

More on sequences in groups

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Abstract

We bring to conclusion the investigation of three problems about sequencings for finite groups: the existence of harmonious sequences in dicyclic groups, the R -sequenceability of dicyclic groups, and the R -sequenceability of the nonabelian groups of order pq , where p and q are primes.

Introduction

Various types of sequences of the elements of a finite group have been studied in connection with questions in combinatorics. In this article we discuss harmonious sequences and R -sequences, both of which are connected to complete mappings of a finite group G .

Definition 1 A *complete mapping* of G is a permutation $g \rightarrow \theta(g)$ of the elements of G such that $\phi : g \rightarrow g\theta(g)$ is again a permutation of the elements of G . In this case, the mapping ϕ is called an *orthomorphism* of G .

The idea of a complete mapping was introduced by H. B. Mann [5] and studied later by L. J. Paige [6]. Results related to this notion and to those that follow are discussed in A. D. Keedwell's recent survey [4].

Definition 2 A group G of order m is called **harmonious** if its m elements can be listed in a sequence

$$g_1, g_2, \dots, g_m$$

such that the products

$$g_1g_2, g_2g_3, \dots, g_{m-1}g_m, g_mg_1$$

of consecutive elements are all distinct.

These sequences were introduced in [1]; if G has a harmonious sequence, then $\theta = (g_1, g_2, \dots, g_m)$ is a complete mapping of G , expressed as a single m -cycle. Harmonious groups include all groups of odd order, the nontrivial finite abelian groups having noncyclic Sylow 2-subgroups (except for the elementary abelian 2-groups), and the dihedral groups D_n of order $2n$ whenever n is divisible by 4 or $n = 6m$, m odd [1]. In Section 1 we discuss harmoniousness in dicyclic groups.

Definition 3 A group of order m is called *R-sequenceable* if its m elements can be listed in a sequence

$$g_1 = 1, g_2, \dots, g_m$$

such that the partial products $g_1, g_1g_2, g_1g_2g_3, \dots, g_1g_2g_3 \dots g_{m-1}$, are all distinct and $g_1g_2g_3 \dots g_{m-1}g_m = 1$.

If G has an R -sequencing, then there is a complete mapping θ of G such that the corresponding orthomorphism ϕ fixes one element and permutes the remaining elements in a single cycle. The dihedral group D_n is R -sequenceable if and only if n is even [3]. We treat this concept for dicyclic groups in Section 2, and for the nonabelian groups of order pq (p and q prime) in Section 3.

In each of the next three sections, the complete result is designated as a Theorem. Previous results are indicated as Propositions; the new contributions are clearly indicated in our discussion.

1 Harmoniousness of dicyclic groups

For an $n \times n$ matrix M , the (i, j) -entry of M is denoted by $M(i, j)$. For a permutation τ of degree n , the collection of n elements $\{M(i, \tau(i)), i = 1, 2, \dots, n\}$ is denoted by $\tau(M)$. The dicyclic group Q_{2n} of order $4n$ is defined by

$$Q_{2n} = \langle \alpha, \beta : \alpha^{2n} = 1, \beta^2 = \alpha^n, \alpha\beta = \beta\alpha^{-1} \rangle.$$

The following proposition was proved in [7].

Proposition 1 *Let A and B be two $n \times n$ matrices defined by*

$$\begin{aligned} A(i, j) &= i + j - 2 \pmod{2n}, \quad i, j = 1, 2, \dots, n \\ B(i, j) &= n - i - j + 1 \pmod{2n}, \quad i, j = 1, 2, \dots, n. \end{aligned}$$

Then the dicyclic group Q_{2n} is harmonious if there exist two permutations π and θ of degree n such that $\theta \circ \pi$ is an n -cycle and $\pi(A) \cup \theta(B)$ is a complete set of residues modulo $2n$.

