proof. We need only show that the lengths of the edges of S_t are all different.

- (1) $|E_I| = t$. $d(E_I) = \{1, 3, 5, \dots, 2t-1\}$.
- (2) $|E_{II}| = t$. $d(E_{II}) = \{r-t+1, r-t+2, \dots, r\}$.
- (3) $|E_{III}| = s$. $d(E_{III}) = \{\infty, 2, 4, 6, \dots, t-2\}$.
- (4) $|E_{IV}| = s$. $d(E_{IV}) = \{t, t+2, t+4, \dots, 2t-2\}$.
- (5) $|E_{V_1}| = u$ and $|E_{V_2}| = v$.

When r is odd, $d(E_{V_1}) = \{2t+1, 2t+3, 2t+5, \dots, r-t\}$ and $d(E_{V_2}) = \{2t, 2t+2, 2t+4, \dots, r-t-1\}$.

When r is even, $d(E_{V_1}) = \{2t+1, 2t+3, 2t+5, \dots, r-t-1\}$ and $d(E_{V_2}) = \{2t, 2t+2, 2t+4, \dots, r-t\}$.

Therefore $d(S_t) = \{\infty, 1, 2, \dots, r\}$, from which it follows that S_t is a starter 1-factor of K_{2n} .

We denote by $N_t K_{2n}$ the 1-factorization induced by S_t in Proposition 3.1 and 4.1.

5. EVEN STARTERS

When the vertices are labeled as $V_{2n} = \{\infty, \infty'\} \cup \{0, 1, 2, \dots, 2n-3\} = \{\infty, \infty'\} \cup Z_{2n-2}$, we define an even starter.

Let σ_1 be the vertex permutation $\sigma_1 = (\infty)(\infty')(0 \ 1 \ 2 \ \dots \ 2n-3)$ and put $\Sigma_1 = \langle \sigma_1 \rangle$. For any edge $\{a, b\}$ of K_{2n} we define the length of the edge with respect to Σ_1 :

$$d(\{a, b\}) = \begin{cases} \min\{|b-a|, (2n-2) - |b-a|\} & (a, b \neq \infty, \infty') \\ \infty & (\text{otherwise}) \end{cases}$$

We note that $1 \leq d(\{a, b\}) \leq n-1$ when $a, b \neq \infty, \infty'$.

An even starter of Z_{2n-2} is a set of edges $T = \{\{v_i, w_i\} \in E_{2n} \mid 1 \leq i \leq n-2\}$ such that the set of vertices of T is $V_{2n} \setminus \{\infty, \infty', 0, a\}$ for some $a \in V_{2n}, a \neq \infty, \infty', 0$, and $d(T) = \{1, 2, \dots, n-2\}$.

If T is an even starter of Z_{2n-2} , $T' = T \cup \{\{\infty, 0\}, \{\infty', a\}\}$ is called an even starter 1-factor of K_{2n} . We may call $\sigma_1^i T'$ ($1 \leq i \leq 2n-3$) an even starter 1-factor. For an even starter 1-factor T' , we obtain a 1-factorization of K_{2n} by rotating T' according to σ_1 and adding the pinwheel P ,

$$P = \{\{0, n-1\}, \{1, n\}, \dots, \{n-2, 2n-3\}\},$$

that is, $\Sigma_1 T' \cup \{P\}$ is a 1-factorization of K_{2n} .

Some starter 1-factors of K_{2n} can be extended to even starter 1-factors of K_{2n+2} ([1], p48). Our starter 1-factors constructed in sections 3 and 4 can be extended to even starter 1-factors, also (Figures 13,14). We denote the induced 1-factorizations by $N'_t K_{2n}$ and $N''_t K_{2n}$, respectively.

When $t \geq 3$, $N_t K_{2n}$ has more variations $N_t^{(1)} K_{2n}$, $N_t^{(1)'} K_{2n}$ and $N_t^{(1)''} K_{2n}$ (Figures 15 to 18).

Finally, we should mention whether the 1-factorizations constructed in this paper are new, i.e., not isomorphic to known 1-factorizations. For example, when $2n = 20, 22$, the t satisfying $6t \leq 2n$ are $t = 1, 2, 3$; so K_{2n} has GK_{2n} , $G'K_{2n}$, AK_{2n} , WK_{2n} , $W'K_{2n}$, $W''K_{2n}$, $W^{(1)}K_{2n}$, $W^{(1)'}K_{2n}$, $W^{(1)''}K_{2n}$, $N_t K_{2n}$, $N'_t K_{2n}$, $N''_t K_{2n}$ ($t = 1, 2, 3$), $N_3^{(1)} K_{2n}$, $N_3^{(1)'} K_{2n}$, $N_3^{(1)''} K_{2n}$. It is shown that these 1-factorizations are not isomorphic each other with the aid of a computer.

