Evenly partite bigraph-factorization of symmetric complete tripartite digraphs

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Abstract

We show that a necessary and sufficient condition for the existence of a $\bar{K}_{p,2q}$ - factorization of the symmetric complete tripartite digraph $K^*_{n_1,n_2,n_3}$ is (i) $n_1 = n_2 = n_3 \equiv 0 \pmod p$ for $p = q$, (ii) $n_1 = n_2 = n_3 \equiv 0 \pmod{dp'q'(p' + 2q')}$ for $p \neq q$ and p' odd, (iii) $n_1 = n_2 = n_3 \equiv 0 \pmod{dp'q'(p' + 2q')/2}$ for $p \neq q$ and p' even, where $d = (p, q)$, $p' = p/d$, $q' = q/d$.

1. Introduction

Let $K^*_{n_1,n_2,n_3}$ denote *the symmetric complete tripartite digraph* with partite sets V_1, V_2, V_3 of n_1, n_2, n_3 vertices each, and let $\bar{K}_{p,2q}$ denote *the evenly partite directed bigraph* in which all arcs are directed away from p start-vertices to $2q$ end-vertices such that the start-vertices are in V_i and q end-vertices are in V_{j_1} and q end-vertices are in V_{j_2} with $\{i, j_1, j_2\} = \{1, 2, 3\}$. A spanning subgraph F of $K^*_{n_1,n_2,n_3}$ is called a $\bar{K}_{p,2q}$ - *factor* if each component of F is $\bar{K}_{p,2q}$. If $K^*_{n_1,n_2,n_3}$ is expressed as an arc-disjoint sum of $\bar{K}_{p,2q}$ - factors, then this sum is called a $\bar{K}_{p,2q}$ - *factorization* of $K^*_{n_1,n_2,n_3}$.
In this paper, it is shown that a necessary and sufficient condition for the existence of such a factorization is (i) $n_1 = n_2 = n_3 \equiv 0 \pmod p$ for $p = q$, (ii) $n_1 = n_2 = n_3 \equiv 0$ $(\text{mod } dp'q'(p'+2q'))$ for $p \neq q$ and p' odd, (iii) $n_1 = n_2 = n_3 \equiv 0$ $(\text{mod } dp'q'(p'+2q')/2)$ for $p \neq q$ and p' even, where $d = (p, q)$, $p' = p/d$, $q' = q/d$.

Let K_{n_1,n_2}, $K^*_{n_1,n_2}$, K_{n_1,n_2,n_3}, $K^*_{n_1,n_2,n_3}$, and $K^*_{n_1,n_2,\ldots,n_m}$ denote the complete bipartite graph, the symmetric complete bipartite digraph, the complete tripartite graph, the symmetric complete tripartite digraph, and the symmetric complete multipartite digraph, respectively. Let $\hat{C}_k$, $\hat{S}_k$, $\hat{P}_k$, and $\hat{K}_{p,q}$ denote the cycle or the directed cycle, the star or the directed star, the path or the directed path, and the complete bipartite graph or the complete bipartite digraph, respectively, on two partite sets V_i and V_j. Let $\bar{S}_k$ and $\tilde{S}_k$ denote the evenly partite star and semi-evenly partite star, respectively, on three partite sets V_i, V_{j_1}, V_{j_2}. Then the problems of giving necessary and sufficient conditions for $\hat{C}_k$ - factorization of K_{n_1,n_2}, $K^*_{n_1,n_2}$, $K^*_{n_1,n_2,n_3}$, and $K^*_{n_1,n_2,\ldots,n_m}$ have been completely solved by Enomoto, Miyamoto and Ushio[3] and Ushio[12,15]. $\hat{S}_k$ - factorizations of K_{n_1,n_2}, $K^*_{n_1,n_2}$, and $K^*_{n_1,n_2,n_3}$ have been studied by Du[2], Martin[5,6], Ushio and Tsuruno[9], Ushio[14], and Wang[18]. Ushio[11] gives a necessary and sufficient condition for $\hat{S}_k$ - factorization of $K^*_{n_1,n_2}$. Ushio[16,17] gives necessary and sufficient conditions of $\bar{S}_k$ - factorization and $\tilde{S}_k$ - factorization for $K^*_{n_1,n_2,n_3}$. $\hat{P}_k$ - factorization of K_{n_1,n_2} and $K^*_{n_1,n_2}$ have been studied by Ushio and Tsuruno[8], and Ushio[7,10]. $\hat{K}_{p,q}$ - factorization of K_{n_1,n_2} has been studied by Martin[5]. Ushio[13] gives a necessary and sufficient condition for $\hat{K}_{p,q}$ - factorization of $K^*_{n_1,n_2}$. For graph theoretical terms, see [1,4].