Sketch of the proof. Let $f = \theta \circ \pi$. Let c be an integer with $1 \leq c \leq n$. For any fixed integer d , we define

$$\begin{aligned} b_{2i-1} &= -f^{i-1}(c) + d, \\ b_{2i} &= f^{i-1}(c) + d - 1, \\ a_{2i-1} &= \pi f^{i-1}(c) - 1, \\ a_{2i} &= \pi f^{i-1}(c) + n - 1. \end{aligned}$$

We notice that

$$\begin{aligned} b_{2i} + a_{2i-1} &= A(f^{i-1}(c), \pi f^{i-1}(c)) + d \text{ and} \\ b_{2i+1} - a_{2i} &= B(\pi f^{i-1}(c), f^i(c)) + d. \end{aligned}$$

Direct calculation shows that

$$\{b_{2i} - b_{2i-1} + n : i = 1, 2, \dots, n\} \cup \{a_{2i} + a_{2i-1} : i = 1, 2, \dots, n\}$$

is a complete set of residues modulo $2n$ and

$$\{b_{2i+1} - a_{2i} : i = 1, 2, \dots, n\} \cup \{b_{2i} + a_{2i-1} : i = 1, 2, \dots, n\} = \pi(A) \cup \theta(B) + d$$

which is also a complete set of residues modulo $2n$ by our hypothesis. Therefore, the following sequence is a harmonious sequence of Q_{2n} :

$$\beta\alpha^{b_1}, \beta\alpha^{b_2}, \alpha^{a_1}, \alpha^{a_2}, \beta\alpha^{b_3}, \beta\alpha^{b_4}, \alpha^{a_3}, \alpha^{a_4}, \dots, \beta\alpha^{b_{2n-1}}, \beta\alpha^{b_{2n}}, \alpha^{a_{2n-1}}, \alpha^{a_{2n}}.$$

■

By using Proposition 1 the following result was proved in [7].

Proposition 2 *If n is a multiple of 4 or 6, the dicyclic group Q_{2n} is harmonious.*

It remains to deal with the case of $n = 4k + 2$. We define two permutations π and θ of degree n by

$$\begin{aligned} \pi(x) &= \begin{cases} x + 2k + 1 & \text{if } 1 \leq x \leq k, \\ x + 2k + 2 & \text{if } k + 1 \leq x \leq 2k, \\ x - 2k & \text{if } 2k + 1 \leq x \leq 3k + 1, \\ 3k + 2 & \text{if } x = 3k + 2, \\ x - 2k - 1 & \text{if } 3k + 3 \leq x \leq 4k + 2, \end{cases} \\ \theta(y) &= \begin{cases} 1 & \text{if } y = 1, \\ 4k + 2 & \text{if } y = 2, \\ y + 2k - 1 & \text{if } 3 \leq y \leq k + 2, \\ k + 1 & \text{if } y = k + 3, \\ y + 2k - 2 & \text{if } k + 4 \leq y \leq 2k + 3, \\ y - 2k - 2 & \text{if } 2k + 4 \leq y \leq 3k + 2, \\ y - 2k - 1 & \text{if } 3k + 3 \leq y \leq 4k + 2. \end{cases} \end{aligned}$$

By the definitions of matrices A and B in Proposition 1, we obtain

$$\begin{aligned} \pi(A) &= \{2k, 2k + 1, 2k + 2, 2k + 3, \dots, 4k, 4k + 2, 4k + 3, 4k + 4, 4k + 5, \dots, 6k, \\ &\quad 6k + 1, 6k + 2\} \end{aligned}$$

and

$$\begin{aligned} \theta(B) &= \{0, 1, 2, 3, \dots, 2k - 3, 2k - 2, 2k - 1, 4k + 1, 6k + 3, 6k + 4, \dots, 8k + 1, \\ &\quad 8k + 2, 8k + 3\}. \end{aligned}$$

Therefore, $\pi(A) \cup \theta(B)$ is a complete set of residues modulo $2n$.

