Semi-evenly partite star-factorization of symmetric complete tripartite digraphs

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Abstract

We show that necessary and sufficient conditions for the existence of a semi-evenly partite star - factorization of the symmetric complete tripartite digraph $K^*_{n_1,n_2,n_3}$ are (i) k is even, $k \geq 4$ and (ii) $n_1 = n_2 = n_3 \equiv 0 \pmod{k(k-1)/3}$ for $k \equiv 0 \pmod 6$ and $n_1 = n_2 = n_3 \equiv 0 \pmod{k(k-1)}$ for $k \equiv 2, 4 \pmod 6$.

1. Introduction

Let $K^*_{n_1,n_2,n_3}$ denote *the symmetric complete tripartite digraph* with partite sets V_1, V_2, V_3 of n_1, n_2, n_3 vertices each, and let $\tilde{S}_k$ denote *the semi-evenly partite directed star* from a center-vertex to $k-1$ end-vertices such that the center-vertex is in V_i and $(k-2)/2$ end-vertices are in V_{j_1} and $k/2$ end-vertices are in V_{j_2} with $\{i, j_1, j_2\}$={1,2,3}. A spanning subgraph F of $K^*_{n_1,n_2,n_3}$ is called an $\tilde{S}_k$ *- factor* if each component of F is $\tilde{S}_k$. If $K^*_{n_1,n_2,n_3}$ is expressed as an arc-disjoint sum of $\tilde{S}_k$ - factors, then this sum is called an $\tilde{S}_k$ *- factorization* of $K^*_{n_1,n_2,n_3}$.
In this paper, it is shown that necessary and sufficient conditions for the existence of such a factorization are (i) k is even, $k \geq 4$ and (ii) $n_1 = n_2 = n_3 \equiv 0 \pmod{k(k-1)/3}$ for $k \equiv 0 \pmod 6$ and $n_1 = n_2 = n_3 \equiv 0 \pmod{k(k-1)}$ for $k \equiv 2, 4 \pmod 6$.

Let K_{n_1,n_2}, $K^*_{n_1,n_2}$, $K^*_{n_1,n_2,n_3}$, and $K^*_{n_1,n_2,\ldots,n_m}$ denote the complete bipartite graph, the symmetric complete bipartite digraph, the symmetric complete tripartite digraph, and the symmetric complete multipartite digraph, respectively. And let $\hat{C}_k$, $\hat{S}_k$, $\hat{P}_k$, and $\hat{K}_{p,q}$ denote the cycle or the directed cycle, the star or the directed star, the path or the directed path, and the complete bipartite graph or the complete bipartite digraph, respectively, on two partite sets V_i and V_j. Then the problems of giving the necessary and sufficient conditions of $\hat{C}_k$ - factorization of K_{n_1,n_2}, $K^*_{n_1,n_2}$, $K^*_{n_1,n_2,n_3}$, and $K^*_{n_1,n_2,\ldots,n_m}$ have been completely solved by Enomoto, Miyamoto and Ushio[2]

Australasian Journal of Combinatorics 20(1999), pp.69–75

and Ushio[11,14]. $\hat{S}_k$ - factorization of K_{n_1,n_2}, $K^*_{n_1,n_2}$, and $K^*_{n_1,n_2,n_3}$ have been studied by Ushio and Tsuruno[8], Ushio[13], and Wang[15]. Recently, Martin[4,5] and Ushio[10] give the necessary and sufficient conditions of $\hat{S}_k$ - factorization of K_{n_1,n_2} and $K^*_{n_1,n_2}$. $\hat{P}_k$ - factorization of K_{n_1,n_2} and $K^*_{n_1,n_2}$ have been studied by Ushio and Tsuruno[7], and Ushio[6,9]. $\hat{K}_{p,q}$ - factorization of K_{n_1,n_2} has been studied by Martin[4]. Ushio[12] gives necessary and sufficient conditions for a $\hat{K}_{p,q}$ - factorization of $K^*_{n_1,n_2}$. For graph theoretical terms, see [1,3].