2. $\bar{K}_{p,2q}$ - factorization of $K^*_{n_1,n_2,n_3}$

Notation. Given a $\bar{K}_{p,2q}$ - factorization of $K^*_{n_1,n_2,n_3}$, let
r be the number of factors
t be the number of components of each factor
b be the total number of components.
Among r components having vertex x in V_i, let r_{ij} be the number of components whose start-vertices are in V_j.
Among t components of each factor, let t_i be the number of components whose start-vertices are in V_i.

We give the following necessary condition for the existence of a $\bar{K}_{p,2q}$ - factorization of $K^*_{n_1,n_2,n_3}$.

Theorem 1. If $K^*_{n_1,n_2,n_3}$ has a $\bar{K}_{p,2q}$ - factorization, then (i) $n_1 = n_2 = n_3 \equiv 0 \pmod p$ for $p = q$, (ii) $n_1 = n_2 = n_3 \equiv 0 \pmod{dp'q'(p' + 2q')}$ for $p \neq q$ and p' odd, (iii) $n_1 = n_2 = n_3 \equiv 0$ (mod $dp'q'(p' + 2q')/2$) for $p \neq q$ and p' even, where $d = (p,q)$, $p' = p/d$, $q' = q/d$.

Proof. Suppose that $K^*_{n_1,n_2,n_3}$ has a $\bar{K}_{p,2q}$ - factorization. Then $b = (n_1n_2 + n_1n_3 + n_2n_3)/pq$, $t = (n_1+n_2+n_3)/(p+2q)$, $r = b/t = (p+2q)(n_1n_2+n_1n_3+n_2n_3)/pq(n_1 + n_2 + n_3)$.
For a vertex x in V_1, we have $r_{11}q = n_2 = n_3$, $r_{12}p = n_2$, $r_{13}p = n_3$, and $r_{11} + r_{12} + r_{13} = r$. For a vertex x in V_2, we have $r_{22}q = n_1 = n_3$, $r_{21}p = n_1$, $r_{23}p = n_3$, and $r_{21}+r_{22}+r_{23} = r$. For a vertex x in V_3, we have $r_{33}q = n_1 = n_2$, $r_{31}p = n_1$, $r_{32}p = n_2$, and $r_{31} + r_{32} + r_{33} = r$. Therefore, we have $n_1 = n_2 = n_3$. Put $n_1 = n_2 = n_3 = n$. Then $r_{ii} = n/q$, $r_{ij} = n/p$ $(j \neq i)$, $b = 3n^2/pq$, $t = 3n/(p+2q)$, and $r = n(p+2q)/pq$. Moreover, in a factor, we have $pt_1 + qt_2 + qt_3 = qt_1 + pt_2 + qt_3 = qt_1 + qt_2 + pt_3 = n$ and $t_1 + t_2 + t_3 = t$. Therefore, we have $t_1 = t_2 = t_3 = n/(p + 2q)$ for $p \neq q$.
So we have (i) $n_1 = n_2 = n_3 \equiv 0 \pmod p$ for $p = q$, (ii) $n_1 = n_2 = n_3 \equiv 0$ (mod $dp'q'(p' + 2q')$) for $p \neq q$ and p' odd, (iii) $n_1 = n_2 = n_3 \equiv 0$ (mod $dp'q'(p' + 2q')/2$) for $p \neq q$ and p' even, where $d = (p,q)$, $p' = p/d$, $q' = q/d$.