Considering k modulo 3, we find that in two cases $\theta \circ \pi$ is a cycle of length n . If $k = 3t$, by direct calculation, we have $\theta \circ \pi = (1 \ 4k \ 4k - 3 \ 4k - 6 \ 4k - 9 \ \dots \ 3k + 6 \ 3k + 3 \ 3k + 1 \ 3k \ 3k - 1 \ 3k - 2 \ \dots \ 2k + 3 \ 2k + 2 \ 4k + 2 \ 4k - 1 \ 4k - 4 \ 4k - 7 \ \dots \ 3k + 5 \ 3k + 2 \ k \ k - 1 \ k - 2 \ k - 3 \ \dots \ 3 \ 2 \ 4k + 1 \ 4k - 2 \ 4k - 5 \ \dots \ 3k + 7 \ 3k + 4 \ k + 1 \ k + 2 \ k + 3 \ \dots \ 2k \ 2k + 1)$ which is an n -cycle.

If $k = 3t + 2$, we have $\theta \circ \pi = (1 \ 4k \ 4k - 3 \ 4k - 6 \ 4k - 9 \ \dots \ 3k + 5 \ 3k + 2 \ k \ k - 1 \ k - 2 \ k - 3 \ \dots \ 3 \ 2 \ 4k + 1 \ 4k - 2 \ 4k - 5 \ \dots \ 3k + 6 \ 3k + 3 \ 3k + 1 \ 3k \ 3k - 1 \ 3k - 2 \ \dots \ 2k + 3 \ 2k + 2 \ 4k + 2 \ 4k - 1 \ 4k - 4 \ 4k - 7 \ \dots \ 3k + 7 \ 3k + 4 \ k + 1 \ k + 2 \ k + 3 \ \dots \ 2k \ 2k + 1)$ which is a cycle of length n .

Therefore, by Proposition 1 we can state

Proposition 3 *If $n = 12t + 2$ or $n = 12t + 10$, the dicyclic group Q_{2n} is harmonious.*

In the remaining we have case $n = 12t + 6$, and this is covered by Proposition 2. It is shown in [1] that Q_{2n} is not harmonious if n is an odd integer or $n = 2$. Hence, by Propositions 2 and 3 the following is true:

Theorem 1 *Q_{2n} is harmonious if and only if n is an even integer greater than 2.*

It is obvious that a harmonious group may have many harmonious sequences. We can, for example, give an alternative construction for the case of $n = 8k + 2$ as follows.

Let π and θ be permutations of degree n defined by

$$\pi(x) = \begin{cases} 4k + 2 + x & \text{if } 1 \leq x \leq 2k - 1, \\ 4k + 4 + x & \text{if } 2k \leq x \leq 4k - 2 \text{ and } x \text{ is even,} \\ 2k + 1 & \text{if } x = 2k + 1, \\ 4k + x & \text{if } 2k + 3 \leq x \leq 4k + 1 \text{ and } x \text{ is odd,} \\ 4k + 2 & \text{if } x = 4k, \\ x - 4k - 1 & \text{if } 4k + 2 \leq x \leq 8k + 2 \text{ and } x \neq 6k + 2, \\ 6k + 2 & \text{if } x = 6k + 2, \end{cases}$$

$$\theta(y) = \begin{cases} y & \text{if } y = 1, \\ 4k + y & \text{if } 2 \leq y \leq 4k + 2, \\ y - 4k - 1 & \text{if } 4k + 3 \leq y \leq 8k + 2. \end{cases}$$

By the definitions of matrices A and B in Proposition 1, we have

$$\pi(A) = \{4k, 4k + 1, 4k + 2, \dots, 8k - 1, 8k, 8k + 2, 8k + 3, \dots, 12k + 2\} \\ \text{mod } 2n \text{ and}$$

$$\theta(B) = \{0, 1, 2, \dots, 4k - 1, 8k + 1, 12k + 3, 12k + 4, \dots, 16k + 3\} \text{mod } 2n.$$

Therefore $\pi(A) \cup \theta(B)$ is a complete set of residues modulo $2n$.