2. $\tilde{S}_k$ - factorization of $K^*_{n_1,n_2,n_3}$

We use the following notation.

Notation. Given an $\tilde{S}_k$ - factorization of $K^*_{n_1,n_2,n_3}$, let
r be the number of factors
t be the number of components of each factor
b be the total number of components.
Among r components having vertex x in V_i, let r_{ij} be the number of components whose center-vertex is in V_j .

We give the following necessary conditions for the existence of an $\tilde{S}_k$ - factorization of $K^*_{n_1,n_2,n_3}$.

Theorem 1. If $K^*_{n_1,n_2,n_3}$ has an $\tilde{S}_k$ - factorization, then (i) k is even, $k \geq 4$ and (ii) $n_1 = n_2 = n_3 \equiv 0 \pmod{k(k-1)/3}$ for $k \equiv 0 \pmod 6$ and $n_1 = n_2 = n_3 \equiv 0 \pmod{k(k-1)}$ for $k \equiv 2, 4 \pmod 6$.

Proof. Suppose that $K^*_{n_1,n_2,n_3}$ has an $\tilde{S}_k$ - factorization. Then $b = 2(n_1n_2 + n_1n_3 + n_2n_3)/(k-1), t = (n_1 + n_2 + n_3)/k, r = b/t = 2(n_1n_2 + n_1n_3 + n_2n_3)k/(n_1 + n_2 + n_3)(k-1)$. By the definition of $\tilde{S}_k$, k is even and $k \geq 4$.
For a vertex x in V_1, we have $r_{11}(k-1) = n_2 + n_3$, $r_{12} = n_2$, $r_{13} = n_3$, and $r_{11} + r_{12} + r_{13} = r$. For a vertex x in V_2, we have $r_{22}(k-1) = n_1 + n_3$, $r_{21} = n_1$, $r_{23} = n_3$, and $r_{21} + r_{22} + r_{23} = r$. For a vertex x in V_3, we have $r_{33}(k-1) = n_1 + n_2$, $r_{31} = n_1$, $r_{32} = n_2$, and $r_{31} + r_{32} + r_{33} = r$. Therefore, we have $n_1 = n_2 = n_3$. Put $n_1 = n_2 = n_3 = n$. Then $r_{11} = r_{22} = r_{33} = 2n/(k-1)$, $r_{12} = r_{13} = r_{21} = r_{23} = r_{31} = r_{32} = n$, $b = 6n^2/(k-1)$, $t = 3n/k$, $r = 2nk/(k-1)$.
Since k is even and $k \geq 4$, we must have $3n \equiv 0 \pmod k$ and $n \equiv 0 \pmod{k-1}$. Therefore, we have $n \equiv 0 \pmod{k(k-1)/3}$ for $k \equiv 0 \pmod 6$ and $n \equiv 0 \pmod{k(k-1)}$ for $k \equiv 2, 4 \pmod 6$.

We prove the following extension theorem, which we use later in this paper.

Theorem 2. If $K^*_{n,n,n}$ has an $\tilde{S}_k$ - factorization, then $K^*_{sn,sn,sn}$ has an $\tilde{S}_k$ - factorization.

Proof. Let $K_{q_1,q_2\oplus q_3}$ denote the tripartite digraph with partite sets U_1,U_2,U_3 of q_1, q_2, q_3 vertices such that q_1 start-vertices in U_1 are adjacent to both q_2 end-vertices in U_2 and q_3 end-vertices in U_3. Then $\tilde{S}_k$ can be denoted by $K_{1,a\oplus(a+1)}$ for $k = 2a+2$. When $K^*_{n,n,n}$ has an $\tilde{S}_k$ - factorization, $K^*_{sn,sn,sn}$ has a $K_{s,sa\oplus s(a+1)}$ - factorization. $K_{s,sa\oplus s(a+1)}$ has an $\tilde{S}_k$ - factorization. Therefore, $K^*_{sn,sn,sn}$ has an $\tilde{S}_k$ - factorization.