We shall need the following lemma.

Lemma 2. Let G, H and K be digraphs. If G has an H - factorization and H has a K - factorization, then G has a K - factorization.

Proof. Let $E(G) = \bigcup_{i=1}^{r} E(F_i)$ be an H - factorization of G. Let $H_j^{(i)}$ $(1 \le j \le t)$ be the components of F_i. And let $E(H_j^{(i)}) = \bigcup_{k=1}^{s} E(K_k^{(i,j)})$ be a K - factorization of $H_j^{(i)}$. Then $E(G) = \bigcup_{i=1}^{r} \bigcup_{k=1}^{s} E(\bigcup_{j=1}^{t} K_k^{(i,j)})$ is a K - factorization of G.

We prove the following extension theorem, which we use throughout the remainder of this paper.

Theorem 3. If $K_{n,n,n}^*$ has a $\bar{K}_{p,2q}$ - factorization, then $K_{sn,sn,sn}^*$ has a $\bar{K}_{p,2q}$ - factorization.

Proof. Let $K_{q_1,q_2 \oplus q_3}$ denote the tripartite digraph with partite sets U_1,U_2,U_3 of q_1, q_2, q_3 vertices such that all arcs are directed away from q_1 start-vertices in U_1 to q_2 end-vertices in U_2 and q_3 end-vertices in U_3. Then $\bar{K}_{p,2q}$ can be denoted by $K_{p,q\oplus q}$. When $K_{n,n,n}^*$ has a $\bar{K}_{p,2q}$ - factorization, $K_{sn,sn,sn}^*$ has a $K_{sp,sq\oplus sq}$ - factorization. Obviously $K_{sp,sq\oplus sq}$ has a $\bar{K}_{p,2q}$ - factorization. Therefore, by Lemma 2 $K_{sn,sn,sn}^*$ has a $\bar{K}_{p,2q}$ - factorization.

We use the following notation for a $\bar{K}_{p,2q}$.

Notation. For a $\bar{K}_{p,2q}$ with start-vertices $u_1, u_2, ..., u_p$ and end-vertices $v_1, v_2, ... v_q, w_1, w_2, ..., w_q$, we denote it by $(u_1, u_2, ..., u_p; v_1, v_2, ... v_q, w_1, w_2, ..., w_q)$.

We give the following sufficient conditions for the existence of a $\bar{K}_{p,2q}$ - factorization of $K_{n,n,n}^*$.

Theorem 4. When $n \equiv 0 \pmod p$, $K_{n,n,n}^*$ has a $\bar{K}_{p,2p}$ - factorization.

Proof. Put $n = sp$. When $s = 1$, let $V_1 = \{1, 2, ..., p\}$, $V_2 = \{1', 2', ..., p'\}$, and $V_3 = \{1'', 2'', ..., p''\}$. Construct $\bar{K}_{p,2p}$ - factors $F_1 = (V_1; V_2, V_3)$, $F_2 = (V_2; V_1, V_3)$, $F_3 = (V_3; V_1, V_2)$. Then they comprise a $\bar{K}_{p,2p}$ - factorization of $K_{p,p,p}^*$. Applying Theorem 3, $K_{n,n,n}^*$ has a $\bar{K}_{p,2p}$ - factorization.

Theorem 5. Let $d = (p, q)$, $p' = p/d$, $q' = q/d$ for $p \ne q$. When $n \equiv 0$ (mod $dp'q'(p' + 2q')$) and p' odd, $K_{n,n,n}^*$ has a $\bar{K}_{p,2q}$ - factorization.

