On the Principal Edge Bipartition of a Graph

Michael A. Henning*

Department of Mathematics
University of Natal
Private Bag X01
Pietermaritzburg, 3209 South Africa

C.H.C. Little†

Institute of Fundamental Sciences
Massey University
Palmerston North, New Zealand

Abstract

A cycle in a graph is a set of edges that covers each vertex an even number of times. An even cycle is cycle of even cardinality. A cocycle is a collection of edges that intersects each cycle in an even number of edges. A coeven is a cocycle or the complement of a cocycle. A bieven is a collection of edges that is both an even cycle and a coeven. The even cycles, coevens, and bievens each form a vector space over the integers modulo two when addition is defined as symmetric difference of sets. An edge is coeven cyclic if it belongs to a coeven C for which $C - \{e\}$ is an even cycle. An edge is bieven cyclic if it belongs to a bieven. We show that any edge in a graph is either coeven cyclic or bieven cyclic.

1 Introduction

Associated with an arbitrary graph are several vector spaces. In each case the vectors are sets and the sum of two vectors is their symmetric difference. Hence the underlying field is the set of integers modulo two. One such space is the *edge space* of a graph G: its elements are the subsets of EG, $\emptyset \subseteq EG$ is the zero, and $F = -F$ for all $F \subseteq EG$. We similarly define the *vertex space*. The dimension of the edge space is $|EG|$ and the dimension of the vertex space is $|VG|$.

*Research supported in part by the University of Natal and the Foundation for Research Development

†Research supported by the Foundation for Research Development

Given two edge sets F, F' of the edge space, $\langle F, F' \rangle = 0$ if and only if F and F' have an even number of edges in common. Given a subspace $\mathcal{F}$ of the edge space of G, we write

$$\mathcal{F}^{\perp} = \{D \subseteq E \mid \langle F, D \rangle = 0 \text{ for all } F \in \mathcal{F}\}.$$

This is again a subspace of the edge space, called the orthogonal subspace of $\mathcal{F}$.

The *cycle space* $\mathcal{C} = \mathcal{C}(G)$ is the subspace of the edge space spanned by all the circuits (connected 2-regular subgraphs) in G–more precisely, by their edge sets. The orthogonal subspace $\mathcal{C}^{\perp}$ of $\mathcal{C}$ is called the *cocycle space*. (The cocycle space is called the *cut space* in [2].) The elements of $\mathcal{C}$ are called *cycles*; those of $\mathcal{C}^{\perp}$ are called *cocycles*. A set of edges is a cycle when it induces a subgraph where the degree of every vertex is even. A set of edges is a cocycle if it is the set $[A, B]$ of all edges joining A and B where $\{A, B\}$ is a partition of the vertex set.

It is possible for a set of edges to be both a cycle and a cocycle. We call such sets *bicycles*. The bicycles form a vector space, called the *bicycle space*, which is precisely the space $\mathcal{C} \cap \mathcal{C}^{\perp}$.

Rosenstiehl and Read [3] have shown the following striking theorem, which classifies the edges of an arbitrary graph into three types.

Theorem 1 The Principal Edge Tripartition. *For any edge e in a graph G, exactly one of the following holds:*

(1) *e belongs to a cycle C for which $C - \{e\}$ is a cocycle,*
(2) *e belongs to a cocycle C for which $C - \{e\}$ is a cycle, or*
(3) *e belongs to a bicycle.*

An edge is called *cyclic*, *cocyclic*, or *bicyclic* according to whether it satisfies conditions (1), (2), or (3) in Theorem 1, respectively.

A cycle C containing an edge e for which $C - \{e\}$ is a cocycle is called a *principal cycle* and $C - \{e\}$ a *principal cocycle* for e. If C is a cocycle containing an edge e with $C - \{e\}$ a cycle, then C is called a *principal cocycle* and $C - \{e\}$ a *principal cycle* for e.