We give the following sufficient conditions for the existence of an $\tilde{S}_k$ - factorization of $K^*_{n,n,n}$.

Theorem 3. When k is even, $k \geq 4$ and $n \equiv 0 \pmod{k(k-1)}$, $K^*_{n,n,n}$ has an $\tilde{S}_k$ - factorization.

Proof. Put $n = k(k-1)s$, $N = k(k-1)$. When $s = 1$, let $V_1 = \{1, 2, \ldots, N\}$, $V_2 = \{1', 2', \ldots, N'\}$, $V_3 = \{1'', 2'', \ldots, N''\}$. For $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, k$, construct $2k^2$ $\tilde{S}_k$ - factors $F^{(1)}_{ij}$, $F^{(2)}_{ij}$ as follows:

$F^{(1)}_{ij}$= { $((A+1);(B+(k-1)+1,\ldots,B+(k-1)+(k-2)/2)', (C+(k-1)+(k-2)/2+1,\ldots,C+2(k-1))'')$
$((A+2);(B+2(k-1)+1,\ldots,B+2(k-1)+(k-2)/2)', (C+2(k-1)+(k-2)/2+1,\ldots,C+3(k-1))'')$
...
$((A+k-1);(B+(k-1)^2+1,\ldots,B+(k-1)^2+(k-2)/2)', (C+(k-1)^2+(k-2)/2+1,\ldots,C+k(k-1))'')$
$((B+1)';(C+(k-1)+1,\ldots,C+(k-1)+(k-2)/2'', (A+(k-1)+(k-2)/2+1,\ldots,A+2(k-1)))$
$((B+2)';(C+2(k-1)+1,\ldots,C+2(k-1)+(k-2)/2)'', (A+2(k-1)+(k-2)/2+1,\ldots,A+3(k-1)))$
...
$((B+k-1)';(C+(k-1)^2+1,\ldots,C+(k-1)^2+(k-2)/2)'', (A+(k-1)^2+(k-2)/2+1,\ldots,A+k(k-1)))$
$((C+1)'';(A+(k-1)+1,\ldots,A+(k-1)+(k-2)/2), (B+(k-1)+(k-2)/2+1,\ldots,B+2(k-1))')$
$((C+2)'';(A+2(k-1)+1,\ldots,A+2(k-1)+(k-2)/2), (B+2(k-1)+(k-2)/2+1,\ldots,B+3(k-1))')$
...
$((C+k-1)'';(A+(k-1)^2+1,\ldots,A+(k-1)^2+(k-2)/2), (B+(k-1)^2+(k-2)/2+1,\ldots,B+k(k-1))')$ },

$F^{(2)}_{ij}$= { $((A+1);(C+(k-1)+1,\ldots,C+(k-1)+(k-2)/2)'', (B+(k-1)+(k-2)/2+1,\ldots,B+2(k-1))')$
$((A+2);(C+2(k-1)+1,\ldots,C+2(k-1)+(k-2)/2)'', (B+2(k-1)+(k-2)/2+1,\ldots,B+3(k-1))')$
...
$((A+k-1);(C+(k-1)^2+1,\ldots,C+(k-1)^2+(k-2)/2)'', (B+(k-1)^2+(k-$