Proof. Put $n = sdp'q'(p' + 2q')$ and $N = dp'q'(p' + 2q')$. When $s = 1$, let $V_1 = \{1, 2, ..., N\}$, $V_2 = \{1', 2', ..., N'\}$, and $V_3 = \{1'', 2'', ..., N''\}$. Construct $(p'+2q')^2$ $\bar{K}_{p,2q}$ - factors F_{ij} $(i = 1, 2, ..., p' + 2q'; j = 1, 2, ..., p' + 2q')$ as follows:

$F_{ij} = \{$ $((A+1,...,A+p);(B+f+1,...,B+f+q),(C+g+1,...,C+g+q))$
$((A+p+1,...,A+2p);(B+f+q+1,...,B+f+2q),(C+g+q+1,...,C+g+2q))$
...
$((A+(p'q'-1)p+1,...,A+p'q'p);(B+f+(p'q'-1)q+1,...,B+f+p'q'q),(C+g+(p'q'-1)q+1,...,C+g+p'q'q))$
$((B+1,...,B+p);(C+f+1,...,C+f+q),(A+g+1,...,A+g+q))$
$((B+p+1,...,B+2p);(C+f+q+1,...,C+f+2q),(A+g+q+1,...,A+g+2q))$
...
$((B+(p'q'-1)p+1,...,B+p'q'p);(C+f+(p'q'-1)q+1,...,C+f+p'q'q),(A+g+(p'q'-1)q+1,...,A+g+p'q'q))$
$((C+1,...,C+p);(A+f+1,...,A+f+q),(B+g+1,...,B+g+q))$
$((C+p+1,...,C+2p);(A+f+q+1,...,A+f+2q),(B+g+q+1,...,B+g+2q))$
...
$((C+(p'q'-1)p+1,...,C+p'q'p);(A+f+(p'q'-1)q+1,...,A+f+p'q'q),(B+g+(p'q'-1)q+1,...,B+g+p'q'q))$ $\}$,
where $f = p'dp'q'$, $g = (p'+q')dp'q'$, $A = (i-1)dp'q'$, $B = (j-1)dp'q'$, $C = (i+j-2)dp'q'$, and the additions are taken modulo N with residues $1, 2, ..., N$, and $(A+x)$, $(B+x)$, $(C+x)$ means $(A+x)$, $(B+x)'$, $(C+x)''$, respectively.
Then they comprise a $\bar{K}_{p,2q}$ - factorization of $K^*_{N,N,N}$. Applying Theorem 3, $K^*_{n,n,n}$ has a $\bar{K}_{p,2q}$ - factorization.

Theorem 6. Let $d = (p,q)$, $p' = p/d$, $q' = q/d$ for $p \neq q$. When $n \equiv 0$ (mod $dp'q'(p'+2q')/2$) and p' even, $K^*_{n,n,n}$ has a $\bar{K}_{p,2q}$ - factorization.

Proof. Put $n = sdp'q'(p'+2q')/2$ and $N = dp'q'(p'+2q')/2$. When $s = 1$, let $V_1 = \{1, 2, ..., N\}$, $V_2 = \{1', 2', ..., N'\}$, and $V_3 = \{1'', 2'', ..., N''\}$. Construct $(p'+2q')^2/2$ $\bar{K}_{p,2q}$ - factors $F^{(1)}_{ij}$, $F^{(2)}_{ij}$ $(i = 1, 2, ..., (p'+2q')/2; j = 1, 2, ..., (p'+2q')/2)$ as follows:
$F^{(1)}_{ij} = \{$ $((A+1,...,A+p);(B+f+1,...,B+f+q),(C+g+1,...,C+g+q))$
$((A+p+1,...,A+2p);(B+f+q+1,...,B+f+2q),(C+g+q+1,...,C+g+2q))$
...
$((A+(p'q'/2-1)p+1,...,A+(p'q'/2)p);(B+f+(p'q'/2-1)q+1,...,B+f+(p'q'/2)q),(C+g+(p'q'/2-1)q+1,...,C+g+(p'q'/2)q))$
$((B+1,...,B+p);(C+f+1,...,C+f+q),(A+g+1,...,A+g+q))$
$((B+p+1,...,B+2p);(C+f+q+1,...,C+f+2q),(A+g+q+1,...,A+g+2q))$
...
$((B+(p'q'/2-1)p+1,...,B+(p'q'/2)p);(C+f+(p'q'/2-1)q+1,...,C+f+(p'q'/2)q),(A+g+(p'q'/2-1)q+1,...,A+g+(p'q'/2)q))$
$((C+1,...,C+p);(A+f+1,...,A+f+q),(B+g+1,...,B+g+q))$
$((C+p+1,...,C+2p);(A+f+q+1,...,A+f+2q),(B+g+q+1,...,B+g+2q))$
...
$((C+(p'q'/2-1)p+1,...,C+(p'q'/2)p);(A+f+(p'q'/2-1)q+1,...,A+f+(p'q'/2)q),(B+g+(p'q'/2-1)q+1,...,B+g+(p'q'/2)q))$ $\}$,
$F^{(2)}_{ij} = \{$ $((A+1,...,A+p);(C+f+1,...,C+f+q),(B+g+1,...,B+g+q))$