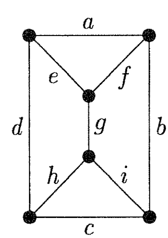

Figure 1: The graph G

In Figure 1, the edges a, c, e, f, h, i are cyclic while the edges b, d, g are cocyclic. For example, the set of edges a, b, d, h, i is a principal cycle for the edge a, the set of edges b, c, f, g, h is a principal cycle for the edge f, and the set of edges $EG - \{f, i\}$ is a principal cocycle for the edge d. The graph G of Figure 1 has no nonempty bicycles. The principal cycle and the principal cocycle are unique if a graph has no nonempty bicyles.

As a direct consequence of the orthogonality of cycles and cocycles, we have the following elementary property of the Principal Edge Tripartition.

Theorem 2 ([3]) *Every principal cycle associated with a cyclic edge e is odd and every principal cocycle associated with e is even. If e is a cocyclic edge, then the above parities are reversed. Every bicycle is even.*

Let $\mathcal{E}(G)$ be the even space of the graph G; that is, $\mathcal{E}(G)$ is the subspace of the edge space spanned by all the even cycles in G. Associated with $\mathcal{E}(G)$ is an edge tripartition analogous to the principal edge tripartition of Rosenstiehl and Read [3]. We propose to study its properties.

We denote the complement of a set T of edges in a graph G by $\overline{T}$ and the complement with respect to G of a subgraph H by $\overline{H}$.

2 The even and coeven spaces

We begin this section by establishing the dimension of $\mathcal{E}(G)$ and $\mathcal{E}^{\perp}(G)$.

Theorem 3 *For any 2-connected graph G,*

$$\dim\, \mathcal{E}(G) = \begin{cases} \dim\, \mathcal{C}(G) & \text{if } G \text{ is bipartite} \\ \dim\, \mathcal{C}(G) - 1 & \text{otherwise.} \end{cases}$$

Proof. The theorem is clear if G is bipartite. We therefore suppose that G is non-bipartite. Then $\dim\, \mathcal{E}(G) < \dim\, \mathcal{C}(G)$. Let $G_0, G_1, \ldots, G_n$ be a sequence of graphs where G_0 is induced by a circuit C_0 of G, $G_n = G$, and, for each $i > 0$, G_i is constructed from G_{i-1} by adding a path P_i joining distinct vertices u_i and v_i of G_{i-1} and having no edges or internal vertices in common with G_{i-1}. For each $i > 0$, let C_i be a circuit which is the union of P_i and a path Q_i in G_{i-1} joining u_i and v_i. Let j be the smallest subscript for which G_j is non-bipartite. Hence C_i is even for every $i < j$ and C_j is odd.

We now show that for each $i > j$, we may choose C_i to be even. Since G_j is non-bipartite, G_{i-1} has an odd circuit C. As G_{i-1} is 2-connected, two independent paths P and Q join u_i and v_i, respectively, to distinct vertices u and v, respectively, of C (see [1]). We may assume that P and Q are chosen to have minimal length. Since C includes both an even and an odd path joining u to v, it follows that G_{i-1} contains both an even and an odd path joining u_i to v_i. Hence Q_i can be chosen to have the same parity as P_i, and so C_i is even.

Let $\mathcal{S} = \{C_0, C_1, \ldots, C_n\}$. Clearly, $\mathcal{S}$ is a basis for the cycle space $\mathcal{C}(G)$. Each circuit of $\mathcal{S} - \{C_j\}$ is even. Since dim $\mathcal{E}(G) <$ dim $\mathcal{C}(G)$, it follows that dim $\mathcal{E}(G) =$ dim $\mathcal{C}(G) - 1$. □

Corollary 4 *For any graph G,*

$$\dim \mathcal{E}(G) = \dim \mathcal{C}(G) - b(G) = |EG| - |VG| + c(G) - b(G),$$

where $c(G)$ denotes the number of components of G and $b(G)$ denotes the number of non-bipartite blocks of G.