It is not easy to demonstrate in general that the 1-factorizations constructed in this paper are new, but it is clear that there are new 1-factorizations among them because the number of the 1-factorizations of K_{2n} constructed in this paper increases as $2n$ increases.

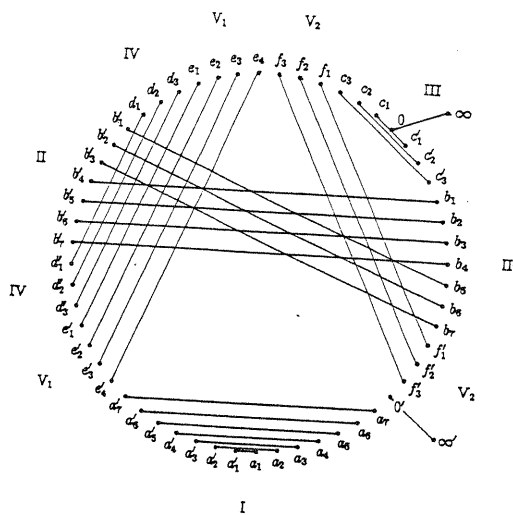


Figure 13: $N'_7 K_{58}$

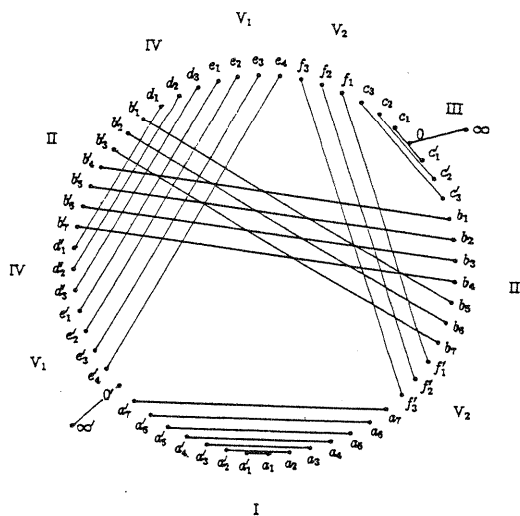


Figure 14: $N''_7 K_{58}$

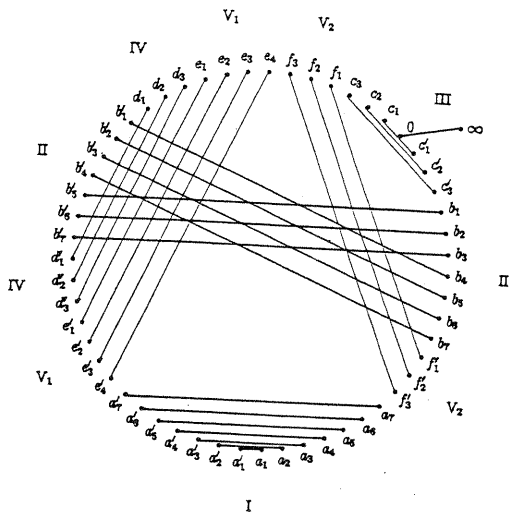


Figure 15: $N_7^{(1)} K_{56}$

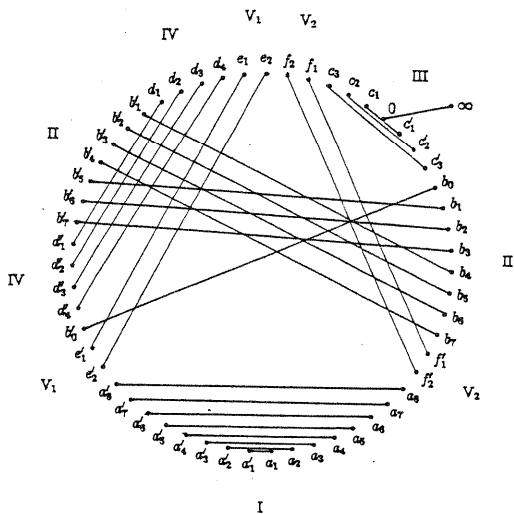


Figure 16: $N_8^{(1)} K_{56}$

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(Received 2/8/99)