$((A+p+1,...,A+2p);(C+f+q+1,...,C+f+2q),(B+g+q+1,...,B+g+2q))$
...
$((A+(p'q'/2-1)p+1,...,A+(p'q'/2)p);(C+f+(p'q'/2-1)q+1,...,C+f+(p'q'/2)q),$
$(B+g+(p'q'/2-1)q+1,...,B+g+(p'q'/2)q))$
$((B+1,...,B+p);(A+f+1,...,A+f+q),(C+g+1,...,C+g+q))$
$((B+p+1,...,B+2p);(A+f+q+1,...,A+f+2q),(C+g+q+1,...,C+g+2q))$
...
$((B+(p'q'/2-1)p+1,...,B+(p'q'/2)p);(A+f+(p'q'/2-1)q+1,...,A+f+(p'q'/2)q),$
$(C+g+(p'q'/2-1)q+1,...,C+g+(p'q'/2)q))$
$((C+1,...,C+p);(B+f+1,...,B+f+q),(A+g+1,...,A+g+q))$
$((C+p+1,...,C+2p);(B+f+q+1,...,B+f+2q),(A+g+q+1,...,A+g+2q))$
...
$((C+(p'q'/2-1)p+1,...,C+(p'q'/2)p);(B+f+(p'q'/2-1)q+1,...,B+f+(p'q'/2)q),$
$(A+g+(p'q'/2-1)q+1,...,A+g+(p'q'/2)q))$ },
where $f=(p'/2)dp'q'$, $g=((p'+q')/2)dp'q'$, $A=(i-1)dp'q'$, $B=(j-1)dp'q'$, $C=(i+j-2)dp'q'$, and the additions are taken modulo N with residues $1,2,...,N$, and $(A+x)$, $(B+x)$, $(C+x)$ means $(A+x)$, $(B+x)'$, $(C+x)''$, respectively. Then they comprise a $\bar{K}_{p,2q}$ - factorization of $K^*_{N,N,N}$. Applying Theorem 3, $K^*_{n,n,n}$ has a $\bar{K}_{p,2q}$ - factorization.

Main Theorem. $K^*_{n_1,n_2,n_3}$ has a $\bar{K}_{p,2q}$ - factorization if and only if (i) $n_1=n_2=n_3\equiv 0 \pmod p$ for $p=q$, (ii) $n_1=n_2=n_3\equiv 0 \pmod{dp'q'(p'+2q')}$ for $p\neq q$ and p' odd, (iii) $n_1=n_2=n_3\equiv 0 \pmod{dp'q'(p'+2q')/2}$ for $p\neq q$ and p' even, where $d=(p,q)$, $p'=p/d$, $q'=q/d$.

Denote $\bar{K}_{1,2q}$ as $\bar{S}_k$ for $k=2q+1$. Then we have the following corollary.

Corollary[16]. $K^*_{n_1,n_2,n_3}$ has an $\bar{S}_k$ - factorization if and only if (i) $n_1=n_2=n_3$ for $k=3$, (ii) $n_1=n_2=n_3\equiv 0 \pmod{k(k-1)/2}$ for $k\geq 5$ and k odd.

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