Corollary 5 *For any 2-connected graph G,*

$$\dim \mathcal{E}^{\perp}(G) = \begin{cases} \dim \mathcal{C}^{\perp}(G) & \text{if } G \text{ is bipartite} \\ \dim \mathcal{C}^{\perp}(G) + 1 & \text{otherwise} \end{cases}$$

Corollary 6 *For any graph G,*

$$\dim \mathcal{E}^{\perp}(G) = \dim \mathcal{C}^{\perp}(G) + b(G) = |VG| - c(G) + b(G),$$

where $c(G)$ denotes the number of components of G and $b(G)$ denotes the number of non-bipartite blocks of G.

Theorem 7 *A cycle of G belongs to $\mathcal{E}(G)$ if and only if it is even.*

Proof. Clearly, any member of $\mathcal{E}(G)$ is even. The result is also clear for bipartite graphs. Suppose, therefore, that G has an odd circuit C. By Theorem 3 it suffices to show that the number of even cycles is equal to the number of odd cycles. We therefore define a bijection ϕ from the set of even cycles to the set of odd cycles by the equation $\phi(A) = A + C$ for each even cycle A. This function is an injection, for if $A + C = B + C$ where A and B are even cycles, then $A = B$. The function is a surjection, for if D is an odd cycle, then $D = \phi(C + D)$ and $C + D$ is even. □

Theorem 8 *The coeven space of a non-bipartite graph consists of the cocycles and their complements.*

Proof. Let G be a non-bipartite graph. It suffices to consider the case when G is 2-connected. By Corollary 5, the cardinality of the coeven space is twice that of the cocycle space. Moreover the cocycle space is a subspace of the coeven space and the set of complements of cocycles is included in the coeven space. It therefore suffices to show that no complement of a cocycle is a cocycle. Let C be an odd circuit and D a cocycle. Then $|D \cap C| \equiv 0\,(mod\,2)$, so that $|(EG - D) \cap C| \equiv 1\,(mod\,2)$. Hence $EG - D$ is not a cocycle. □

We call the elements of $\mathcal{E}^{\perp}(G)$ *coevens.* Adapting the terminology of [3], we define an *eventree* in a graph G as a minimal subset of EG which meets every nonzero coeven. A *coeventree* is a minimal subset of EG which meets every nonzero even cycle.

For each $S \subseteq VG$ we write $\partial_G S = [S, VG - S]$.

Theorem 9 *A subgraph of a non-bipartite graph is an eventree if and only if it is a connected spanning subgraph with no even circuit and just one odd circuit.*

Proof. Let G be a non-bipartite graph and T a connected spanning subgraph with no even circuit and just one odd circuit. Since T contains a spanning tree, T meets every nonzero cocycle of G. Since T contains an odd circuit, T meets the complement of every cocycle. Furthermore, for every bridge e of T there is a cocycle containing e but no other edge of T. For every edge f of T which belongs to the circuit there is a cocycle (namely, $\partial_G S$ where $\partial_{T-\{f\}} S = ET - \{f\}$) whose complement contains f but no other edge of T. Hence T is minimal and therefore an eventree.

Conversely, let T be an eventree of G. Then T contains a spanning tree of G since ET meets every nonzero cocycle. Suppose T does not contain an odd circuit. Then T is bipartite. Hence $ET = \partial_T S$ for some subset S of VG. Therefore, ET does not meet the complement of $\partial_G S$. Hence T must contain an odd circuit. By the minimality of T, it follows that T is a connected spanning subgraph with no even circuit and just one odd circuit. □

Theorem 10 *A subgraph of a graph is an coeventree if and only if it is the complement of an eventree.*