By direct calculation, we have $\theta \circ \pi = (1 \ 2 \ 3 \ \dots \ 2k - 1 \ 2k \ 2k + 3 \ 2k + 2 \ 2k + 5 \ 2k + 4 \ 2k + 7 \ 2k + 6 \ \dots \ 4k + 1 \ 4k \ 8k + 2 \ 8k + 1 \ 8k \ 8k - 1 \ \dots \ 6k + 3 \ 6k + 2 \ 2k + 1 \ 6k + 1 \ 6k \ 6k - 1 \ \dots \ 4k + 2)$ which is an n -cycle.

2 The R-sequenceability of dicyclic groups

The following two propositions were proved in [8].

Proposition 4 Q_{2n} is R-sequenceable if there are integers $a_2, a_3, \dots, a_{2n-1}$ and $b_1, b_2, \dots, b_{2n}$ satisfying

- (1) $0, a_2, a_3, \dots, a_{2n-1}$ are distinct mod $2n$,
- (2) $b_1, b_2, \dots, b_{2n}$ are distinct mod $2n$,
- (3) $0, a_2, a_3 - a_2, \dots, a_n - a_{n-1}, b_{n+1} - b_n, b_{n+2} - b_{n+1}, \dots, b_{2n} - b_{2n-1}$ are distinct mod $2n$,
- (4) $b_1 + a_n, b_1 + a_{n+1} + n, b_2 + a_{n+1}, b_2 + a_{n+2} + n, b_3 + a_{n+2}, \dots, b_{n-1} + a_{2n-1} + n, b_n + a_{2n-1}, b_{2n} + n$ are distinct mod $2n$.

Proposition 5 Let A and B be two $n \times n$ matrices defined by

$$A(i, j) = \begin{cases} 3n/2 + i + j - 1 & \text{mod } 2n \text{ if } i \leq n/2 \\ 3n/2 + i + j & \text{mod } 2n \text{ if } i > n/2 \end{cases}$$

$$B(i, j) = \begin{cases} n/2 + i + j - 1 & \text{mod } 2n \text{ if } i \leq n/2 \\ n/2 + i + j & \text{mod } 2n \text{ if } i > n/2. \end{cases}$$

Then the dicyclic group Q_{2n} is R-sequenceable if there exist two permutations π and θ of degree n such that $\pi \circ \theta^{-1}$ is a cycle of length n with $\theta(1) = n$, and $\pi(A) \cup \theta(B)$ is a complete set of residues modulo $2n$.

Theorem 2 Q_{2n} is R-sequenceable if and only if n is an even integer greater than 2.

Proof. It was shown in [8] that for $n = 2$, and for n odd, Q_{2n} is not R-sequenceable, and for $n \equiv 0 \pmod{4}$, Q_{2n} is R-sequenceable. Thus we assume that $n = 4k - 2$ where $k > 1$ is an integer. We modify the proof of the case of $n = 4k$ in [8] by defining the permutations of π and θ by

$$\pi(x) = \begin{cases} x + 2k - 1 & \text{if } 1 \leq x \leq 2k - 1, \\ x - 2k + 1 & \text{if } 2k \leq x \leq 4k - 2, \end{cases}$$

$$\theta(y) = \begin{cases} y + 2k - 2 & \text{if } 2 \leq y \leq 2k - 1, y \neq k + 1, \\ y - 2k + 2 & \text{if } 2k \leq y \leq 4k - 3, y \neq 3k - 2, \end{cases} \text{ together with}$$

$$\theta(1) = 4k - 2, \theta(4k - 2) = 1, \theta(k + 1) = k, \text{ and } \theta(3k - 2) = 3k - 1.$$

By the definitions of A and B in Proposition 5, we have $\pi(A) = \{1, 2, 3, \dots, 4k - 2\}$ and $\theta(B) = \{4k - 1, 4k, 4k + 1, \dots, 8k - 4\}$. Hence $\pi(A) \cup \theta(B)$ is a complete set of residues modulo $2n$.