$2)/2+1,\dots,B+k(k-1))')$
$((B+1)';(A+(k-1)+1,\dots,A+(k-1)+(k-2)/2),\ (C+(k-1)+(k-2)/2+1,\dots,C+2(k-1))'')$
$((B+2)';(A+2(k-1)+1,\dots,A+2(k-1)+(k-2)/2),\ (C+2(k-1)+(k-2)/2+1,\dots,C+3(k-1))'')$
...
$((B+k-1)';(A+(k-1)^2+1,\dots,A+(k-1)^2+(k-2)/2),\ (C+(k-1)^2+(k-2)/2+1,\dots,C+k(k-1))'')$
$((C+1)'';(B+(k-1)+1,\dots,B+(k-1)+(k-2)/2)',\ (A+(k-1)+(k-2)/2+1,\dots,A+2(k-1)))$
$((C+2)'';(B+2(k-1)+1,\dots,B+2(k-1)+(k-2)/2)',\ (A+2(k-1)+(k-2)/2+1,\dots,A+3(k-1)))$
...
$((C+k-1)'';(B+(k-1)^2+1,\dots,B+(k-1)^2+(k-2)/2)',\ (A+(k-1)^2+(k-2)/2+1,\dots,A+k(k-1)))\ \}$,
where $A=(i-1)(k-1)$, $B=(j-1)(k-1)$, $C=(i+j-2)(k-1)$, and the additions are taken modulo N with residues $1,2,\dots,N$.
Then we claim that they comprise an $\tilde{S}_k$ - factorization of $K^*_{N,N,N}$.
We can see that each of them is an $\tilde{S}_k$ - factor, because it spans all vertices of $K^*_{N,N,N}$.
We show that they are arc-disjoint.
Suppose that they are not arc-disjoint. In the following, we consider $A=(i-1)(k-1)$, $B=(j-1)(k-1)$, $C=(i+j-2)(k-1)$, $D=(h-1)(k-1)$, $E=(l-1)(k-1)$, $F=(h+l-2)(k-1)$, $1\le i,j,h,l\le k$. Note that A,B,C,D,E,F,N are integral multiples of $k-1$.
Let (X,Y') be an arc joining from V_1 to V_2 and let x and y be the residues of X and Y modulo $k-1$, respectively. Then the arc (X,Y') can appear only in the x-th components of $F^{(1)}_{ij}$,$F^{(2)}_{hl}$ according as $1\le y\le (k-2)/2$,$(k-2)/2+1\le y\le k-1$, respectively.
First, we assume that the common arc joining from V_1 to V_2 appears in the x-th component $((A+x);(B+x(k-1)+1,\dots,B+x(k-1)+(k-2)/2)',(C+x(k-1)+(k-2)/2+1,\dots,C+(x+1)(k-1))'')$ of $F^{(1)}_{ij}$ and the x-th component $((D+x);(E+x(k-1)+1,\dots,E+x(k-1)+(k-2)/2)',(F+x(k-1)+(k-2)/2+1,\dots,F+(x+1)(k-1))'')$ of $F^{(1)}_{hl}$.
Say $((A+x),(B+x(k-1)+y)')=((D+x),(E+x(k-1)+y)')$, where $1\le y\le (k-2)/2$. Then $A+x\equiv D+x \pmod N$ and $B+x(k-1)+y\equiv E+x(k-1)+y \pmod N$. From the congruences, we have $A=D$ and $B=E$, which implies $i=h$ and $j=l$. This contradicts the assumption.
Next, we assume that the common arc joining from V_1 to V_2 appears in the x-th component $((A+x);(C+x(k-1)+1,\dots,C+x(k-1)+(k-2)/2)'',(B+x(k-1)+(k-2)/2+1,\dots,B+(x+1)(k-1))')$ of $F^{(2)}_{ij}$ and the x-th component $((D+x);(F+x(k-1)+1,\dots,F+x(k-1)+(k-2)/2)'',(E+x(k-1)+(k-2)/2+1,\dots,E+(x+1)(k-1))'')$ of $F^{(2)}_{hl}$.
Say $((A+x),(B+x(k-1)+y)')=((D+x),(E+x(k-1)+y)')$, where $(k-2)/2+1\le y\le k-1$.

Then $A+x \equiv D+x \pmod N$ and $B+x(k-1)+y \equiv E+x(k-1)+y \pmod N$. From the congruences, we have $A=D$ and $B=E$, which implies $i=h$ and $j=l$. This contradicts the assumption.
Thus, there is no common arc joining from V_1 to V_2.
Similarly, there are no common arcs joining from V_1 to V_3, from V_2 to V_1, from V_2 to V_3, from V_3 to V_1, or from V_3 to V_2.
Therefore, $2k^2$ $\tilde{S}_k$ - factors $F_{ij}^{(1)}$, $F_{ij}^{(2)}$ comprise an $\tilde{S}_k$ - factorization of $K^*_{N,N,N}$. Applying Theorem 2, $K^*_{n,n,n}$ has an $\tilde{S}_k$ - factorization.