Proof. A set S of edges in a graph G meets every even circuit if and only if its complement induces a spanning subgraph H where every block is K_2 or an odd circuit. In fact, if S is a coeventree, then H has at most one odd circuit, for the sum of two odd circuits is an even cycle. The converse is obvious. □

Let e be an edge in an eventree T in a non-bipartite graph G. If e is a bridge, then there is a unique cocycle which meets T in e alone but no complement of a cocycle which meets T in e alone because of the presence of the odd circuit. If e belongs to the odd circuit, then there is a unique cocycle whose complement meets T in e alone but there is no cocycle which meets T in e alone. In both cases, we let $T(e)$ be the unique coeven which meets T in e alone. The family $\{T(e) \mid e \in ET\}$ is a basis for $\mathcal{E}^{\perp}(G)$. We call this basis a *fundamental basis* for $\mathcal{E}^{\perp}(G)$.

Let f be an edge in the coeventree $\overline{T}$. Then $T + \{f\}$ contains either a unique even circuit or exactly two odd circuits. Thus, in either case $T + \{f\}$ contains a unique even cycle $\overline{T}(f)$ which meets $\overline{T}$ in f alone. The family $\{\overline{T}(f) \mid f \in E\overline{T}\}$ is a basis for $\mathcal{E}(G)$. We call this basis a *fundamental basis* for $\mathcal{E}(G)$.

3 The principal edge bipartition of a graph

Theorem 1 in fact holds for any vector space associated with a graph. In particular, we have the following edge tripartition associated with $\mathcal{E}(G)$. A *bieven* is an even cycle which is also a coeven.

Theorem 11 *For any edge e in a graph G, exactly one of the following holds:*

(1) *e belongs to an even cycle C for which $C - \{e\}$ is a coeven,*
(2) *e belongs to a coeven C for which $C - \{e\}$ is an even cycle, or*
(3) *e belongs to a bieven.*

We call an edge *even cyclic*, *coeven cyclic*, or *bieven cyclic* according to whether it satisfies conditions (1), (2), or (3) in Theorem 11, respectively. The next results show that every edge is either coeven cyclic or bieven cyclic.

Lemma 12 *If C is an even cycle containing an edge e, then $C - \{e\}$ is not the complement of a cocycle.*

Proof. Suppose $C - \{e\}$ is the complement of a cocycle X. Then $C - \{e\} = EG - X$, and so $X \cap C = \{e\}$. This contradicts the fact that X is a cocycle and therefore meets every cycle in an even number of edges. □

Lemma 13 *No edge is even cyclic.*

Proof. Let e be an edge of a graph G. Then by the Principal Edge Tripartition, e is either cyclic, cocyclic, or bicyclic. If e is a cyclic edge, then, by Theorem 2, every principal cycle associated with e is odd. Furthermore, by Lemma 12, if e belongs to an even cycle C, then $C - \{e\}$ is not the complement of a cocycle. Hence, e is not even cyclic. If e is a cocyclic edge, then, by Theorem 2, e is also coeven cyclic. If e is a bicyclic edge, then, by Theorem 2, e is also bieven cyclic. In all cases, the edge e is not even cyclic. □

Theorem 11 and Lemma 13 thus classify the edges of an arbitrary graph into two types.

Theorem 14 The Principal Edge Bipartition. *Any edge e in a graph is either coeven cyclic or bieven cyclic.*

In Figure 1, the edges b, d, g are cocyclic and therefore coeven cyclic, while the edges in $EG - \{b, d, g\}$ are bieven cyclic since $EG - \{b, d, g\}$ is both an even cycle and a complement of a cocycle and is therefore a bieven.

References

[1] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, Third Edition, Chapman & Hall, London, 1996.

[2] R. Diestel, *Graph Theory*, Springer-Verlag, New York, Inc., 1997.

[3] P. Rosenstiehl and R.C. Read, On the principal edge tripartition of a graph. *Annals of Discrete Math.* **3** (1978), 195–226.

(Received 16/7/98)