Notice $k > 1$. By direct calculation, we have $\pi \circ \theta^{-1} = (2k - 1 \ 2k - 2 \ \dots \ k + 1 \ k \ 3k \ 3k + 1 \ 3k + 2 \ \dots \ 4k - 2 \ 2k \ 2k + 1 \ 2k + 2 \ \dots \ 3k - 1 \ k - 1 \ k - 2 \ \dots \ 2 \ 1)$ which is a cycle of length n . Hence by Proposition 5, Q_{2n} is R-sequenceable. ■

By using Proposition 4 we can give a distinct R -sequencing of Q_{2n} from the construction indicated in the proof of Theorem 2. This construction can serve as an alternative proof to Theorem 2 for the case $n = 8k + 2$.

We define the sequences (1) and (2) in Proposition 4 as follows:

Sequence (1), of $2n - 1$ elements, can be given in seven segments, with numbers of elements and rule of construction given by

- (i) $8k + 2$ elements: $0, 8k + 1, 1, 8k, 2, 8k - 1, \dots, 4k, 4k + 1$;
- (ii) $2k$ elements: $16k + 3, 16k + 2, 16k + 5, \dots, 14k + 5, 14k + 4$;
- (iii) $2k - 2$ elements: $14k + 1, 14k + 2, 14k - 1, 14k, 14k - 3, 14k - 2, \dots, 12k + 5, 12k + 6$;
- (iv) $2k$ elements: $12k + 3, 12k + 2, 12k + 1, 12k, \dots, 10k + 4$;
- (v) The single element $14k + 3$;
- (vi) $2k + 1$ elements: $10k + 3, 10k + 2, \dots, 8k + 3$;
- (vii) The final element $12k + 4$.

Similarly sequence (2), of $2n$ elements, is given by

- (i) $2k + 1$ elements: $8k + 1, 8k, 8k - 1, \dots, 6k + 1$;
- (ii) $2k - 2$ elements: $6k - 2, 6k - 1, 6k - 4, 6k - 3, 6k - 6, 6k - 5, \dots, 4k + 2, 4k + 3$;
- (iii) $2k$ elements: $4k, 4k - 1, 4k - 2, \dots, 2k + 1$;
- (iv) the single element $6k$;
- (v) $2k + 1$ elements: $2k, 2k - 1, 2k - 2, \dots, 1, 0$;
- (vi) the single element $4k + 1$;
- (vii) $8k + 2$ elements: $12k + 3, 12k + 2, 12k + 4, 12k + 1, 12k + 5, 12k, 12k + 6, 12k - 1, \dots, 16k + 3, 8k + 2$.

In the following examples, semicolons separate the segments in the listing of sequence elements. $k = 1$, one segment of each sequence is vacuous (noted by a repeated semicolon).

When $k = 1$, so that $n = 8k + 2 = 10$, the sequences are

- (1₁): $0, 9, 1, 8, 2, 7, 3, 6, 4, 5; 19, 18;; 15, 14; 17; 13, 12, 11; 16$, and
- (2₁): $9, 8, 7;; 4, 3; 6; 2, 1, 0; 5; 15, 14, 16, 13, 17, 12, 18, 11, 19, 10$.

When $k = 2$, so that $n = 8k + 2 = 18$, the sequences are

- (1₂): $0, 17, 1, 16, 2, 15, 3, 14, 4, 13, 5, 12, 6, 11, 7, 10; 8, 9; 35, 34, 33, 32; 29, 30; 27, 26, 25, 24; 31; 23, 22, 21, 20, 19; 28$, and
- (2₂): $17, 16, 15, 14, 13; 10, 9; 8, 7, 6, 5; 12; 4, 3, 2, 1, 0; 9; 27, 26, 28, 25, 29, 24, 30, 23, 31, 22, 32, 21, 33, 20, 34, 19, 35, 18$.