Theorem 4. When $k \equiv 0 \pmod 6$ and $n \equiv 0 \pmod{k(k-1)/3}$, $K^*_{n,n,n}$ has an $\tilde{S}_k$ - factorization.

Proof. Put $k=6p$, $n=2p(6p-1)s$, $N=2p(6p-1)$. When $s=1$, let $V_1=\{1,2,\dots,N\}$, $V_2=\{1',2',\dots,N'\}$, $V_3=\{1'',2'',\dots,N''\}$. For $i=1,2,\dots,2p$ and $j=1,2,\dots,2p$, construct $24p^2$ $\tilde{S}_k$ - factors $F_{ij}^{(1)}$, $F_{ij}^{(2)}$, $F_{ij}^{(3)}$, $F_{ij}^{(4)}$, $F_{ij}^{(5)}$, $F_{ij}^{(6)}$ as follows:
First, construct $F_{ij}^{(1)}$.
$F_{ij}^{(1)} = \{((A+x);\ (B+(6p-1)p-(x-1)(3p-1)+3p+1,\dots,B+(6p-1)p-(x-1)(3p-1)+6p-1),\ (C+(6p-1)+(x-1)3p+1,\dots,C+(6p-1)+(x-1)3p+3p)) \mid 1 \le x \le 2p\}$
$\cup\,\{((B+x);\ (C+(6p-1)(p+1)+(x-2p-1)(3p-1)+p+1,\dots,C+(6p-1)(p+1)+(x-2p-1)(3p-1)+4p-1),\ (A+(6p-1)p-(x-2p-1)3p-2p+1,\dots,A+(6p-1)p-(x-2p-1)3p+p)) \mid 2p+1 \le x \le 3p\}$
$\cup\,\{((B+x);\ (C+(6p-1)(p+1)+(x-3p-1)3p+3p^2+1,\dots,C+(6p-1)(p+1)+(x-3p-1)3p+3p^2+3p),\ (A-(x-3p-1)(3p-1)+3p^2-3p+2,\dots,A-(x-3p-1)(3p-1)+3p^2)) \mid 3p+1 \le x \le 4p-1\}$
$\cup\,\{(C+x);\ (A+(6p-1)p+p+1,\dots,A+(6p-1)p+4p-1),\ (B+(6p-1)-2p+1,\dots,B+(6p-1)+p)) \mid x=4p\}$
$\cup\,\{((C+x);\ (A+(6p-1)2p-(x-4p-1)3p-3p+1,\dots,A+(6p-1)2p-(x-4p-1)3p),\ (B+(6p-1)(p+1)+(x-4p-1)(3p-1)+1,\dots,B+(6p-1)(p+1)+(x-4p-1)(3p-1)+3p-1)) \mid 4p+1 \le x \le 6p-2\}$
$\cup\,\{((C+x);\ (A+2p+1,\dots,A+4p-1,A+(6p-1)p+4p,\dots,A+(6p-1)p+5p),\ (B+(6p-1)2p-p+2,\dots,B+(6p-1)2p+2p)) \mid x=6p-1\}$,
where $(A+u)$,$(B+u)$,$(C+u)$ mean $(A+u)$,$(B+u)'$,$(C+u)''$, respectively, and $A=(i-1)(6p-1)$, $B=(j-1)(6p-1)$, $C=(i+j-2)(6p-1)$, and the additions are taken modulo N with residues $1,2,\dots,N$.
Next, construct $F_{ij}^{(2)}$, $F_{ij}^{(3)}$, $F_{ij}^{(4)}$, $F_{ij}^{(5)}$, $F_{ij}^{(6)}$ by applying all possible permutations of A,B,C in $F_{ij}^{(1)}$.
Then we claim that they comprise an $\tilde{S}_k$ - factorization of $K^*_{N,N,N}$.
We can see that each of them is an $\tilde{S}_k$ - factor, because it spans all vertices of $K^*_{N,N,N}$.
We show that they are arc-disjoint.
Suppose that they are not arc-disjoint. Let (X,Y') be a common arc joining from V_1 to V_2 and let x be the residue of X modulo $6p-1$. Then the arc (X,Y') can appear only in the x-th components of $F_{ij}^{(1)}$, $F_{ij}^{(2)}$, $F_{ij}^{(3)}$, $F_{ij}^{(4)}$, $F_{ij}^{(5)}$, or $F_{ij}^{(6)}$. But in the same