3 R -sequenceability of groups of order pq

First we state an alternative definition of R -sequenceability. It is easily seen that this definition is equivalent to Definition 3.

Definition 4 (a) A group G of order m is called R -sequenceable if its nonidentity elements can be listed in a sequence

$$g_1, g_2, \dots, g_{m-1}$$

such that

$$g_1^{-1}g_2, g_2^{-1}g_3, \dots, g_{m-2}^{-1}g_{m-1}, g_{m-1}^{-1}g_1$$

are all distinct.

(b) If, further, $g_1 = g_2g_{m-1} = g_{m-1}g_2$, G is called R^* -sequenceable.

The nonabelian group G of order pq where p, q are primes with $q \equiv 1 \pmod{p}$ is defined, using r such that $r^p \equiv 1 \pmod{q}$, $r \not\equiv 1 \pmod{q}$, as follows:

$$G = \{(u, v) \mid u \in \mathbf{Z}/p\mathbf{Z}, v \in \mathbf{Z}/q\mathbf{Z}\}$$

with multiplication defined by

$$(u, v)(x, y) = (u + x, vr^x + y).$$

The following result was proved by Keedwell [3].

Proposition 6 *The nonabelian group of order pq is R -sequenceable if p has 2 as a primitive root.*

This result can be extended, essentially by means of a modification of the direct product theorem in [2].

Theorem 3 *The nonabelian group of order pq is R -sequenceable.*

Proof. In view of Proposition 6 we assume $p > 5$. The (additive) cyclic group $\mathbf{Z}/p\mathbf{Z}$ is R^* -sequenceable [2]; let $g_1, \dots, g_{p-1}$ be an R^* -sequencing with $g_1 = g_2 + g_{p-1}$.

In order to list the nonidentity elements of G , we introduce $p \times q$ matrices A and B , defined over $\mathbf{Z}/p\mathbf{Z}$ and $\mathbf{Z}/q\mathbf{Z}$, respectively. Entries a_{11} and b_{11} are left blank; group elements are given as (a_{ij}, b_{ij}) , and are sequenced by reading column-by-column from A and B .

Let $a_{ij} = g_1$ for $i = 1$, $2 \leq j \leq (q+1)/2$, and for $i = 2$, $1 \leq j \leq (q+1)/2$. For $i = 1, 2$ and $j > (q+1)/2$, let $a_{ij} = 0$. For $i > 2$ and $1 \leq j \leq q$, let $a_{ij} = g_{i-1}$. The definition of B uses the nonzero element $c = -r^{g_1 - g_{p-2} - g_2}$ of $\mathbf{Z}/q\mathbf{Z}$; all arithmetic is in $\mathbf{Z}/q\mathbf{Z}$. For $j \geq 2$, let $b_{1j} = b_{3j} = (j-1) \cdot c$. Complete row 3 by letting $b_{31} = 0$, and define row 2 by letting $b_{2j} = -b_{3j}$, $1 \leq j \leq q$. For $4 \leq i \leq p-1$ (except as noted below), row i is the q -tuple $(0, q-1, q-2, \dots, 2, 1)$. The final row