way as the proof of Theorem 3, there is no common arc in those components. Thus, there is no common arc joining from V_1 to V_2.
Similarly, there is no common arc joining from V_1 to V_3, from V_2 to V_1, from V_2 to V_3, from V_3 to V_1, or from V_3 to V_2. Therefore, those $24p^2$ $\tilde{S}_k$ - factors $F_{ij}^{(1)}$, $F_{ij}^{(2)}$, $F_{ij}^{(3)}$, $F_{ij}^{(4)}$, $F_{ij}^{(5)}$, $F_{ij}^{(6)}$ are arc-disjoint.
The total number of arcs in the factors is equal to the total number of arcs in $K^*_{N,N,N}$. So the factors do indeed constitute a decomposition of $K^*_{N,N,N}$ and comprise an $\tilde{S}_k$ - factorization of $K^*_{N,N,N}$. Applying Theorem 2, $K^*_{n,n,n}$ has an $\tilde{S}_k$ - factorization.

We give the following example of Theorem 4.

Example. *An $\tilde{S}_6$ - factorization of $K^*_{10,10,10}$.* We have 24 $\tilde{S}_6$ - factors as follows:

$$F_{11}^{(1)} = \{(1;9',10',6'',7'',8'')(2;7',8',1'',9'',10'')(3';2'',3'',4,5,6)(4'';7,8,4',5',6')(5'';3,9,10,1',2')\}$$
$$F_{11}^{(2)} = \{(1';9'',10'',6,7,8)(2';7'',8'',1,9,10)(3'';2,3,4',5',6')(4;7',8',4'',5'',6'')(5;3',9',10',1'',2'')\}$$
$$F_{11}^{(3)} = \{(1'';9,10,6',7',8')(2'';7,8,1',9',10')(3;2',3',4'',5'',6'')(4';7'',8'',4,5,6)(5';3'',9'',10'',1,2)\}$$
$$F_{11}^{(4)} = \{(1;9'',10'',6',7',8')(2;7'',8'',1',9',10')(3'';2',3',4,5,6)(4';7,8,4'',5'',6'')(5';3,9,10,1'',2'')\}$$
$$F_{11}^{(5)} = \{(1';9,10,6'',7'',8'')(2';7,8,1'',9'',10'')(3;2'',3'',4',5',6')(4'';7',8',4,5,6)(5'';3',9',10',1,2)\}$$
$$F_{11}^{(6)} = \{(1'';9',10',6,7,8)(2'';7',8',1,9,10)(3';2,3,4'',5'',6'')(4;7'',8'',4',5',6')(5;3'',9'',10'',1',2')\}$$

$$F_{12}^{(1)} = \{(1;4',5',1'',2'',3'')(2;2',3',4'',5'',6'')(8';7'',8'',4,5,6)(9'';7,8,1',9',10')(10'';3,9,10,6',7')\}$$
$$F_{12}^{(2)} = \{(1';4'',5'',1,2,3)(2';2'',3'',4,5,6)(8'';7,8,4',5',6')(9;7',8',1'',9'',10'')(10;3',9',10',6'',7'')\}$$
$$F_{12}^{(3)} = \{(1'';4,5,1',2',3')(2'';2,3,4',5',6')(8;7',8',4'',5'',6'')(9';7'',8'',1,9,10)(10';3'',9'',10'',6,7)\}$$
$$F_{12}^{(4)} = \{(1;4'',5'',1',2',3')(2;2'',3'',4',5',6')(8'';7',8',4,5,6)(9';7,8,1'',9'',10'')(10';3,9,10,6'',7'')\}$$
$$F_{12}^{(5)} = \{(1';4,5,1'',2'',3'')(2';2,3,4'',5'',6'')(8;7'',8'',4',5',6')(9'';7',8',1,9,10)(10'';3',9',10',6,7)\}$$
$$F_{12}^{(6)} = \{(1'';4',5',1,2,3)(2'';2',3',4,5,6)(8';7,8,4'',5'',6'')(9;7'',8'',1',9',10')(10;3'',9'',10'',6',7')\}$$