Use $r = 7$ (which implies $c = 6$). Since $g_3 - g_2 + g_{p-1} - g_{p-2} \equiv 0 \pmod{7}$, we modify row 4 and obtain the following matrix as B :

$$\begin{bmatrix} - & 06 & 12 & 18 & 24 & 01 & 07 & 13 & 19 & 25 & 02 & 08 & 14 & 20 & 26 & 03 & 09 & 15 & 21 & 27 & 04 & 10 & 16 & 22 & 28 & 05 & 11 & 17 & 23 \\ 0 & 23 & 17 & 11 & 05 & 28 & 22 & 16 & 10 & 04 & 27 & 21 & 15 & 09 & 03 & 26 & 20 & 14 & 08 & 02 & 25 & 19 & 13 & 07 & 01 & 24 & 18 & 12 & 06 \\ 0 & 06 & 12 & 18 & 24 & 01 & 07 & 13 & 19 & 25 & 02 & 08 & 14 & 20 & 26 & 03 & 09 & 15 & 21 & 27 & 04 & 10 & 16 & 22 & 28 & 05 & 11 & 17 & 23 \\ 0 & 01 & 02 & 03 & 04 & 05 & 06 & 07 & 08 & 09 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 \\ 0 & 28 & 27 & 26 & 25 & 24 & 23 & 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 09 & 08 & 07 & 06 & 05 & 04 & 03 & 02 & 01 \\ 0 & 28 & 27 & 26 & 25 & 24 & 23 & 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 09 & 08 & 07 & 06 & 05 & 04 & 03 & 02 & 01 \\ 23 & 17 & 11 & 05 & 28 & 22 & 16 & 10 & 04 & 27 & 21 & 15 & 09 & 03 & 26 & 20 & 14 & 08 & 02 & 25 & 19 & 13 & 07 & 01 & 24 & 18 & 12 & 06 & 0 \end{bmatrix}$$

By the definitions of the elements (a'_{ij}, b'_{ij}) we have A' as follows:

$$\begin{bmatrix} - & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 2 & 0 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 \\ 1 & 1 \\ 5 & 5 \\ 4 & 4 \end{bmatrix}$$

and, finally, B' is as follows:

$$\begin{bmatrix} - & 10 & 20 & 01 & 11 & 21 & 02 & 12 & 22 & 03 & 13 & 23 & 04 & 14 & 24 & 20 & 02 & 13 & 24 & 06 & 17 & 28 & 10 & 21 & 03 & 14 & 25 & 07 & 18 \\ 0 & 17 & 05 & 22 & 10 & 27 & 15 & 03 & 20 & 08 & 25 & 13 & 01 & 18 & 06 & 23 & 11 & 28 & 16 & 04 & 21 & 09 & 26 & 14 & 02 & 19 & 07 & 24 & 12 \\ 0 & 11 & 22 & 04 & 15 & 26 & 08 & 19 & 01 & 12 & 23 & 05 & 16 & 27 & 09 & 05 & 15 & 25 & 06 & 16 & 26 & 07 & 17 & 27 & 08 & 18 & 28 & 09 & 19 \\ 0 & 02 & 04 & 06 & 08 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 01 & 03 & 05 & 07 & 09 & 11 & 13 & 15 & 17 & 19 & 21 & 23 & 25 & 27 \\ 0 & 21 & 13 & 05 & 26 & 18 & 10 & 02 & 23 & 15 & 07 & 28 & 20 & 12 & 04 & 25 & 17 & 09 & 01 & 22 & 14 & 06 & 27 & 19 & 11 & 03 & 24 & 16 & 08 \\ 0 & 15 & 01 & 16 & 02 & 17 & 03 & 18 & 04 & 19 & 05 & 20 & 06 & 21 & 07 & 22 & 08 & 23 & 09 & 24 & 10 & 25 & 11 & 26 & 12 & 27 & 13 & 28 & 16 \\ 23 & 11 & 28 & 16 & 04 & 21 & 09 & 26 & 14 & 02 & 19 & 07 & 24 & 12 & 0 & 17 & 05 & 22 & 10 & 27 & 15 & 03 & 20 & 08 & 25 & 13 & 01 & 18 & 06 \end{bmatrix}$$

The listing of the elements (a_{ij}, b_{ij}) by columns gives the indicated R^* -sequencing of G , as can be checked by examining the (a'_{ij}, b'_{ij}) .

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