$$F_{21}^{(1)} = \{(6;9',10',1'',2'',3'')(7;7',8',4'',5'',6'')(3';7'',8'',1,9,10)(9'';2,3,4',5',6')(10'';4,5,8,1',2')\}$$
$$F_{21}^{(2)} = \{(6';9'',10'',1,2,3)(7';7'',8'',4,5,6)(3'';7,8,1',9',10')(9;2',3',4'',5'',6'')(10;4',5',8',1'',2'')\}$$
$$F_{21}^{(3)} = \{(6'';9,10,1',2',3')(7'';7,8,4',5',6')(3;7',8',1'',9'',10'')(9';2'',3'',4,5,6)(10';4'',5'',8'',1,2)\}$$
$$F_{21}^{(4)} = \{(6;9'',10'',1',2',3')(7;7'',8'',4',5',6')(3'';7',8',1,9,10)(9';2,3,4'',5'',6'')(10';4,5,8,1'',2'')\}$$
$$F_{21}^{(5)} = \{(6';9,10,1'',2'',3'')(7';7,8,4'',5'',6'')(3;7'',8'',1',9',10')(9'';2',3',4,5,6)(10'';4',5',8',1,2)\}$$
$$F_{21}^{(6)} = \{(6'';9',10',1,2,3)(7'';7',8',4,5,6)(3';7,8,1'',9'',10'')(9;2'',3'',4',5',6')(10;4'',5'',8'',1',2')\}$$

$$F_{22}^{(1)} = \{(6;4',5',6'',7'',8'')(7;2',3',1'',9'',10'')(8';2'',3'',1,9,10)(4'';2,3,1',9',10')(5'';4,5,8,6',7')\}$$
$$F_{22}^{(2)} = \{(6';4'',5'',6,7,8)(7';2'',3'',1,9,10)(8'';2,3,1',9',10')(4;2',3',1'',9'',10'')(5;4',5',8',6'',7'')\}$$
$$F_{22}^{(3)} = \{(6'';4,5,6',7',8')(7'';2,3,1',9',10')(8;2',3',1'',9'',10'')(4';2'',3'',1,9,10)(5';4'',5'',8'',6,7)\}$$
$$F_{22}^{(4)} = \{(6;4'',5'',6',7',8')(7;2'',3'',1',9',10')(8'';2',3',1,9,10)(4';2,3,1'',9'',10'')(5';4,5,8,6'',7'')\}$$
$$F_{22}^{(5)} = \{(6';4,5,6'',7'',8'')(7';2,3,1'',9'',10'')(8;2'',3'',1',9',10')(4'';2',3',1,9,10)(5'';4',5',8',6,7)\}$$
$$F_{22}^{(6)} = \{(6'';4',5',6,7,8)(7'';2',3',1,9,10)(8';2,3,1'',9'',10'')(4;2'',3'',1',9',10')(5;4'',5'',8'',6',7')\}$$

We have the following main theorem.

Main Theorem. $K^*_{n_1,n_2,n_3}$ has an $\tilde{S}_k$ - factorization if and only if (i) k is even, $k \geq 4$ and (ii) $n_1 = n_2 = n_3 \equiv 0 \pmod{k(k-1)/3}$ for $k \equiv 0 \pmod 6$ and $n_1 = n_2 = n_3 \equiv 0 \pmod{k(k-1)}$ for $k \equiv 2, 4 \pmod 6$.

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(Received 7/7/98)